UNIT 5

Time-dependent perturbation theory

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Introduction and Method

In the former chapter, we talked about the Time-independent Perturbation Theory. In this chapter, we will work on a more complicated problem, the Time-dependent Perturbation Theory. Now let’s consider the same problem as we did in the chapter three: a system whose Hamiltonian can be expressed in the form

\[ H_0|n> = E_n|n> \]  

(1)

\( H_0 \) is the unperturbed Hamiltonian like what we did before. However, the perturbation in this case is \( \lambda W(t) \), which depends on time. So if we want to solve the Schrodinger equation, we will have the equation:

\[ H = H_0 + \lambda W(t) \]  

(2)

If we want to find the eigenvalue of \( H \), which is time dependent, we need to solve time-depended Schrodinger equation:

\[ H|\psi(t)> = i\hbar \frac{d}{dt}|\psi(t)> \]  

(3)

In this case, we have the initial condition \( t = 0 \), and we have the eigenstate of the original Hamiltonian:

\[ |\psi(0)> = |i> \]  

(4)

where \( i \) is some values of \( n \). After we measure the system at time \( t > 0 \), the system will be in a different state of Hamiltonian \( |f> \), where the \( f \) is another number of \( n \). If we are given the initial state \( |i> \), the probability of the transition to \( |f> \) is:

\[ P_{if}(t) = |<f|\psi(t)>|^2 \]  

(5)
So, let’s calculate the probability and solve the Schrodinger equation. First we need to split the function $\psi$ into some states, which is the eigenstate of the Hamiltonian. Therefore we can have

$$|\psi(t)\rangle = \sum_n c_n(t)|n\rangle$$

where we have the coefficients $c_n$ are:

$$C_n(t) = \langle n|\psi(t)\rangle$$

Then we go back to the Schrodinger equation, and split the Hamiltonian $H$. We will have:

$$H_0|\psi\rangle + \lambda w|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle$$

Since the equation 6 defined the $|\psi\rangle$ before, so we can have

$$\sum_n c_n E_n(t)|n\rangle + \lambda \sum_n c_n w|n\rangle = i\hbar \sum_n \frac{dc_n}{dt}|n\rangle$$

Next, we can select one state of $n$, which we can write as $|k\rangle$, to multiply the equation 9, so we can have:

$$C_k E_k + \lambda \sum_n c_n \langle k|w|n\rangle = i\hbar \frac{dc_k}{dt}$$

In this case, the $w$ is a matrix which we can write as $w_k n$. Let’s simplify the complex equation. Firstly, let assume $\lambda = 0$, which means there is no perturbation, so we have:

$$i\hbar \frac{dc_k}{dt} = E_k c_k$$

We can solve this equation seriately. Then we have:

$$C_k(t) = C_k(0)e^{-i\frac{E_k}{\hbar}t}$$

Next, we calculate the condition that $\lambda \neq 0$. Then we have

$$C_k(t) = b_k(t)e^{-i\frac{E_k}{\hbar}t}$$

In this case, we got a complex function. However, we can separate the complex function to two parts. The first part is $b_k(t)$, which we don’t know, while the second part $e^{-i\frac{E_k}{\hbar}t}$ that we have already talked about in former chapters. We substitute the equation 13 into 10, then we can get:

$$C_k(t) = b_k(t)e^{-i\frac{E_k}{\hbar}t} + \lambda \sum_n W_{kn}b_n e^{-i\frac{E_n}{\hbar}t}$$

which equals to

$$i\hbar \left[ \frac{db_k}{dt} e^{-i\frac{E_k}{\hbar}t} + b_k(-\frac{i E_k}{\hbar}) e^{-i\frac{E_k}{\hbar}t} \right]$$
and then we can get
\[ b_k E_k + \lambda \sum_n W_{kn} b_n e^{-i\frac{E_n - E_k}{\hbar}t} = i\hbar \frac{db_k}{dt} - \frac{i}{\hbar} E_k b_k \] (16)
so we can get:
\[ \frac{db_k}{dt} = \lambda i\hbar \sum_n W_{kn} e^{-i\frac{E_k - E_n}{\hbar}t} b_n \] (17)
This is the exact equation which we didn’t use any approximation, and this equation is equivalent to the Schrodinger equation. In addition, if we write \( \frac{E_k - E_n}{\hbar} = \omega_{kn} \), then we can have the frequency \( \omega_{kn} \), which we get from the transition between state \( k \) to state \( n \). We call the \( \omega_{kn} \) Born Frequency.

**Approximate Solution Using Perturbation Theory**

In last section, we calculate the exact solution of the Schrodinger equation for a time-dependent perturbation question. However, the equation is very complicated and we can not solve it in many cases. In this section, we find the approximation solution through perturbation theory. Firstly, under the perturbation theory, we will writer the equation \( b_n(t) \) in the form:
\[ b_n(t) = b_n^0(t) + \lambda b_n^1(t) + \lambda^2 b_n^2(t) + \ldots \] (18)
First, we have to look the terms in the equation when there is no \( \lambda \), then we can have the \( \lambda^0 \), which means the 0th order, term:
\[ \frac{db_k^0}{dt} = 0 \] (19)
Then we know that \( b_k^0 \) is a constant. In this case, there is only one term which we write as \( b_k^0 \) equal to 1. Other \( b_k^0 \) where \( k \neq i \) will be 0. So if we write equation of \( \psi \) at \( t = 0 \), then we have:
\[ |\psi(0)\rangle = \sum_n b_n^0 |n\rangle = |i\rangle \] (20)
Now we consider the first order, we can get:
\[ \lambda^1 = i\hbar \frac{db_k^1}{dt} = e^{iW_{ki}t} W_{ki} \] (21)
Form the former equation, we can integrate the \( b_k^1(t) \) as
\[ b_k^1(t) = \frac{1}{i\hbar} \int_0^t dt' e^{iW_{ki}t'} W_{ki}(t') \] (22)
In this case, when \( t = 0 \), we have \( b_k^0(t) = 0 \). Because \( b_k^0 \) equal to 1, Other \( b_k^0 \) where \( k \neq i \) will be 0. That is also the reason why all the other coefficients will be vanish at \( t = 0 \). Since we have get the \( b_n \), then we can calculate the \( c_n \) and function \(|\psi|\).

\[
|\psi(t)| = \sum_n b_n(t)e^{-iE_nt/\hbar} |n| >
\]  \hspace{1cm} (23)

Then we can substitute the \( \lambda^0 \) and \( \lambda^1 \) we get before into this equation, we will have:

\[
|\psi(t)| = e^{-iE_0t/\hbar} |i| + \lambda \sum_n b_n^1(t)e^{-iE_nt/\hbar} |n| > ...
\]  \hspace{1cm} (24)

So, if we measure the system at \( t > 0 \), I will find the system will stay at the state \(|f| >\). Then we have probability of the transition from \(|i| >\) to \(|f| >\) is \( P_{if}(t) = |<f|\psi(t)|i>|^2 \), and it equals to

\[
P_{if}(t) = \lambda^2 \sum_n b_n^1(t)e^{-iE_nt/\hbar} <f|n| >^2 = \lambda^2 |b_f^1(t)e^{-iE_f t/\hbar}|^2
\]  \hspace{1cm} (25)

As what we did for the Born frequency, we can write the probability as:

\[
P_{if}(t) = \frac{\lambda^2}{\hbar^2} \int_0^t dt' e^{iW_{if}t'} W_{if}(t')
\]  \hspace{1cm} (26)

where we have \( W_{if} = \frac{E_f - E_i}{\hbar} \). That is the probability for the first order. Now we will work on some examples.

**Example 1:** \( W(t) = A \sin(\omega t) \)

In this case, we will have the probability of \( P_{if}(t) \) as:

\[
P_{if}(t) = \frac{|A|^2}{\hbar^2} \int_0^t dt' e^{iW_{if}t'} \sin(\omega t')^2
\]  \hspace{1cm} (27)

\[
= \frac{|A|^2}{\hbar^2} \left[ \frac{e^{i(\omega+\omega_f)t} + e^{-i(\omega-\omega_f)t}}{2(\omega + \omega_f)} + \frac{e^{-i(\omega+\omega_f)t} - e^{-i(\omega-\omega_f)t}}{2(\omega - \omega_f)} \right]^2
\]

**Example 2:** \( W(t) = A \cos(\omega t) \)

This example is very familiar with the example 1, so we will have:

\[
P_{if}(t) = \frac{|A|^2}{4\hbar^2} \left[ \frac{e^{i(\omega+\omega_f)t} - e^{i(\omega-\omega_f)t}}{\omega + \omega_f} + \frac{e^{-i(\omega+\omega_f)t} - e^{-i(\omega-\omega_f)t}}{\omega - \omega_f} \right]^2
\]  \hspace{1cm} (28)

**Example 3:** \( W(t) = A \)

In this case, the perturbation term equals to a constant. We will have:

\[
P_{if}(t) = \frac{|A|^2}{4\hbar^2} \left[ \frac{e^{i\omega_f t} - 1}{\omega_f} + \frac{e^{-i\omega_f t} - 1}{\omega_f} \right]^2 = \frac{|A|^2}{\hbar^2} \left[ \frac{e^{i\omega_f t} - 1}{\omega_f} \right]^2 = \frac{|A|^2}{\hbar^2} \left[ \frac{e^{i\omega_f t} - e^{-i\omega_f t}}{\omega_f} \right]^2
\]  \hspace{1cm} (29)
so we can have \( P_{fi} = \frac{4|A|^2 \sin^2(\frac{\omega_{fi} t}{2})}{\hbar^2} \).

In the example 2, we have the probability of the transition is \( P_{fi}(t) = \frac{|A|^2}{4\hbar^2} \left[ \frac{e^{i(\omega + \omega_{fi})t} - 1}{\omega + \omega_{fi}} \right]^2 \) which we can write as \( P_{fi} = \frac{|A|^2}{4\hbar^2} \left| A_+ + A_- \right|^2 \)

So if we ignore the second term \((A_-)\), we can calculate the rest part through the same way we used in the example 3. Then we can get:

\[
P_+ = \frac{|A|^2}{4\hbar^2} \left| A_+ \right|^2 = \frac{|A|^2}{4\hbar^2} \left| A |^2 \sin^2 \frac{\omega_{fi} + \omega}{2} \frac{\omega_{fi} + \omega}{2 (\omega_{fi} + \omega)^2} \right| (30)
\]

Next, if we ignore the fist term \( A_+ \), then we will have

\[
P_- = \frac{|A|^2}{4\hbar^2} \left| A_- \right|^2 = \frac{|A|^2}{4\hbar^2} \left| A |^2 \sin^2 \frac{\omega_{fi} - \omega}{2} \frac{\omega_{fi} - \omega}{2 (\omega_{fi} - \omega)^2} \right| (31)
\]

In this case, if we assume the state is near resonance, which means the distance from the resonance is much smaller than the resonance, we will have \(|\omega - \omega_{fi}| \ll \omega_{fi}\), then we get \( P_+ \approx 0 \) and \( P_{fi} = P_- \). The width \( \Delta \omega = \frac{4\pi}{T} \ll 2\omega_{fi} \), so we have \( t \gg \frac{2\pi}{\omega_{fi}} \approx \frac{1}{\omega_{fi}} \). This means we need to wait long enough, which is much longer than a couple of oscillations, to the resonance. However, on the other hand, I can not wait too long, for the probability can not bigger than 1. So we have \( P_{fi} \ll 1 \), and because \( \frac{|A|^2}{4\hbar^2} \ll 1 \), then we can get \( t \ll \frac{1}{|A|} \). At last, we can get \( \frac{1}{\omega_{fi}} \ll t \ll \frac{\hbar}{|A|} \). If we write it in difference of energy levels, we will have \( E_f - E_i = \hbar \omega_{fi} \gg |A| \).

Secular Approximation

In last section, we discussed the situation near the resonance, which means \( t \ll \frac{1}{|A|} \). On the other hand, if we have \( t \gg \frac{1}{|A|} \), the schrodinger equation will be:

\[
i\hbar \frac{db_k}{dt} = \sum_n e^{iW_{kn}t}W_{kn}b_n
\]

(32)

in this case, we assume \( \lambda = 1 \) in the equation. Then we the example we did in last section again. We assume \( W = A \sin(\omega t) \), then \( W_{kn} = A_{kn} \sin(\omega t) \), in which \( A_{kn} = \langle k|A|n \rangle \). If we go near the resonance, we will have the condition \( \omega \approx \omega_{fi} \). We assume the initial and final state as \( |i \rangle \) and \( |f \rangle \), and we only consider the situation when the system in these two states. Then we will have:

\[
i\hbar \frac{db_i}{dt} = A_{if}b_i \sin(\omega t) + e^{-iW_{if}t}A_{if}b_f \sin(\omega t)
\]

(33)

Another equation is:

\[
i\hbar \frac{db_f}{dt} = e^{iW_{if}t}A_{if}b_i \sin(\omega t) + A_{ff}b_f \sin(\omega t)
\]

(34)
Because we have \( \sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \), so we can have equations:

\[
\frac{i\hbar}{2} \frac{db_i}{dt} = \frac{1}{2i} e^{-i(\omega f_i - \omega)t} A_{if} b_f
\]

(35)

\[
\frac{i\hbar}{2} \frac{db_f}{dt} = -\frac{1}{2i} e^{i(\omega f_i - \omega)t} A_{fi} b_i
\]

(36)

If we calculate the second derivative, then we can get:

\[
\frac{i\hbar}{2} \frac{d^2 b_f}{dt^2} = \frac{1}{2} (\omega f_i - \omega) e^{i(\omega f_i - \omega)t} A_{fi} b_i - \frac{1}{2i} e^{i(\omega f_i - \omega)t} A_{fi} \frac{db_i}{dt}
\]

(37)

so we can transform the equation to:

\[
\frac{d^2 b_f}{dt^2} + i(\omega f_i - \omega) \frac{db_f}{dt} + \frac{1}{4\hbar} |A_{fi}|^2 b_f = 0
\]

(38)

If we assume \( b_f = e^{\lambda t} \), and we plug it into the former equation, then we have:

\[
\lambda^2 - i(\omega f_i - \omega) \lambda + \frac{|\lambda f_i|^2}{\hbar^2} = 0
\]

(39)

Then we can solve the equation to get the \( \lambda \) is:

\[
\lambda = \lambda_\pm = \frac{1}{2} [i(\omega f_i - \omega) \pm i \sqrt{(\omega f_i - \omega)^2 + \frac{|A_{fi}|^2}{\hbar^2}}]
\]

(40)

It is important to note that \( \lambda \) is pure imaginary, so that there is no damping. We have also seen that

\[
b_f = e^{\lambda_\pm t}
\]

(41)

so the most general solution will be a linear combination of the two.

\[
b_f(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}
\]

(42)

To fix the coefficients \( A \) and \( B \) we look at the initial conditions, \( t = 0 \). At \( t = 0 \) the initial state is the state \( |i > \) so that \( b_i(0) = 1 \) and \( b_f(0) = 0 \). The condition that \( b_f(0) = 0 \) tells us that \( A = -B \). The condition \( b_i(0) = 1 \) is useful if we know \( b_i(0) \). From our equations above

\[
\frac{i\hbar}{2} \frac{db_i}{dt} = \frac{1}{2i} e^{i(\omega f_i - \omega)t} A_{if} b_f
\]

(43)

and using (42) we can show that

\[
b_i(0) = \frac{2\hbar}{A_{fi}} (A\lambda_+ + B\lambda_-) = 1
\]

(44)
and
\[ A = \frac{A_{fi}}{2\hbar(\lambda_+ - \lambda_-)} = \frac{A_{fi}}{2i\hbar\sqrt{(\omega - \omega_{fi})^2 + |A_{fi}|^2}} \] (45)

This allows us to calculate the probability of a transition from the state \( |i \rangle \) to the state \( |f \rangle \). It is given by

\[ P_{if}(t) = |b_f(t)|^2 = |A|^2|e^{\lambda_+ t} - e^{\lambda_- t}|^2 \]
\[ = \frac{|A_{if}|^2}{\hbar^2(\omega - \omega_{fi})^2 + |A_{if}|^2} \sin^2\left(\frac{\sqrt{(\omega - \omega_{fi})^2 + |A_{if}|^2}t}{2}\right) \] (46)

Since we are near the resonance the terms \((\omega - \omega_{fi}) \approx 0\) so we can ignore them. This gives

\[ P_{if}(t) = \sin^2\left(\frac{|A_{if}|t}{2\hbar}\right) \] (47)

This is Rabi’s formula, which we studied last semester.

**Interaction of an atom with light**

When considering the interaction between an atom and an electromagnetic wave, we first choose a gauge. It is always possible to choose a gauge such that
\[ \phi = A_0 = 0, \] (48)

The vector potential is

\[ \vec{A}(\vec{r}, t) = \hat{A}_z \sin(\vec{k} \cdot \vec{r} - \omega t) \] (49)
\[ \omega = c|\vec{k}| \] (50)
\[ \vec{A}_z = A_z \hat{z} \] (51)
\[ \vec{k} = k\hat{y} \] (52)

The electric field is given by

\[ \vec{E} = -\vec{\nabla} \phi - \frac{d\vec{A}}{dt} = \omega A_z \cos(ky - \omega t)\hat{z} = E_0 \cos(ky - \omega t)\hat{z} \] (53)

and the magnetic field by

\[ \vec{B} = \vec{\nabla} \times \vec{A} = kA_z \cos(ky - \omega t)\hat{x} = B_0 \cos(ky - \omega t)\hat{x} \] (54)

so that

\[ \frac{E_0}{B_0} = \frac{\omega}{k} = c \] (55)

The time averaged Poynting vector is given by

\[ \langle \vec{S} \rangle = \frac{c\epsilon_0 E_0^2}{2} \hat{y} \] (56)
Interaction of an atom with light

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(58)

\[ \omega = c |\vec{k}| \]  

(59)

\[ A_z = A_z \hat{z} \]  

(60)

\[ \vec{k} = k \hat{y} \]  

(61)

The electric field is given by

\[ \vec{E} = -\nabla \phi - \frac{d\vec{A}}{dt} = \omega A_z \cos(ky - \omega t) \hat{z} = E_0 \cos(ky - \omega t) \hat{z} \]  

(62)

and the magnetic field by

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(63)

so that

\[ \frac{E_0}{B_0} = \frac{\omega}{k} = c \]  

(64)

The time averaged Poynting vector is given by

\[ < \vec{S} > = \frac{\epsilon_0 E_0^2}{2} \hat{y} \]  

(65)

Interaction with an Hydrogen Atom: Low-Intensity Limit

The hamiltonian for the electron in a Hydrogen atom interacting with an electromagnetic wave is:

\[ H = (\vec{p} - q \vec{A})^2 \frac{1}{2m} + V(r) - \frac{q}{m} \vec{S} \cdot \vec{B} - q \phi \]  

(66)

This can be rewritten as

\[ H = H_0 - \frac{q}{m} p_z A_z - \frac{q}{m} S_z B_0 + \frac{q^2 A_z^2}{2m} \]  

(67)

where \( H_0 \) is the atomic hamiltonian and

\[ W(t) = W_1 + W_2 + W_3 = -\frac{q}{m} p_z A_z + \frac{q^2 A_z^2}{2m} - \frac{q}{m} S_z B_0 \]  

(68)
is the perturbation. We will consider $W_1$ here since $W_1$ is much larger than $W_3$ or $W_2$. This can be seen by considering the order of the ratio

$$\frac{W_3}{W_1} \approx \frac{\hbar k}{p} \approx \frac{1}{1000}$$

(69)

for everyday light, and noting that at low intensity the quadratic terms of $A_z$ are insignificant. So,

$$H \approx H_0 + W_1 \approx H_0 - \frac{q}{m} \overrightarrow{p} \cdot \overrightarrow{A}$$

(70)

The Electric Dipole Hamiltonian

Now let us consider a gauge transformation given by

$$\overrightarrow{A}' = \overrightarrow{A} + \nabla \chi$$

(71)

$$\phi' = \phi - \frac{d\chi}{dt}$$

(72)

If we let

$$\chi = zA_0 \sin(\omega t)$$

(73)

then

$$\phi' = -zA_0 \omega \cos(\omega t)$$

(74)

$$A'_x = A'_y = 0$$

(75)

$$A'_z = A_0 \sin(ky - \omega t) + A_0 \sin(\omega t)$$

(76)

In the electric dipole approximation we have $ky \approx 0$ so that

$$A_z \approx 0$$

(77)

In this case the interaction hamiltonian is

$$H = H_0 - q\phi' = H_0 - qE_0 \omega \cos(\omega t) = H_0 - \overrightarrow{d} \cdot \overrightarrow{E} - \frac{q}{m} \overrightarrow{S} \cdot \overrightarrow{B}$$

(78)

Again, as before, we ignore the last term because its contribution is insignificant and we have

$$W_1 = -\overrightarrow{d} \cdot \overrightarrow{E}$$

(79)

$W_1$ is called the electric dipole hamiltonian. The gauge transformation used here illustrates that the $W_1$ we found before((70)) is, in fact, the electric dipole hamiltonian.
The Matrix Elements of the Electric Dipole Hamiltonian

In order to describe a transition from some initial state, \( i \), to some final state, \( f \),

\[
|i \rangle \rightarrow |f \rangle
\]  

we need to calculate the matrix elements

\[
<f | W_1 | i \rangle = \frac{qE_0}{m\omega} \sin(\omega t) < f | p_z | i \rangle
\]  

In order to calculate the matrix elements we first consider the commutator between \( H_0 \) and \( Z \). Using

\[
H_0 = \frac{p_z^2}{2m} + V(r)
\]

we get

\[
[Z, H_0] = i\hbar \frac{p_z}{m}
\]

On the other hand,

\[
<f | [z, H_0] | i \rangle = < f | zH_0 | i \rangle - < f | H_0 z | i \rangle = -(E_f - E_i) < f | z | i \rangle
\]  

Therefore

\[
<f | W_1 | i \rangle = iq \frac{\omega f_i}{\omega} E_0 \sin(\omega t) < f | z | i \rangle
\]

\[
\omega f_i = (E_f - E_i)/\hbar
\]

We see that the matrix elements of the electric dipole Hamiltonian are proportional to those of \( z \).

Electric Dipole Selection Rules

Now let us consider transitions in an hydrogen atom. The wavefunctions will be

\[
\phi_{nlm}(\vec{r}) = R_{ni}(r)Y_{lm}(\theta, \phi)
\]

The matrix elements can then be represented in integral form as

\[
<f | z | i \rangle = I \int d\Omega \overline{Y_{lj}^{m_j} * (\theta, \phi)Y_{l_i}^0(\theta, \phi)Y_{li}^{m_i}(\theta, \phi)}
\]

where

\[
I = \sqrt{\frac{4\pi}{3}} \int_0^\infty dr r^2 R_{n_l f}(r)R_{n_i l}(r)
\]
The angular integral, which was calculated in our discussion of “Addition of Spherical Harmonics” (see Cohen-Tannoudji), is nonzero if and only if
\[ \Delta l = l_f - l_i = \pm 1 \]  
\[ m_f = m_i \]  
(91) (92)
The equations (91) (92) are the selection rules for the electric dipole transition. We can also consider the case where the polarization of the electric field is in the x-direction. In this case, the matrix elements are proportional to the integral
\[ \int d\Omega Y_{l_f m_f}^* (\theta, \phi)[Y_{l_i}^0 (\theta, \phi) - Y_{l_i}^{-1} (\theta, \phi)]Y_{l_i m_i} (\theta, \phi) \]  
(93)
and the selection rules are
\[ \Delta l = l_f - l_i = \pm 1 \]  
\[ \Delta m = m_f - m_i = \pm 1 \]  
(94) (95)
We can generalize this to any direction of polarization. In this case, the direction will be some linear combination of x, y, and z. Our angular integral will then have some combination of the corresponding spherical harmonics. Evaluation of this integral will give us, as was shown previously, the selection rules for electric dipole transitions.

**Transition Probability Associated with Natural Light**

Let us now consider an hydrogen atom placed in an electromagnetic field with a spectrum of angular frequencies corresponding to natural light. In this case our expression for \( E_0 \) is a function of \( \omega \). If we consider the Poynting vector
\[ \langle \vec{S} \rangle = \frac{c\epsilon_0 E_0^2}{2} \]  
(96)
\[ \langle \vec{S} \rangle = I(\omega) \Delta \omega \]  
(97)
This gives us a relation between \( E_0 \) and \( I(\omega) \), explicitly
\[ E_0^2 = \frac{2I(\omega) \Delta \omega}{c\epsilon_0} \]  
(98)
The probability of a transition, corresponding to emission or absorption, is given by
\[ P_{if} = \frac{q^2}{\hbar^2} \left( \frac{\omega f_i}{\omega} \right)^2 |E_0|^2 \langle f | Z | i \rangle \left| \frac{\sin \left( \frac{(\omega f_i - \omega) t}{2} \right)}{(\omega f_i - \omega)^2} \right|^2 \]  
(99)
For natural light we replace $E_0^2$ using equation 42 and get

$$P_{if} = \int_0^\infty \frac{q^2}{\hbar^2} \frac{\omega_i}{\omega} \frac{2I(\omega)\delta \omega}{\epsilon_0} |< f|Z|i>|^2 \sin\left(\frac{\omega_i - \omega}{2}\right) \delta \omega (\omega_i - \omega)^2$$  \hspace{1cm} (100)$$

To solve this integral we first cast it in the form

$$P_{if} = \frac{2q^2}{\epsilon_0 \hbar^2} |< f|Z|i>|^2 \int_0^\infty \delta \omega \frac{\omega_i}{\omega} \omega f(\omega)$$  \hspace{1cm} (101)$$

$$f(\omega) = \frac{\sin^2\left(\frac{\omega_i - \omega}{2}\right)}{(\omega_i - \omega)^2}$$  \hspace{1cm} (102)$$

If we plot $I(\omega)$ and $f(\omega)$ against $\omega$ we see that the largest contribution to the product $I(\omega)f(\omega)$ comes from $\omega \approx \omega_i$. Replacing $I(\omega)$ with $I(\omega_i)$ and $(\frac{\omega_i}{\omega})$ with 1 and noting that integrating from $-\infty$ to 0 adds almost nothing to the integral, we can express the transition probability as

$$P_{if} \approx \frac{2q^2}{\epsilon_0 \hbar^2} |< f|Z|i>|^2 I(\omega_i) \int_{-\infty}^\infty \delta \omega f(\omega)$$  \hspace{1cm} (103)$$

Using substitution and an integral table or solving with a computer we find

$$P_{if} = \frac{\pi q^2}{\epsilon_0 \hbar^2} |< f|z|i>|^2 I(\omega_i)$$  \hspace{1cm} (104)$$

or

$$\frac{P_{if}}{t} = \frac{\pi q^2}{\epsilon_0 \hbar^2} |< f|z|i>|^2 I(\omega_i)$$  \hspace{1cm} (105)$$

When considering natural light we also need to average over all polarizations. Our electric dipole matrix elements are then given by

$$< W_1 > = \frac{1}{3} (|< f|x|i>|^2 + |< f|y|i>|^2 + |< f|z|i>|^2) = \frac{1}{3} < f|\mathbf{r}|i>|^2$$  \hspace{1cm} (106)$$

giving

$$\frac{P_{if}}{t} = \frac{\pi q^2}{\epsilon_0 \hbar^2} |< f|\mathbf{r}|i>|^2 I(\omega_i) = B$$  \hspace{1cm} (107)$$

This is known as Einstein’s B coefficient, it is the probability per unit time for a transition corresponding to absorption or induced emission.

**Einstein’s A and B coefficients**

For spontaneous emission Einstein calculated his A coefficient using a rather clever thought experiment. He considered a volume in which a finite number of atoms are confined. He assumed that if the rate of absorption is given by $N_i B$, the rate of induced emission by $N_f B$, and the rate of spontaneous emission by
$N_f A$, where $N_i$ and $N_f$ are the number of particles in the initial or final state respectively, then the following equality must hold

$$N_f B + N_f A = N_i B$$  \hspace{1cm} (108)

which gives

$$A = \left( \frac{N_i}{N_f} - 1 \right) B = (e^{E_f - E_i} - 1) B$$  \hspace{1cm} (109)

In this case, we can get the rate of absorption can be written as

$$rate\ of\ absorption = \begin{cases} \text{induced or} & \text{stimulated} \\ \end{cases} \text{emission}$$

Einstein’s A and B coefficients are defined by

$$A = \left( e^{\frac{\hbar \omega}{kT}} - 1 \right) B$$  \hspace{1cm} (110)

$$B = \frac{\pi q^2}{3\epsilon_0 c^3 h} |\langle f | \vec{r} | i \rangle|^2 I(\omega_f)$$  \hspace{1cm} (111)

In case of black body radiation, we have the intensity of the photons $I(\omega)$ as

$$I(\omega) = \frac{\hbar^2 \omega^3}{\pi^2 c^2 \left( e^{\frac{\hbar \omega}{kT}} - 1 \right)}$$  \hspace{1cm} (112)

Thus, we get

$$A = \frac{\pi q^2}{3\epsilon_0 c^3 h} \frac{\hbar^2 \omega_f^3}{\pi^2 c^2} |\langle f | \vec{r} | i \rangle|^2$$

$$= \frac{q^2 \omega_f^3}{3\pi \epsilon_0 c^3 h} |\langle f | \vec{r} | i \rangle|^2$$  \hspace{1cm} (113)

Let’s look at asymptotic behavior of $I(\omega)$ in detail. For small $\omega$, we can expand the exponential term as

$$e^{\frac{\hbar \omega}{kT}} - 1 = 1 + \frac{\hbar \omega}{kT}.$$

So, we get $I(\omega)$ as

$$I(\omega) \approx \frac{\hbar \omega^3}{\pi^2 c^2 \left( \frac{\hbar \omega}{kT} \right)^2} = \frac{\omega^2}{\pi^2 c^2} kT$$  \hspace{1cm} (115)

This result is well known from Maxwell equation.

However, if the radiation comes in ”packet” (such as photons with $\hbar \omega$), we must identify how many photons are in a certain $E$. We know that

$$N(nE) \approx e^{-\frac{nE}{\pi \tau}}, P(n) = P_n = e^{-\frac{nE}{\pi \tau}} \frac{e^{-nE}}{Z}$$  \hspace{1cm} (116)
, where

$$Z = \sum_{n=0}^{\infty} e^{-\frac{nE}{kT}} = \frac{1}{1 - e^{-\frac{E}{kT}}}$$

Thus, we find \( \langle n \rangle \) as

$$\langle n \rangle = \sum nP(n) = \frac{1}{Z} \sum_{n=0}^{\infty} n e^{-\frac{nE}{kT}}$$ (117)

Let’s define \( \alpha \) as \( \alpha = e^{-\frac{E}{kT}} \). Then, we get \( Z = \sum_{n=0}^{\infty} \alpha^n \). Hence, we can rewrite \( \langle n \rangle \) as

$$\langle n \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} n \alpha^n$$ (118)

, as well as

$$\frac{dZ}{d\alpha} = \sum_{n=0}^{\infty} n \alpha^{n-1}$$

$$\Rightarrow \frac{dZ}{d\alpha} = \sum_{n=0}^{\infty} n \alpha^n$$

$$\Rightarrow \langle n \rangle = \frac{\alpha}{Z} \frac{dZ}{d\alpha}$$

$$= \alpha \frac{1 - \alpha}{(1 - \alpha)^2}$$

$$= \frac{\alpha}{1 - \alpha}$$

$$= \frac{1}{e^{\frac{E}{kT}} - 1}$$ (119)

Average # of photons with \( \vec{k} = 2 \langle n \rangle \frac{d^3k}{(2\pi)^3} \), whose 2 is the number of polarization.

Thus, the average # of photons with \( \omega \) can be written as

$$\int 2\langle n \rangle \frac{d^3k}{(2\pi)^3} = \int 2\langle n \rangle \frac{d\Omega}{(2\pi)^3}$$

$$= \frac{8\pi}{(2\pi)^3} \langle n \rangle d\Omega$$

$$= \frac{1}{\pi^2} \langle n \rangle d\Omega$$

$$= \frac{1}{e^{\frac{E}{kT}}} \langle n \rangle d\omega$$, from \( \omega = ck \) (120)
Finally, we get the total energy $E_{total}$ as

$$E_{total} = E \frac{1}{c^2 \pi^2} \langle n \rangle d\omega \omega^2$$

$$= I(\omega) d\omega = cE_{total}$$

Also, we get $I(\omega) d\omega$ as

$$\Rightarrow I(\omega) d\omega = E \frac{1}{c^2 \pi^2} \langle n \rangle \omega^2 d\omega$$

$$\Rightarrow I(\omega) d\omega = \hbar \omega \frac{1}{c^2 \pi^2} \frac{1}{e^{\frac{\hbar \omega^3}{\hbar \omega}} - 1}$$

**Dipole moment**

Let’s consider an electron subject to a restoring force directed towards the origin and proportional to the displacement. $|\Psi(0)\rangle$ can be written as

$$|\Psi(0)\rangle = |i\rangle = |0\rangle \text{ (ground state)}$$

and we have $|\Psi(t)\rangle$ as

$$|\Psi(t)\rangle = e^{-i\frac{E_0 t}{\hbar}} |0\rangle + \lambda \sum_{n \neq 0} b_n^{(1)} e^{-i\frac{E_n t}{\hbar}} |n\rangle + \cdots$$

where

$$\lambda b_n^{(1)}(t) = \frac{W_{f0}}{\hbar} \int_0^t dt' e^{i\omega_0 t'} \sin(\omega t')$$

$$= \frac{W_{n0}}{2i\hbar} \left[ e^{i(\omega_0 + \omega) t} - \frac{1}{\omega_n + \omega} - e^{i(\omega_0 - \omega) t} - \frac{1}{\omega_n - \omega} \right]$$

and

$$W_{n0} = \frac{qE_0}{m\omega} \langle n | p_z | 0 \rangle$$

$$= iqE_0 \frac{\omega_0}{\omega} \langle n | z | 0 \rangle$$

Hence, we can rewrite $|\Psi(t)\rangle$ as

$$|\Psi(t)\rangle = e^{-i\frac{E_0 t}{\hbar}} \left[ |0\rangle + \lambda \sum_{n \neq 0} b_n^{(1)} e^{-i\omega_n t} |n\rangle + \cdots \right]$$

(121)
where
\[ \omega_{n0} = \frac{E_n - E_0}{\hbar} \]

However, we know that \( e^{-i\omega_{n0}t} \) goes to zero in nature due to the damping factor \( e^{-\Gamma t} \), in such that
\[ e^{-i\omega_{n0}t} = e^{-i(\omega_{n0} - \Gamma)t} = e^{-\Gamma t}e^{-i\omega_{n0}t} \rightarrow 0, \text{ as } t \rightarrow \infty \] (122)

Thus, we get
\[
|\Psi (t)\rangle = e^{-\frac{iE_0 t}{\hbar}} \left[ |0\rangle + \sum_{n \neq 0} \frac{q\xi_0 \omega_{n0}}{2\hbar} \frac{\omega_n}{\omega} \langle n | z | 0 \rangle | n \rangle \right] \left[ \frac{e^{i\omega t}}{\omega_{n0} + \omega} - \frac{e^{-i\omega t}}{\omega_{n0} - \omega} \right] \]

(123)

Therefore, the projection of the dipole moment \( \vec{D} (= q\vec{z}) \) becomes
\[
D_z (t) = \langle \Psi (t) | qz | \Psi (t) \rangle \\
= \langle 0 | qz | 0 \rangle + \sum_{n \neq 0} \frac{q\xi_0 \omega_{n0}}{2\hbar} \frac{\omega_n}{\omega} \langle n | z | 0 \rangle \left[ \frac{e^{i\omega t}}{\omega_{n0} + \omega} - \frac{e^{-i\omega t}}{\omega_{n0} - \omega} \right] \langle 0 | qz | n \rangle + C.C.
\]

The first term becomes zero due to \( \int \text{even} \ast \text{odd} \ast \text{even} \), and we get
\[
D_z (t) = \sum_{n \neq 0} \frac{q^2 \xi_0 \omega_{n0}}{2\hbar} \frac{\omega_n}{\omega} \langle n | z | 0 \rangle \left[ \frac{1}{\omega_{n0} + \omega} - \frac{1}{\omega_{n0} - \omega} \right] \cos (\omega t)
\]
\[
= \sum_{n \neq 0} \frac{q^2 \xi_0 \omega_{n0}}{2\hbar} \frac{\omega_n}{\omega} \langle n | z | 0 \rangle \left[ \frac{2\omega}{\omega_{n0}^2 - \omega^2} \right] \cos (\omega t)
\]
\[
= \sum_{n \neq 0} \frac{q^2 \xi_0}{2\hbar} \omega_{n0} \langle n | z | 0 \rangle \left[ \frac{2\omega}{\omega_{n0}^2 - \omega^2} \right] \cos (\omega t) \] (124)

Finally, we find the electric susceptibility \( \chi_e \) as,
\[
\chi_e = -\sum_{n \neq 0} \frac{2q^2 \omega_{n0}}{\hbar} \frac{\omega_n}{\omega_{n0}^2 - \omega^2} \langle n | z | 0 \rangle \] (125)

**Spring Problem**

For the spring problem, the differential equation can be written as
\[
m \frac{d^2z}{dt^2} = -m\omega_0^2 z + q\xi_0 \cos (\omega t). \] (126)
And, the ansatz of this equation can be written as

\[ z = A \cos(\omega t) \]  \hspace{1cm} (127)

By inserting this ansatz to find the coefficient \( A \), we get

\[-m\omega^2 A = -m\omega_0^2 A + q\mathcal{E}_0 \]
\[ \Rightarrow A = \frac{q\mathcal{E}_0}{m(\omega_0^2 - \omega^2)} \]  \hspace{1cm} (128)

Hence, we find the projection of the dipole moment \( d_z = qz \) as

\[ d_z = \frac{q^2\mathcal{E}_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) \]  \hspace{1cm} (129)

Also, we find \( \mathcal{X}_e \) as

\[ \mathcal{X}_e = \frac{q^2}{m(\omega_0^2 - \omega^2)} \]  \hspace{1cm} (130)

Let’s compare two results. We found two \( \mathcal{X}_e \) as

\[ \mathcal{X}_e = -\sum_{n \neq 0} \frac{2q^2\omega_{n0}}{\hbar} \frac{|\langle n | z | 0 \rangle|^2}{\omega_{n0}^2 - \omega^2} \]
\[ \Rightarrow \mathcal{X}_e = \sum_{n \neq 0} \frac{2m\omega_{n0}}{\hbar} \frac{|\langle n | z | 0 \rangle|^2}{m(\omega_{n0}^2 - \omega^2)} \frac{q^2}{m(\omega_{n0}^2 - \omega^2)} \]
\[ = \sum_{n \neq 0} f_{n0} \frac{q^2}{m(\omega_{n0}^2 - \omega^2)} \]  \hspace{1cm} (131)

where

\[ f_{n0} = \frac{2m\omega_{n0}}{\hbar} \frac{|\langle n | z | 0 \rangle|^2}{m(\omega_{n0}^2 - \omega^2)} \]

It turns out \( \sum f_{n0} = 1 \), which means the sum of the probability is equal to 1. Let’s prove it.

Since we find that

\[ \omega_{n0} \langle n | z | 0 \rangle = \frac{1}{\hbar} \langle n | [H_0, z] | 0 \rangle \]
\[ = -\frac{i}{m} \langle n | p_z | 0 \rangle, \]

We can calculate \( \sum f_{n0} \) as

\[ \sum f_{n0} = \frac{2m}{\hbar} \sum -\frac{i}{m} \langle n | p_z | 0 \rangle \langle 0 | z | n \rangle \]  \hspace{1cm} (132)
Since we know \( C = \frac{1}{2} (C + C^*) \) when \( C \) is a complex number,
\[
\sum f_{n0} = \frac{2m}{\hbar} \sum -\frac{i}{2m} \langle n \mid p_z \mid 0 \rangle \langle 0 \mid z \mid n \rangle + C.C.
\]
\[
= -\frac{i}{\hbar} \sum [(0 \mid z \mid n) \langle n \mid p_z \mid 0 \rangle - (0 \mid p_z \mid n) \langle n \mid z \mid 0 \rangle]
\]
\[
= -\frac{i}{\hbar} (0 \mid [z,p_z] \mid 0)
\]
\[
= 1 \ (q.e.d.) \quad (133)
\]

**Blackbody Radiation**

In case of the blackbody radiation, we have \( \langle n (E) \rangle \) of the photon as
\[
\langle n (E) \rangle = \frac{1}{e^{\frac{E}{kT}} - 1} \quad (134)
\]

The average number with \( \vec{k} \) is, then,
\[
\langle n (E) \rangle \ast \frac{d^3k}{(2\pi)^3} \quad (135)
\]

Since we can write \( \vec{k} \) as
\[
\vec{k} = (k_x, k_y, k_z)
\]
and using periodic conditions of the boundary condition, we find
\[
\Psi(\vec{k}, \vec{r}) = e^{i k_x x} e^{i k_y y} e^{i k_z z} \quad (136)
\]
and
\[
e^{i k_x L} = 1 \Rightarrow k_x L = 2\pi L \\
\Rightarrow k_x = \frac{2\pi}{L} n \Rightarrow \Delta k_x = \frac{2\pi}{L} \quad (137)
\]

Thus, we get
\[
\langle n (E) \rangle \left( \frac{L}{2\pi} \right)^3 \Delta k_x \left( \frac{L}{2\pi} \right)^3 \Delta k_y \left( \frac{L}{2\pi} \right)^3 \Delta k_z = \frac{L^3}{(2\pi)^3} d^3k \quad (138)
\]

So, the energy \( E \) can be found by
\[
E = \hbar \omega \int \langle n(E) \rangle \frac{L^3}{(2\pi)^3} d^3k \\
= \frac{V \hbar \omega^3 d\omega}{\pi^2 c^3 (e^{\hbar \omega/kT} - 1)} \quad (139)
\]
And, we find the energy density $du(= E/V)$ by
\[
du = \frac{\hbar \omega^3 d\omega}{\pi^2 c^3 (e^{\hbar \omega/kT} - 1)} \tag{140}
\]
Therefore, we find $I(\omega)$ as
\[
I(\omega) = c \frac{du}{d\omega} = \frac{\hbar \omega^3}{\pi^2 c^3 (e^{\hbar \omega/kT} - 1)} \tag{141}
\]

**Continuous Spectrum: $|i\rangle \rightarrow |f\rangle$**

We will consider the case that has a transition from discrete states to continuous states. Let’s say the energy of the final states can be written as
\[
E_f = \frac{p_f^2}{2m} \tag{142}
\]
In case of continuous spectrum, we may write the probability of the transition as $P_i(E_f)$ instead of $P_{if}$, where $E_f < E < E_f + \Delta E_f$. Also, the probability can be written as
\[
Probability = P_i(E_f) \Delta E_f \rho(E_f) \tag{143}
\]
, where the density of states $\rho(E)$ is
\[
\rho(E) = \frac{\# of states}{\Delta E} \tag{144}
\]
In order to find $\rho(E)$, we will introduce a projection operator $|\vec{p}\rangle \langle \vec{p}|$.

In the range of $E_f < E(= \frac{p_f^2}{2m}) < E_f + \Delta E_f$, we find
\[
\int d^3p \ |\vec{p}\rangle \langle \vec{p}| = \int dp^2 \int d\Omega \ |\vec{p}\rangle \langle \vec{p}| \\
= \int \frac{m}{\sqrt{2} mE} \frac{dE}{2mE} \int d\Omega \ |\vec{p}\rangle \langle \vec{p}| \\
= \int dE \int d\Omega \ m\sqrt{2mE} \ |\vec{p}\rangle \langle \vec{p}| \\
= \int dE \int d\Omega \ \rho(E, \Omega) \ |\vec{p}\rangle \langle \vec{p}| \tag{144}
\]
, where
\[
\rho(E, \Omega) = \frac{\# of states}{\Delta E \Delta \Omega} .
\]
We know that
\[
| f\rangle = |\vec{p}\rangle = |E, \Omega\rangle \tag{145}
\]
So,
\[
P_i(E_f) = |\langle \Psi(t) \ | f\rangle|^2 = |\langle \Psi(t) \ | E, \Omega\rangle|^2 \tag{146}
\]
Let's look at examples.

Example 0) To measure the energy of electrons.
Assuming $E = 10\text{eV}$, with $\sigma_E = 1\text{eV}$, we have the probability as

$$\text{Probability} = \int P(E, \Omega) \rho(E, \Omega) d\lambda dE$$  \hspace{1cm} (147)

Example 1) Constant perturbation.

$$P(E, \Omega) = | \langle E, \Omega | \psi(t) \rangle |^2 = | b_E^{(1)}(t) |^2$$

$$= \frac{1}{\hbar^2} \int_0^t dt' e^{\frac{E-E_i}{\hbar} t} \langle E, \Omega \mid W \mid i \rangle |^2$$

$$= \frac{| \langle E, \Omega \mid W \mid i \rangle |^2 \sin^2 \frac{E-E_i}{2\hbar} t \rho(E_i, \Omega)}{(E-E_i)^2}$$  \hspace{1cm} (148)

Since we know

$$\frac{\text{Probability}}{\Delta \Omega} = \int dE \rho(E, \Omega) P(E, \Omega)$$

and by assuming $\rho(E, \Omega)$ is approximately constant, which is $\rho(E_i, \Omega)$, we find

$$\frac{\text{Probability}}{\Delta \Omega} \simeq \rho(E_i, \Omega) \int_0^\infty dE | \langle E, \Omega \mid W \mid i \rangle |^2 \frac{\sin^2 \frac{E-E_i}{2\hbar} t \rho(E_i, \Omega)}{(E-E_i)^2}$$

$$= \rho(E_i, \Omega) | \langle E, \Omega \mid W \mid i \rangle |^2 \int_0^\infty dE \frac{\sin^2 \frac{E-E_i}{2\hbar} t}{(E-E_i)^2}$$  \hspace{1cm} (149)

For $E < 0$, the integral is very small ($\Rightarrow 0 \rightarrow -\infty$.) By defining a new variable $u$ as

$$u \equiv \frac{E - E_i}{2\hbar} t, \; du = \frac{dE}{2\hbar}$$

we find

$$\frac{\text{Probability}}{\Delta \Omega} = \rho(E_i, \Omega) | \langle E, \Omega \mid W \mid i \rangle |^2 \int_{-\infty}^{\infty} \frac{2\hbar \sin^2 ut^2}{u^2}$$

$$= \frac{2\pi t}{\hbar} \rho(E_i, \Omega) | \langle E, \Omega \mid W \mid i \rangle |^2$$  \hspace{1cm} (150)

Hence, we get $w$ as

$$w \equiv \frac{\text{Probability}}{\Delta \Omega \Delta t} = \frac{2\pi}{\hbar} \rho(E_i, \Omega) | \langle E, \Omega \mid W \mid i \rangle |^2 = \text{constant}!!$$  \hspace{1cm} (151)

This result, $w = \text{constant}$, is so called "Fermi’s Golden Rule.”

Now, let's consider the scattering theory to describe with the time perturbation theory. We have an electron $e^-$ for incoming particle having a mass $m$ and a
momentum $\vec{p}_i$, as well as a heavy target having a potential $V$. Due to this potential $V$, the momentum of the electron is deflected to $\vec{p}_f$. From this situation, we get

$$E_i = \frac{p_i^2}{2m}, \quad E_f = \frac{p_f^2}{2m}, \quad E_i = E_f, \quad p_i^2 = p_f^2$$  \hspace{1cm} (152)

The hamiltonian $H$ can be written as

$$H = \frac{p^2}{2m} + V = H_0 + W$$  \hspace{1cm} (153)

, where

$$H_0 = \frac{p^2}{2m}, \quad W = V.$$

We define an initial state and a final state as

$$| i \rangle = | \vec{p}_i \rangle, \quad | f \rangle = | \vec{p}_f \rangle = | E_i, \Omega \rangle$$  \hspace{1cm} (154)

Then, $w$ becomes

$$w = \frac{2\pi}{\hbar} m \sqrt{2mE} \left| \langle \vec{p}_f | V | \vec{p}_i \rangle \right|^2$$  \hspace{1cm} (155)

We need to find the form of $| \vec{p}_i \rangle$. Since we know that

$$| \vec{p} \rangle = C e^{i\vec{p}\cdot \vec{r}/\hbar} \text{ and } \langle \vec{p} | \vec{p}' \rangle \propto \delta^3(\vec{p} - \vec{p}')$$,

we need to find the coefficient $C$. Let’s work on it.

We have an inner product $\langle \vec{p} | \vec{p}' \rangle$ as

$$\langle \vec{p} | \vec{p}' \rangle = C^2 \int d^3r e^{i(\vec{p} - \vec{p}') \cdot \vec{r}/\hbar}$$

\hspace{1cm}

$$= \int_{-\infty}^{\infty} dx e^{i(p'_x - p_x)x/\hbar}$$

\hspace{1cm}

$$= \int_{0}^{\infty} dx e^{i(p'_x - p_x)x/\hbar} + \int_{-\infty}^{0} dx e^{i(p'_x - p_x)x/\hbar}$$

\hspace{1cm}

$$= \int_{0}^{\infty} dx e^{i(p'_x - p_x)x/\hbar} + \int_{0}^{\infty} dx e^{-i(p'_x - p_x)x/\hbar}$$  \hspace{1cm} (156)

By multiplying $e^{-\epsilon x}$ with $\epsilon > 0$ but $\epsilon \sim 0$, we get

$$\langle \vec{p} | \vec{p}' \rangle = \frac{1}{\frac{\epsilon}{\hbar} - i\frac{(p'_x - p_x)}{\hbar}} + \frac{1}{\frac{\epsilon}{\hbar} + i\frac{(p'_x - p_x)}{\hbar}}$$

\hspace{1cm}

$$= \frac{2\epsilon \hbar}{\epsilon^2 + (p'_x - p_x)^2}$$  \hspace{1cm} (157)

Let’s define $f(q)$ as

$$f(q) = \frac{2\epsilon \hbar}{\epsilon^2 + q^2}$$  \hspace{1cm} (158)
When $\epsilon \to 0$, $f(q)$ behaves in such that
\begin{align}
  f(q) & \to 0, \text{ if } q \neq 0 \quad (159) \\
  f(q) & \to \infty, \text{ if } q = 0 \quad (160)
\end{align}

Thus, we find that
\begin{align}
  \int_{-\infty}^{\infty} f(q) dq &= \int_{-\infty}^{\infty} \frac{2\epsilon \hbar}{\epsilon^2 + q^2} dq \\
  \int_{-\infty}^{\infty} f(q) dq &= \int_{-\infty}^{\infty} \frac{2\epsilon \hbar \cos^2 \theta}{\epsilon \cos^2 \theta} d\theta \\
  &= 2\hbar \int_{-\pi/2}^{\pi/2} d\theta = 2\pi \hbar \quad (161)
\end{align}

Let $q = \epsilon \tan \theta$. Then, $dq = \frac{\epsilon}{\cos^2 \theta} d\theta$. So, we get
\begin{align}
  \int_{-\infty}^{\infty} f(q) dq &= \int_{-\infty}^{\infty} \frac{2\epsilon \hbar \cos^2 \theta}{\epsilon \cos^2 \theta} d\theta \\
  &= 2\hbar \int_{-\pi/2}^{\pi/2} d\theta = 2\pi \hbar \quad (162)
\end{align}

Thus, we find $f(q)$ as
\begin{align}
  f(q) &= 2\pi \hbar \delta(q) \quad (163)
\end{align}

And, we get
\begin{align}
  \int_{-\infty}^{\infty} dx e^{i(p'_x - p_x)x/\hbar} &= 2\pi \hbar \delta(p'_x - p_x) \\
  \Rightarrow \int_{-\infty}^{\infty} d^3r e^{i(\vec{p}' - \vec{p}) \cdot \vec{r}/\hbar} &= (2\pi \hbar)^3 \delta^3(\vec{p}' - \vec{p}) \quad (164)
\end{align}

By recalling the result of $\langle \vec{p}' | \vec{p} \rangle$, we finally find $C$ as
\begin{align}
  \langle \vec{p}' | \vec{p} \rangle &= C (2\pi \hbar)^3 \delta^3(\vec{p}' - \vec{p}) \\
  \Rightarrow C &= \frac{1}{(2\pi \hbar)^{3/2}} \quad (165)
\end{align}

Therefore, we find $w(\vec{p}'_i, \vec{p}'_f)$ as
\begin{align}
  w(\vec{p}'_i, \vec{p}'_f) &= \frac{2\pi}{\hbar} m \sqrt{2mE} \frac{1}{(2\pi \hbar)^3} \left| \int d^3r V(r) e^{i(\vec{p}'_i - \vec{p}')} \cdot \vec{r}/\hbar \right|^2 \quad (166)
\end{align}

We can write the probability current density $J_i$ as
\begin{align}
  J_i &= \frac{\hbar}{2mi} (\Psi^* \partial_i \Psi - C.C.) \\
  &= \frac{p_i}{m} \frac{1}{(2\pi \hbar)^3} \quad (167)
\end{align}

where
\begin{align}
  \Psi = \frac{1}{(2\pi \hbar)^{3/2}} e^{ip_i z/\hbar}
\end{align}
Hence, the differential cross section $d\sigma/d\Omega$ can be found

$$
\frac{d\sigma}{d\Omega} = \frac{w}{J_i} = \frac{2\pi m\sqrt{2mE}m}{\hbar} \int d^3r \, V(r) e^{i(\vec{p}_i - \vec{p}_f) \cdot \vec{r}/\hbar} \left| \frac{1}{w\pi\hbar} \right|^2 = \frac{m^2}{4\pi^2\hbar^2} \int d^3r \, V(r) e^{i(\vec{p}_i - \vec{p}_f) \cdot \vec{r}/\hbar} \left| \frac{1}{w\pi\hbar} \right|^2 .
$$

(168)

= same as Born cross section!!

Example 2) Sinusoidal Perturbation.

We can state the transition probability $P_{if}(t, \omega)$ as

$$
P_{if}(t, \omega) = |\langle E, \Omega | \Psi(t) \rangle|^2 = \left| \frac{w_{fi}}{4\hbar^2} \right|^2 = \frac{\sin^2\left(\frac{(E-E_i-h\omega) t}{2\hbar}\right)}{\left(\frac{(E-E_i-h\omega)}{2\hbar}\right)^2} .
$$

(169)

Since we have a condition in such that

$$
E \sim E_i + h\omega,
$$

we get $w$ as

$$
w = \frac{\pi}{2\hbar} \left| \langle E_i + h\omega, \Omega | W | i \rangle \right|^2 \rho(\Omega, E_i + h\omega). \tag{170}
$$

Example 3) Decay; nucleus $\rightarrow \alpha + nucleus'$.

Again, in this problem we also have a transition from a discrete state(nucleus, $| i \rangle$) to two continuous states($\alpha$ and nucleus', $| E, \Omega \rangle$). We will talk about the decay rate $\Gamma$ at this time, which is defined as

$$
deay rate \Gamma \equiv \text{probability/time} = \int d\Omega w(\Omega) .
$$

(171)

We also define the probability of no decay $P_{ii}$ as

$$
P_{ii} = 1 - \Gamma t .
$$

(172)

In order to make it true, we must have a condition in such that

$$
\Gamma t \ll 1 \Rightarrow t \ll \frac{1}{\Gamma} .
$$

(173)

Also, if you look at this $\Gamma$ as a function of $E$, then

$$
\Gamma(E) = \frac{2\pi}{\hbar} \left| \langle E, \Omega | W | i \rangle \right|^2 \rho(\Omega, E) .
$$

(174)

Since we have a majority of $\Gamma(E)$ in the range of the small width centered by a peak, we have another condition in such that

$$
t \gg \frac{1}{\Delta \Gamma} .
$$

(175)
Therefore, we have a condition,
\[ \frac{1}{\Delta \Gamma} \ll t \ll \frac{1}{\Gamma} \quad (176) \]

Example 4) Beyond perturbation theory; \( t \geq \frac{1}{\Gamma} \).
We can consider a Schrodinger equation, in such that
\[ i\hbar \frac{db_i}{dt} = \sum_n e^{i\omega_{kn}t} W_{kn} b_n \quad (177) \]

, where
\[ \omega_{kn} = \frac{E_i - E}{\hbar} \]
and \( W_{kn} = \langle i \mid W \mid E, \Omega \rangle \)

In the case of the continuous state and \( k = i \), we can rewrite this equation as
\[ i\hbar \frac{db_i}{dt} = \int d\Omega dE \rho(\Omega, E) e^{i\frac{(E_i - E)}{\hbar}t} \langle i \mid W \mid E, \Omega \rangle b(\Omega, E) \quad (178) \]

When \( k = k(\Omega, E) \), it becomes
\[ i\hbar \frac{db(\Omega, E)}{dt} = e^{i\frac{(E_i - E)}{\hbar}t} \langle E, \Omega \mid W \mid i \rangle b_i \quad (179) \]

At \( t = 0 \), \( b_i = 0 \) and \( b(\Omega, E) = 0 \). So,
\[ \frac{db_i}{dt} = -\frac{1}{\hbar^2} \int d\Omega dE \rho(\Omega, E) e^{i\frac{(E_i - E)}{\hbar}t} \langle i \mid W \mid E, \Omega \rangle \]
\[ \times \int_0^t dt' e^{i\frac{(E_i - E)}{\hbar}t'} \langle \Omega, E \mid W \mid i \rangle b_i(t') \]
\[ = -\frac{1}{\hbar^2} \int d\Omega dE \rho(\Omega, E) \int_0^t dt' \langle i \mid W \mid E, \Omega \rangle^2 e^{i\frac{(E_i - E)}{\hbar}(t-t')} b_i(t') \]
\[ = -\frac{1}{\hbar^2} \hbar \int dE \Gamma(E) \int_0^t dt' e^{i\frac{(E_i - E)}{\hbar}(t-t')} b_i(t') \quad (180) \]

When \( (t - t') \) is very large, we get
\[ \int dE \Gamma(E) e^{i\frac{(E_i - E)}{\hbar}(t-t')} \propto \sin \frac{E(t-t')}{\hbar} \]
\[ \Rightarrow \frac{E(t-t')}{\hbar} = \pi \]
\[ \Rightarrow E = \frac{\pi \hbar}{(t-t')} \quad (181) \]
Thus, we have a rapid oscillation and it leads to
\[ \int_0^t dt' \cdots = 0!! \] (182)
When \( t - t' \) is very small, we get
\[ \int_0^t dt' \cdots \neq 0, \] (183)
compared to \( \frac{1}{\Delta \Gamma} \).
For \( (t - t') \), we have a condition that
\[ \text{large: } t - t' \gg \frac{1}{\Delta \Gamma} \]
\[ \text{small: } t - t' \leq \frac{1}{\Delta \Gamma} \]
\[ \Rightarrow (t - \frac{1}{\Delta \Gamma}) < t' < t \text{ only for a reasonable contribution.} \]
\[ \Rightarrow b_i(t') \sim b_i(t) \sim b_i \]
Finally, we find
\[ \frac{db_i}{dt} = - \left[ \frac{1}{2\pi \hbar} \int dE \int_0^t dt' \Gamma(E) e^{\frac{(E_i - E)}{\hbar}(t - t')} \right] b_i \]
\[ = - \left[ \frac{1}{2\pi \hbar} \int dE \Gamma(E) \frac{e^{\frac{(E_i - E)}{\hbar}t} - 1}{i \frac{E_i - E}{\hbar}} \right] b_i \]
\[ = - \left[ \frac{1}{2\pi \hbar} \int dE \Gamma(E) f(E) \right] b_i \] (184)
where
\[ f(E) = \frac{e^{\frac{(E_i - E)}{\hbar}t} - 1}{i \frac{E_i - E}{\hbar}} \]
Note that \( |f(E)|^2 \) becomes
\[ |f(E)|^2 = \frac{\sin^2 \frac{E_i - E t}{2\hbar}}{(\frac{E_i - E}{\hbar})^2}. \] (185)
Since \( \Gamma(E) \) is approximately constant, we can rewrite \( \frac{db_i}{dt} \) as
\[ \frac{db_i}{dt} = - \left[ \frac{1}{2\pi \hbar} \Gamma(E_i) \int dE f(E) \right] b_i \] (186)
In addition, we want to look at \( \int dE f(E) \) in detail.
For the real part of $\int dE \, f(E)$, we find
\[ \int dE \frac{1}{2} [f(E) + F^*(E)] = \frac{1}{2} \int dE e^{i \frac{(E_i - E)}{\hbar} t} - e^{-i \frac{(E_i - E)}{\hbar} t} = \int_{-\infty}^{\infty} dE \sin \frac{E_i - E}{\hbar} t \] (187)

Let’s define $u$ as $u = \frac{E - E_i}{\hbar} t$ to solve this integral.
\[ \int dE \frac{1}{2} [f(E) + F^*(E)] = \hbar \int_{-\infty}^{\infty} du \frac{\sin u}{u} = \pi \hbar \] (188)

On the other hand, for the imaginary part of $\int dE \, f(E)$, we find
\[ \int dE \frac{1}{2i} [f(E) - F^*(E)] = \text{constant} \sim \delta E. \] (189)

Thus, we find
\[ \frac{db}{dt} = - \left[ \frac{1}{2} \Gamma(E_i) + i \frac{\delta E}{\hbar} \right] b_i \]
\[ \Rightarrow b_i(t) = e^{-\left( \frac{1}{2} + i \frac{\delta E}{\hbar} \right) t} \] (190)

In case of no decay, we have
\[ P_{ii} = |b_i(t)|^2 = e^{-\Gamma t} = 1 - \Gamma t + \ldots \] (191)

In the time dependent perturbation theory, we had
\[ P_{ii} = 1 - \Gamma t \] (192)

Thus, we can see good agreement between the exact solution and the perturbation theory.

For the decay probability, we have
\[ b(\Omega, E) = \frac{1}{i\hbar} \int_0^t dt' e^{i \frac{E_i - E}{\hbar} t'} \langle \Omega E \mid W \mid i \rangle b_i(t') \]
\[ = \frac{1}{i\hbar} \langle \Omega E \mid W \mid i \rangle \int_0^t dt' e^{-\frac{\Gamma t'}{2}} e^{i \frac{E_i - E}{\hbar} t'} \]
\[ = \langle \Omega E \mid W \mid i \rangle \frac{1 - e^{-\frac{\Gamma t}{2}} e^{i \frac{E_i - E}{\hbar} t}}{E - E_i - \delta E + i\hbar \frac{\Gamma}{2}} \] (193)

Since the probability density is defined by $|b|^2$, we finally get
\[ |b|^2 \sim \frac{1}{(E - E_i - \delta E)^2 + \hbar^2 \frac{\Gamma^2}{4}} \] (194)

We can find the uncertainty principle, in such that
\[ \Delta E \geq \hbar \Gamma = \frac{\hbar}{\tau} \Rightarrow \tau \Delta E \geq \hbar. \] (195)