Problem 1
Consider a system consisting of a spin 1/2 particle and a spin 3/2 particle governed by the Hamiltonian
\[ H = a \vec{S}_1 \cdot \vec{S}_2 \]
where \( \vec{S}_1 \) and \( \vec{S}_2 \) are the two spin operators.

(a) Find the energy levels of the system and their degeneracies.

(b) Express the eigenvectors of \( H \) in terms of the common eigenvectors of \( \{ \vec{S}_2, S_{1z}, \vec{S}_2, S_{2z} \} \).

Problem 2
Consider a two-level system governed by the Hamiltonian
\[ H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \]
where \( E_1 < E_2 \).

Apply a perturbation \( \lambda W \), where
\[ W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

The Hamiltonian of the system is now
\[ H = H_0 + \lambda W \]

Assume \( \lambda \ll E_2 - E_1 \).

(a) Find the exact energy levels of the perturbed system (eigenvalues of \( H \)) and corresponding eigenvectors.

(b) Use second order perturbation theory to calculate the energy levels to second order in \( \lambda \) and corresponding eigenvectors to first order in \( \lambda \). Compare your results to the exact expressions obtained in part (a).
Problem 3

A particle of mass $m$ is moving in the $x$-direction under the influence of the potential

$$V(x) = g|x|, \quad g > 0$$

Estimate the ground state energy $E_0$ by using the variational method with the trial function

$$\phi_\alpha(x) = \begin{cases} \alpha - |x|, & |x| < \alpha \\ 0, & |x| > \alpha \end{cases}$$

Compare your result with the exact value

$$E_0 = a \left( \frac{\hbar^2 g^2}{2m} \right)^{1/3}, \quad a = 1.019 \ldots$$

[CAUTION: $\phi_\alpha''(x)$ is not defined when $\phi_\alpha'(x)$ is discontinuous. Integrate by parts to get rid of second derivatives before you evaluate any integrals.]

Problem 4

An electron-positron pair is created. They are both spin 1/2 particles. Suppose that the system has total spin $S = 0$ and the two particles travel in opposite directions. Observer A measures the spin of the electron whereas observer B measures the spin of the positron.

(a) What is the state of the system?

(b) If B makes no measurement, calculate the probability that A will find the spin of the electron to be pointing in the positive $z$-direction.

(c) If B makes a measurement and finds that the spin of the positron is in the direction of the unit vector

$$\hat{n} = \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix}$$

calculate the probability that $A$ will find the spin of the electron to be in the positive $z$-direction.