Angular momentum

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Angular Momentum

The angular momentum classically is given as:

\[ \vec{L} = \vec{r} \times \vec{p} \]  \hspace{1cm} (1)

When we write this as an operator is it shown that the components satisfy the commutation relation:

\[ [L_x, L_y] = i\hbar L_z \]  \hspace{1cm} (2)

If you want to understand nature, you have to be able to ask what values of \( L \) can you get? When we call the \( L, J \), we see the same commutation relations are satisfied.

\[ [J_x, J_y] = i\hbar J_z \]  \hspace{1cm} (3)

We can find all the possible combinations by writing:

\[ J_{\pm} = J_x \pm i\hbar J_y \]  \hspace{1cm} (4)

and if writing \( J^2 \) as a vector, it is the sum of the squares of the components:

\[ J^2 = J_x^2 + J_y^2 + J_z^2 \]  \hspace{1cm} (5)

\( J^2 \) commutes with all the components, in particular let's look at \( J_z \):

\[ [J^2, J_z] = 0 \]  \hspace{1cm} (6)

Since these commute we can label them with common eigenvalues. Calling the eigenvalue \( \lambda \) and pulling out the \( \hbar^2 \) for dimensional reasons we get:

\[ J^2 \ket{\lambda, m} = \lambda^2 \hbar^2 \ket{\lambda, m} \]  \hspace{1cm} (7)

Now as a state of \( J_z \):

\[ J_z \ket{\lambda, m} = \hbar \ket{\lambda, m} \]  \hspace{1cm} (8)
When acting with $J_-$ our $m$ state is lowered by 1.

$$J_- | \lambda, m \rangle \propto | \lambda, m - 1 \rangle$$

(9)

In order to avoid the negative norm states we take the norm of a state and square it as:

$$|| J_- | \lambda, m \rangle ||^2 = [\lambda^2 - m(m - 1)]\hbar^2 \geq 0$$

(10)

Since it has to be positive we see:

$$m^2 - m - \lambda^2 \leq 0$$

(11)

When this is thought of a polynomial of $m$ it must satisfy the relation:

$$m_- \leq m \leq m_+$$

(12)

Solving the polynomial we get the following roots:

$$m_\pm = \frac{1}{2}(1 \pm \sqrt{1 + 4\lambda^2})$$

(13)

$$m = m_- + integer$$

(14)

Now acting with $J_+$ we go through the same story.

$$J_+ | \lambda, m \rangle \propto | \lambda, m + 1 \rangle$$

(15)

Writing the norm:

$$|| J_+ | \lambda, m \rangle ||^2 = [\lambda^2 - m(m + 1)]\hbar^2$$

(16)

It follows that:

$$m^2 + m - \lambda^2 \leq 0$$

(17)

Where the roots to our polynomial are:

$$m_\pm = \frac{1}{2}(-1 \pm \sqrt{1 + 4\lambda^2})$$

(18)

$$m_- \leq m \leq m_+$$

(19)

$$m = m_+ + integer$$

(20)

Rearranging gives:

$$m_+ - m_- = integer$$

(21)

Writing what these numbers are in terms of $\lambda$ we get:

$$\frac{1}{2}(-1 + \sqrt{1 + 4\lambda^2}) - \frac{1}{2}(1 - \sqrt{1 + 4\lambda^2}) = integer$$

(22)
This integer is referred to as $2j$.

$$-1 + \sqrt{1 + 4\lambda^2} = \text{integer} = 2j$$  \hspace{1cm} (23)

Then:

$$\sqrt{1 + 4\lambda^2} = \text{integer} = 2j + 1$$  \hspace{1cm} (24)

for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. 

Squaring we get:

$$1 + 4\lambda^2 = (2j + 1)^2 = 4j^2 + 4j + 1$$  \hspace{1cm} (25)

Solving we get:

$$\lambda^2 = j(j + 1)$$  \hspace{1cm} (26)

Now we will present the first non-trivial case of $j = \frac{1}{2}$. We all consist of spin $\frac{1}{2}$ particles. Its not angular momentum anymore because we can never have values of halves. This has no classical analog to explain this, for this case we have:

$$m = \pm \frac{1}{2}$$  \hspace{1cm} (27)

We have two states given as:

$$|j, m\rangle = \begin{cases} 
|\frac{1}{2}, \frac{1}{2}\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
|\frac{1}{2}, -\frac{1}{2}\rangle = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{cases}$$  \hspace{1cm} (28)

The particles that carry that spin are called spinors. Such examples are electrons (point particles), protons, and neutrons which are made of quarks (point particles).

**ROTATIONS**

We want to understand what happens when you rotate something that is a spinor. Under rotation the vector components of $\mathbf{r}$ are transformed as:

$$\vec{r}' = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$  \hspace{1cm} (29)
by using the transformations for rotation about the Z-axis by an angle $\theta$ we have:

\[
\begin{align*}
  x' &= \cos(\theta)x - \sin(\theta)y \\
  y' &= \sin(\theta)x + \cos(\theta)y \\
  z' &= z
\end{align*}
\]  

(30)

We get the rotation matrix which is multiplied by our original vector components:

\[
\begin{pmatrix}
  \cos(\theta) & -\sin(\theta) & 0 \\
  \sin(\theta) & \cos(\theta) & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]  

(31)

For small $\theta$, we make a Taylor expansion that gives:

\[
\begin{pmatrix}
  \theta & -\theta & 0 \\
  \theta & \theta & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]  

(32)

We can separate the 1’s from the $\theta$’s we can write this as

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]  

(33)

The angular momentum is held responsible for generating such rotations, to see this we express this as:

\[
\varphi(\vec{r}) \rightarrow \varphi(x - \theta y, y + \theta x, z) = \varphi(x, y, z) + \theta \left[-y \frac{\partial \varphi}{\partial x} + x \frac{\partial \varphi}{\partial y}\right]
\]

(34)

where are coordinates are transformed in our function. Since we are rotating about the Z-axis, the proper component would be the Z component of the angular momentum. This is shown and then where it is converted to its operator form. Since $\theta$ is small we need to Taylor expand this function. Expanding in
Angular momentum

\[ L_z = xp_y - yp_x \rightarrow -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \]  

(35)

If we act on \( \varphi \):

\[ L_z \varphi = -i\hbar \left( x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} \right) \]  

(36)

Pulling out common factors we see:

\[ \varphi (\vec{r}) + \frac{i\theta}{\hbar} L_z \varphi + ... = (I + \frac{i\theta}{\hbar} L_z) \varphi + ... = e^{\frac{i\theta}{\hbar} L_z} \varphi \]  

(37)

This shows us that the wave equation is being operated on by an operator. If we choose another axis for rotation we can always just redefine our axis to meet the requirements. We can write the exponent as \( \theta \hat{n} \cdot \vec{L} = \hat{\theta} \cdot \vec{L} \) when we have an arbitrary direction. \( \theta \) is not the angle about the Z-axis, but the angle we rotate. Thinking of this as an operator, if we have a spinor, we can replace L by S:

\[ \vec{S} = \frac{\hbar}{2} \vec{\sigma} \]  

(38)

where \( \sigma \) is just our familiar Pauli matrices.

We can call this matrix R, and in terms of \( \sigma \) we have:

\[ R = e^{\frac{i\theta}{\hbar} \vec{S}} = e^{\frac{i\theta}{\hbar} \vec{\sigma}} \]  

(39)

When we act on a spinor we get a 2x2 matrix. This matrix operator on a spinor rotates operator A like \( A \rightarrow RAR^{-1} \). We get a 2 component object.

Let’s choose \( \hat{n} = \hat{z} \) to identify the Z-axis as the direction we are rotating about. Then if \( \theta \) is small, keeping just the first two terms:

\[ R = e^{\frac{i\theta}{\hbar} S_z} = I + \frac{i}{\hbar} \theta S_z + ... \]  

(40)

We now want to find out what happens to S, in particular taking the component \( S_x \) and rotating we see:

\[ S_x \rightarrow RS_x R^{-1} \]

\[ = (I + \frac{i}{\hbar} \theta S_z) S_x (I - \frac{i}{\hbar} \theta S_z) + ... \]

\[ = S_x + \frac{i}{\hbar} \theta S_z S_x - \frac{i}{\hbar} \theta S_z S x + ... \]

\[ = S_x + \frac{i}{\hbar} \theta [S_z, S_x] + ... = S_x - \theta S_y + ... \]  

(41)

So now, going back to the matrix, we want to see how to write it. We want to keep \( \hat{n} \) in the Z-direction. R is expressed with an exponent which can be
expanded in a series to yield:

\[ R = e^{i\theta \sigma_z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i\theta}{2} \right)^n \sigma^n_z \]

\[ = II \left( 1 + \frac{1}{2!} \left( \frac{i\theta}{2} \right)^2 + \ldots \right) + \sigma_z \left( \frac{i\theta}{2} + \frac{1}{3!} \left( \frac{i\theta}{2} \right)^3 + \ldots \right) \]

\[ R = \cos \frac{\theta}{2} II + i \sin \frac{\theta}{2} \sigma_z \]

This last equation for \( R \) is a single matrix and is what we get when we have rotation around the Z-axis. All the even terms are proportional to the identity and the odd terms are proportional to \( \sigma_z \). If we replace \( \sigma_z = \hat{n} \cdot \vec{\sigma} \) we get the rotation around an arbitrary axis.

The table below shows what \( \sigma \)'s behavior is for different values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \sigma^n_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>II</td>
</tr>
<tr>
<td>1</td>
<td>( \sigma_z )</td>
</tr>
<tr>
<td>2</td>
<td>II</td>
</tr>
<tr>
<td>3</td>
<td>( \sigma_z )</td>
</tr>
<tr>
<td>..</td>
<td>..</td>
</tr>
</tbody>
</table>

If we rotate by \( \theta = 2\pi \) we get:

\[ R(2\pi) = \cos \pi II + i \sin \pi \sigma_z = -II \]

\[ R(4\pi) = II \]

Therefore we see we need to rotate by \( 4\pi \) to get back to where we started. Our everyday experiences would suggest \( 2\pi \), but our everyday experience uses operator and observables. \( A \rightarrow RAR^{-1} = (-II)A(-II) = A \) The physical consequences of what we see for \( R \) is that if we operate on a spinor by rotating \( 2\pi \), we get: \( R(2\pi) \mid \varphi \rangle = - \mid \varphi \rangle \). Which gives us \(-\varphi \), but with a phase, and as far as the probability is concerned this sign doesn’t matter, but the phase is very important when dealing with interference effects.

### Two Spinors

Two Spinors has four states: \( \mid ++\rangle, \mid --\rangle, \mid +\rangle, \mid -\rangle \). They are eigenstate of spin operators: \( \vec{S}_1 \), and \( \vec{S}_2 \), so we have a total of six components. For the particles we now get states: \( S^2_1, S^z_1, S^2_2, S^z_2 \).

If we operate \( S^2_1 \) we get:

\[ S^2_1 \mid \pm\pm \rangle = \frac{3}{4}\hbar^2 \mid \pm\pm \rangle \]
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They form a basis so, $S_1^2$ is 4x4 diagonalizable matrix and we get this same result acting on any of our 4 initial states as indicated by the ± signs. Since they form a basis we always get the same eigenvalue we get the same result for $S_1^2$ and $S_2^2$.

$$S_1^2 = \frac{3}{4} \hbar^2 I = S_2^2$$ \hspace{1cm} (45)

If we want to write $S_{1z}$, we act on the 1st component (1st particle) of our four initial vectors and get:

$$S_{1z} = \frac{\hbar}{2} \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ \hspace{1cm} (46)

It is diagonizable with eigenvalues 1, 1, -1, -1.

If we do the same thing for $S_{2z}$, we act on the 2nd component (2nd particle) of each of our four states.

$$S_{2z} = \frac{\hbar}{2} \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ \hspace{1cm} (47)

**TOTAL SPIN**

If we want to define total spin then $\vec{S} = \vec{S}_1 + \vec{S}_2$ which follows the commutation relation $[S^2, S_z] = 0$. This means that they commute. We can ask what are their common eigenstates? To find them we write $S_z$ in its matrix form and again it is diagonizable.

$$S_z = S_{1z} + S_{2z} = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ \hspace{1cm} (48)

Writing $S^2$, we already know $S_1^2$ and $S_2^2$ are the identity matrix.

$$S^2 = (\vec{S}_1 + \vec{S}_2)$$
$$= S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$$ \hspace{1cm} (49)

We expand this dot product and use $S_+$ and $S_-$,

$$S_{\pm} = (S_x \pm iS_y)$$ \hspace{1cm} (50)

to get:

$$2\vec{S}_1 \cdot \vec{S}_2 = 2S_{1x}S_{2x} + 2S_{1y}S_{2y} + 2S_{1z}S_{2z}$$
$$= S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}$$ \hspace{1cm} (51)
By acting on a state with $S_+$ and $S_-$, we see:

\[
\begin{align*}
S_+ |+\rangle &= 0 \\
S_- |+\rangle &= \hbar |\rangle \\
S_+ |-\rangle &= \hbar |+\rangle \\
S_- |-\rangle &= 0
\end{align*}
\]

(52)

Now if we act with $S^2$ on the first state $|++\rangle$:

\[
S^2 |++\rangle = \frac{3}{2} \hbar^2 |++\rangle + 2(\frac{\hbar}{2})^2 |++\rangle = 2\hbar^2 |++\rangle
\]

(53)

And now the second state and third states. We get the same result as the $|+-\rangle$ when acting on $|+-\rangle$.

\[
S^2 |+-\rangle = S^2 |+-\rangle = \hbar^2 |+-\rangle + \hbar^2 |-\rangle
\]

(54)

Acting on $|--\rangle$ we get the same result as $|++\rangle$

\[
S^2 |--\rangle = 2\hbar^2 |--\rangle
\]

(55)

Since we know how to act on the basis we now know the matrix of $S^2$. We are no longer diagonal.

\[
S^2 = \hbar^2 \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
\]

(56)

To find the common eigenstates we need to diagonalize this matrix. So we first need to find the eigenvalues of this matrix.

\[
0 = \text{det}(S^2 - \lambda I) = \begin{vmatrix}
2 - \lambda & 0 & 0 & 0 \\
0 & 1 - \lambda & 1 & 0 \\
0 & 1 & 1 - \lambda & 0 \\
0 & 0 & 0 & 2 - \lambda
\end{vmatrix}
\]

(57)

Solving for the eigenvalues by find the determinant gives:

\[
= (2 - \lambda)^2 [(1 - \lambda)^2 - 1] = (2 - \lambda)^2(\lambda^2 - 2\lambda) = \lambda(\lambda - 2)^3
\]

(58)

with eigenvalues $\lambda = 2, 0$ The degeneracy is $2l + 1$, such that $\lambda = l(l + 1)$. So for $\lambda = 0 \rightarrow l = 0$ and for $\lambda = 2 \rightarrow l = 1$

2 spinors

A spinor means the particle which has spin half and when we have two spinors we have 4 states. The can be written as:

\[
|++\rangle
\]

(59)
\( |+\rangle \)
\( |-\rangle \)
\( |\pm\rangle \)
\( |\mp\rangle \)
\( S_{\pm} = (S_x \pm iS_y) \)
\( \vec{S} = \vec{S}_1 + \vec{S}_2 \)
\( [S^2, S_z] = 0 \)

I want to find the common eigenstates of these two.

\[ S^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \]

The eigenvalues are 1 and 0. The \( l = 1 \) has degeneracy of three. The \( l = 0 \) has the degeneracy of one. We want to figure out the eigenstates and:

\( |++\rangle = |11\rangle \)
\( |--\rangle = |1-1\rangle \)
\( |\mp\rangle = |1-1\rangle \)

\( |10\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) \)
\( |1\mp\rangle = \frac{1}{\sqrt{2}} (|+\rangle - |\mp\rangle) \)

\( m_1 = -\frac{1}{2}, \frac{1}{2} \)
\( m_1 = -\frac{1}{2}, \frac{1}{2} \)
\( m_2 = -\frac{1}{2}, \frac{1}{2} \)

The original number of states, which is \( 3 + 1 = 4 \). All three of these states have a spin of one.

\( S_+ |11\rangle = 0 \)
\( S_- |11\rangle = |10\rangle \)
\( S_- |10\rangle = |1-1\rangle \)

The states correspond to: \( S = 1 \) being all symmetric and \( S = 0 \) being antisymmetric.
Alternative Derivation:

\[ M = m_1 + m_2 \]  
\[ |++\rangle = |11\rangle \]

what we should do is to select highest \( m_1 \) and the highest \( m_2 \)

\[ m_1 = m_2 = \pm \frac{1}{2} \]

Therefore \( M = 1 \) and \( S \) must be equal to 1. How we can find the forth state?

We know the 4th state should be orthogonal to all the previous \( S=1 \) states. Therefore:

\[ |00\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |+-\rangle) \]

So lets generalize our results: Imagine instead of two spin half particles we have one angular momentum \( J_1 \) and another angular momentum \( J_2 \). \( J \) could be the spin or a combination of angular momentum and spin. We know the eigenstates of the \( J_1 \) which is:

\[ |j_1m_1\rangle \]

and for \( J_2 \) which is

\[ |j_2m_2\rangle \]

and when we put the two particle in one system then we will have:

\[ |j_1j_2; m_2m_2\rangle \]

for the eigenstate. Now we want to build states which belong to total angular momentum and we define the total angular momentum as:

\[ \vec{J} = \vec{J}_1 + \vec{J}_2 \]

And the set of commuting observables \( \{ J^2, J_z \} \). So we should be able to find states which have quantum numbers appropriate for these two and they will be:

\[ |J; M\rangle \]

So we want to find the common eigenstates of \( J^2 \) and \( J_z \) :

\[ J^2 \ | J; M\rangle = \hbar^2 J(J + 1) \ | J; M\rangle \]

\[ J_z \ | J; M\rangle = \hbar M \ | J; M\rangle \]

Now we want to find all the eigenvalues. We know that \( m_1 = -j_1, \ldots, j_1 \) and \( m_2 = -j_2, \ldots, j_2 \) so we will have space with \( (2j_1 + 1) \times (2j_2 + 1) \) different eigenvalues and that is how many states we have the highest possible \( M \) is \( j_1 + j_2 \) and there is only one state which has the highest \( M \).
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\[ | j_1 j_2; j_1 j_2 \rangle = | j_1 + j_2; j_1 + j_2 \rangle \]  

(87)

and

\[ M = -J, ..., J \]  

(88)

By acting \( J_- \) we should be able to produce other states. Let's count the states which we have here.

Example:
Let's choose \( j_1 = 3 \) and \( j_3 = 2 \) how many states do we have originally:

\[(2j_1 + 1) \times (2j_2 + 1) = (2 \times 3 + 1) \times (2 \times 2 + 1) = 7 \times 5 = 35 \]  

(89)

So \( J = 5, 4, 3, 2, 1, 0 \) but can we really have all these numbers? The degeneracy of \( J = 5 \) is \( 11 = 2 \times 5 + 1 \) and for the other \( J \):

<table>
<thead>
<tr>
<th>( J )</th>
<th>Degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

If we add all these degeneracies we will have \( 11 + 9 + 7 + 5 + 3 + 1 = 36 \) and this tells us that we can not have \( J = 0 \). So the \( J_{\max} = 5 \) and the \( J_{\min} = 1 \).

We can say:

\[ J_{\max} = j_1 + j_2 \]  

(90)

\[ J_{\min} = j_1 - j_2 \]  

(91)

By counting states if \( j_1 \geq j_2 \) and we do know that for each \( J \) we have \( 2J + 1 \) multiplicity

\[ \sum_{j_1 + j_2} (2J + 1) = 2 \sum_{j_1 - j_2} J + \sum_{j_1 - j_2} 1 \]  

(92)

\[ = 2(\sum_1^{j_1+j_2} J - \sum_1^{j_1-j_2-1} J) + j_1 + j_2 - (j_1 - j_2) + 1 \]  

(93)

by using:

\[ \sum_1^N J = \frac{N(N+1)}{2} \]  

(94)

We will have:
\((j_1 + j_2) (j_1 + j_2 + 1) - (j_1 + j_2 - 1) (j_1 - j_2) + 2j_2 + 1\)  
(95)

By doing simple algebra here we will have:
\[= (j_1 + j_2)^2 + j_1 + j_2 - (j_1 - j_2)^2 + j_1 - j_2 + 2j_2 + 1\]  
(96)

So we can write it as:
\[= 4j_1j_2 + 2j_1 + 2j_2 + 1\]  
(97)

By factoring this:
\[= (2j_1 + 1) (2j_2 + 1)\]  
(98)

That is exactly the number of states we began with and tells us that our guess was very good. So:
\[j_1 + j_2 \geq J \geq |j_1 - j_2|\]  
(99)

This gives the angular momentum a nice vector sense and in this picture we can visualize it as a vector. Let's construct all those eigenstates by starting from the highest eigenstate and acting \(J^-\) on that state.

\[M_{\text{highest}} = j_1 + j_2\]  
(100)

The general formula for \(J^-\) is given as:
\[J^- |J, M\rangle = \hbar \sqrt{J (J + 1) - M (M - 1)} | J, M - 1\rangle\]  
(102)

acting \(J^-\) on the highest state we have:
\[J^- | j_1 + j_2, j_1 + j_2\rangle = \hbar \frac{\sqrt{(j_1 + j_2) (j_1 + j_2 + 1) - (j_1 + j_2 - 1)}}{|j_1 + j_2, j_1 + j_2 - 1\rangle} \]  
(103)

\[= \hbar \sqrt{(j_1 + j_2)} | j_1 + j_2, j_1 + j_2 - 1\rangle\]  
(104)

So this tells us we can have new state by acting \(J^-\) on the highest one the new state:
\[| \psi \rangle = | j_1 + j_2, j_1 + j_2 - 1\rangle = \frac{1}{\hbar \sqrt{2 (j_1 + j_2)}} J^- | j_1 + j_2, j_1 + j_2\rangle\]  
(106)

To figure out the new state we write it as:
\[= \frac{1}{\hbar \sqrt{2 (j_1 + j_2)}} (J_1^- + J_2^-) | j_1j_2, j_1j_2\rangle\]  
(107)

Then we write it using \(J_1^-\) and \(J_2^-\).
\[ \text{Angular momentum} \]

\[ = \frac{1}{\hbar \sqrt{2 \left( j_1 + j_2 \right)}} \left( \hbar \sqrt{j_1 (j_1 + 1)} - j_1 (j_1 - 1) \left| j_1 j_2; j_1 - 1 j_2 \right\rangle \\
+ \hbar \sqrt{j_2 (j_2 + 1)} - j_2 (j_2 - 1) \left| j_1 j_2; j_1 j_2 - 1 \right\rangle \right) \] (108)

So by summarizing it we have:

\[ = \sqrt{j_1 \left| j_1 j_2; j_1 - 1 j_2 \right\rangle} + \sqrt{j_2 \left| j_1 j_2; j_1 j_2 - 1 \right\rangle} \] (109)

So this is our new state. By repeating this procedure we can have all the M values. We started with \( M = j_1 + j_2 \) and we have all of them now:

\[ M = j_1 + j_2, j_1 + j_2 - 1, \ldots, -j_1 - j_2 \] (110)

So we know two states which have the M that we want. Do we know any other states? There are no other states and we can find that by a simple arithmetic. Here we only have two choices and we want to have the state with highest M and would have to be a linear combination of these two and also it has to be orthogonal to these two states and this is:

\[ = \sqrt{j_1 \left| j_1 j_2; j_1 - 1 j_2 \right\rangle} - \sqrt{j_2 \left| j_1 j_2; j_1 j_2 - 1 \right\rangle} \] (111)

and this tells us

\[ \left| j_1 + j_2 + 1 j_1 + j_2 + 1 \right\rangle = \sqrt{j_1 \left| j_1 j_2; j_1 - 1 j_2 \right\rangle} - \sqrt{j_2 \left| j_1 j_2; j_1 j_2 - 1 \right\rangle} \] (112)

So we constructed the next state by acting \( J_+ \) on the highest M.

We can repeat this until we have all the states. All these has been summarized by the Clebsch-Gordan Coefficients.

**Clebsch-Gordan Coefficients**

We start with:

\[ \left| j_1 j_2; m_1 m_2 \right\rangle \implies \left| JM \right\rangle \] (113)

RHS and LHS of the above form an orthonormal bases.

\[ \left| JM \right\rangle = \sum_{m_1 = -j_1}^{j_1} \sum_{m_2 = -j_2}^{j_2} a_{JM m_1 m_2} \left| j_1 j_2; m_1 m_2 \right\rangle \] (114)

most of the coefficients \( (a) \) are 0 and if you multiply both sides by bra of the RHS we will have the coefficients:

\[ a_{JM m_1 m_2} = \langle j_1 j_2; m_1 m_2 | JM \rangle \] (115)
and these coefficients are all real (and remember that most of them are zero) and so if we take the complex conjugate we would have the same number:

\[ = \langle JM \mid j_1 j_2; m_1 m_2 \rangle \]  

(116)

and we know that:

\[ M = m_1 + m_2 \]  

(117)

otherwise the coefficient is zero and this reduces the number of coefficients and also we figure out:

\[ |j_1 - j_2| \leq J \leq j_1 + j_2 \]  

(118)

and that relation also reduces the number of coefficient to.

**Example:** We have

\[ \langle j_1 j_2 + j_1 j_2 \mid j_1 + j_2; j_1 + j_2 \rangle = 1 \]  

(119)

So here it should be 1 and if we have:

\[ \langle j_1 j_2; j_1 - 1 j_2 \mid j_1 + j_2; j_1 + j_2 - 1 \rangle = \sqrt{\frac{j_1}{j_1 + j_2}} \]  

(120)

the coefficient is \( \sqrt{\frac{j_1}{j_1 + j_2}} \) and also for:

\[ \langle j_1 j_2; j_1 j_2 - 1 \mid j_1 + j_2; j_1 + j_2 - 1 \rangle = \sqrt{\frac{j_2}{j_1 + j_2}} \]  

(121)

So these are the coefficients.

\[ \sum_{m_1 = -j_1}^{j_1} \sum_{m_2 = -j_2}^{j_2} | j_1 j_2; m_1 m_2 \rangle \langle j_1 j_2; m_1 m_2 | = 1 \]  

(122)

\[ \sum_{J = |j_1 - j_2|}^{j_1 + j_2} \sum_{M = -J}^{J} | JM \rangle \langle JM | = 1 \]  

(124)

\[ \langle j_1 j_2 m_1 m_2 \mid \sum_{J = |m_1 - m_2|}^{j_1 + j_2} \sum_{M = -J}^{J} \delta_{m_1 m_2} a_{JM m_1 m_2} \rangle = \delta_{m_1 m_2} \delta_{m_2 m_2} \]  

(125)

because a’s are orthonormals to each other.

Last time we spoke of Clebsch-Gordan Coefficients. You obtain these from transforming basis.

\[ | JM \rangle = \sum A_{JM \mid j_1 j_2 m_1 m_2} | j_1 j_2; m_1 m_2 \rangle \]  

(126)
Angular momentum

\[ A_{JMj_1j_2m_1m_2} = \langle j_1j_2; m_1m_2 | JM \rangle = \langle JM | j_1j_2; m_1m_2 \rangle \quad (127) \]

\[ |j_1j_2; m_1m_2\rangle = \sum_{J=|j_1-j_2|}^{J} \sum_{M=-J}^{J} A_{JMj_1j_2m_1m_2} |JM\rangle \quad (128) \]

and

\[ \sum \sum |j_1j_2; m_1m_2\rangle \langle j_1j_2; m_1m_2| = II \quad (129) \]

\[ \sum \sum |JM\rangle \langle JM| = II \quad (130) \]

Recurrence Relations

Act on equation (1.1) with the \( J_− \) operator.

\[ \vec{J} = \vec{J}_1 + \vec{J}_2 \quad (131) \]

From last semester we know:

\[ J_− |JM\rangle = \hbar \sqrt{J(J+1) - M(M-1)} |JM_{-1}\rangle \quad (132) \]

But now our state, \(|JM\rangle\), is made up of two states. Introduce a sum to include the lowering operator acting on \( m_1 \) and \( m_2 \).

\[ J_− |JM\rangle = \sum A_{JMj_1j_2m_1m_2} \left[ \hbar \sqrt{j_1(j_1+1) - m_1(m_1-1)} |j_1j_2; m_1 - 1m_2\rangle \\
+ \hbar \sqrt{j_2(j_2+1) - m_2(m_2-1)} |j_1j_2; m_1m_2 - 1\rangle \right] \]

Take the inner product:

\[ \langle j_1j_2; m_1' m_2' | j_1j_2; m_1m_2 - 1 \rangle \quad (133) \]

Using equation (1.3) we obtain

\[ \langle j_1j_2; m_1' m_2' | j_1j_2; m_1m_2 - 1 \rangle \]

Using equation (1.3) we obtain

\[ \hbar \sqrt{J(J+1) - M(M-1)} a_{JM - l_1 l_2 m_1' m_2'} \\
= \sum a_{JMj_1j_2m_1m_2} \hbar \sqrt{j_1(j_1+1) - m_1(m_1-1)} \delta_{m_1 - 1m_1'} \delta_{m_2m_2'} \\
+ \hbar \sqrt{j_2(j_2+1) - m_2(m_2-1)} \delta_{m_1m_1'} \delta_{m_2 - 1m_2'} \\
= \hbar \sqrt{j_1(j_1+1) - m_1(m_1+1)} a_{JMj_1j_2m_1' + 1m_2'} \\
+ \hbar \sqrt{j_2(j_2+1) - m_2(m_2+1)} a_{JMj_1j_2m_1' m_2' + 1} \quad (134) \]
Example 1: \( j_1 = j_2 \)

(Remember we can rewrite \(|lm\rangle\) as \(Y_l^m(\theta, \psi)\))

If \( j_1 = j_2 = 1 \) and \( m_1, m_2 = -1, 0, 1 \), then I'm interested in the state

\[
|11; m_1 m_2 \rangle \rightarrow Y_{l_1}^{m_1}(\theta, \psi) Y_{l_2}^{m_2}(\theta, \psi)
\]

(135)

Where \( J \) can take on three values: 0, 1, 2

First case: \( J = 2 \)

Start with the highest state \(|22\rangle = |11; 11\rangle\) and act on it with \( J_- \).

\[
J_- |22\rangle = \hbar \sqrt{2(2+1) - 2(2-1)} |21\rangle = 2\hbar |21\rangle
\]

(136)

We can also write (1.11) as:

\[
|21\rangle = A |11; 01\rangle + B |11; 10\rangle
\]

(137)

By normalization we can determine the coefficients are \( \sqrt{2} \). Therefore our normalized state is:

\[
|21\rangle = \frac{1}{\sqrt{2}} \left[ |11; 01\rangle + |11; 10\rangle \right]
\]

Repeat this process for \(|21\rangle\), \(|20\rangle\), and \(|2-1\rangle\).

\[
|20\rangle = \frac{1}{\sqrt{6}} \left[ |11; 1 - 1\rangle + 2 |11; 00\rangle + |11; -11\rangle \right]
\]

(138)

\[
|2-1\rangle = \frac{1}{\sqrt{2}} \left[ |11; 0 - 1\rangle + |11; -10\rangle \right]
\]

(139)

\[
|2-2\rangle = |11; -1 - 1\rangle
\]

(140)

Take time to notice that the \(|2-1\rangle\) and \(|2-2\rangle\) states are mirror images of the \(|21\rangle\) and \(|22\rangle\) states (with \( m = -m \)).

Second case: \( J = 1 \)

From inspection we can deduce \(|11\rangle = \frac{1}{\sqrt{2}} (|11; 10\rangle - |11; 01\rangle)\).

By applying the lowering operator \( J_- \) to \(|11\rangle\) we see:

\[
J_- |11\rangle = |10\rangle = \frac{1}{\sqrt{2}} \left[ (|11; 1 - 1\rangle - |11; -11\rangle) - 0 |11; 00\rangle \right]
\]

(141)

Therefore,

\[
|10\rangle = \frac{1}{\sqrt{2}} \left( |11; 1 - 1\rangle - |11; -11\rangle \right)
\]

(142)

\[
|1 - 1\rangle = \frac{1}{\sqrt{2}} \left( |11; 0 - 1\rangle - |11; 0 - 1\rangle \right)
\]

(143)
Third case: \( J = 0 \)

\[
|00\rangle = a|11; 1 - 1\rangle + b|11; -11\rangle - c|11; 00\rangle \tag{144}
\]

We can deduce \( a, b, \) and \( c \) by exploiting orthogonality.

\[
\langle 00 | 10 \rangle = \frac{1}{\sqrt{2}}(a - b) = 0 \\
\langle 00 | 20 \rangle = \frac{1}{\sqrt{6}}(a + 2c - b) = 0 \\
a = b \rightarrow 2b + 2c = 0 \rightarrow a = b = -c
\]

Normalize: \( a^2 + b^2 + c^2 = 1 \)

\[
|00\rangle = \frac{1}{\sqrt{3}}(|11; 1 - 1\rangle + |11; -11\rangle - |11; 00\rangle) \tag{145}
\]

Example 2: \( j_1 = l_1 \) and \( j_2 = l_2 \)

\[
|l_1l_2; m_1m_2\rangle \rightarrow Y_{l_1}^{m_1}(\theta_1, \psi_1)Y_{l_2}^{m_2}(\theta_2, \psi_2) \\
|LM\rangle \rightarrow \Phi(\theta_1, \psi_1; \theta_2, \psi_2) = \\
\sum \sum \langle l_1l_2; m_1m_2 |LM\rangle Y_{l_1}^{m_1}(\theta_1, \psi_1)Y_{l_2}^{m_2}(\theta_2, \psi_2)
\]

\[
Y_{l_1}^{m_1}(\theta_1, \psi_1)Y_{l_2}^{m_2}(\theta_2, \psi_2) = \sum \sum \langle l_1l_2; m_1m_2 |LM\rangle \Phi(\theta_1, \psi_1; \theta_2, \psi_2) \tag{146}
\]

Two Coincident particles

\( \theta = \theta_1 = \theta_2 \) and \( \psi = \psi_1 = \psi_2 \)

\( \Phi_{LM} = A(\theta, \psi) \) represents the probability amplitude (wavefunction).

Where \( |A|^2 \) = probability of particles "coinciding" in a given direction.

Think of \( A \) as a single particle: \( A_{LM} = \sum a_{lm}Y_l^m \)

Apply \( L_z \) to \( A_{LM} \):

\[
L_zA_{LM}(\theta, \psi) = -i\hbar \frac{\partial}{\partial \psi} A_{LM} \tag{147}
\]

\[
-i\hbar \frac{\partial}{\partial \psi} A_{LM} = -i\hbar \frac{\partial}{\partial \psi_1} A_{LM} + -i\hbar \frac{\partial}{\partial \psi_2} A_{LM} \tag{148}
\]

\[
L_zA_{LM}(\theta, \psi) = (m_1\hbar + m_2\hbar)A_{LM} = M\hbar A_{LM} \tag{149}
\]

This tells us that \( A_{LM} \) must be a single spherical harmonic and shares common eigenstates.
Therefore $A_{LM}$ is no longer a sum.

$$A_{LM} = a_{LM} Y^M_L$$

and

$$L_z A_{LM} = M \hbar A_{LM}$$

$$L^2 A_{LM} = \hbar (L+1) A_{LM}$$

If we apply the lowering operator to $A_{LM}$ we get

$$L^- A_{LM} = \hbar \sqrt{L(L+1) - M(M-1)} A_{LM-1}$$

Which is the same as when we apply the lowering operator to a spherical harmonic.

$$L^- A_{LM} = \hbar \sqrt{L(L+1) - M(M-1)} Y^M_{L-1}$$

From this we see $a_{LM}$ is independent of $M$ and we have this final expression:

$$A^M_L = a_L Y^M_L$$

Consider the forward direction. Set $\theta_1 = \theta_2 = 0$ or $\psi_l^m(0, \psi)$

The sum will collapse to:

$$\Phi_{LM}(0, \psi; 0, \psi) = \langle l_1 l_2 00 | L0 \rangle Y^0_{l_1}(0, \psi) Y^0_{l_2}(0, \psi)$$

$$\Phi_{LM}(0, \psi; 0, \psi) = A^0_L(0, \psi) = a_L Y^0_{l_1}(0, \psi) = a_L \sqrt{\frac{2L+1}{4\pi}}$$

$$a_L = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}} \langle l_1 l_2 00 | L0 \rangle$$

Set $\theta_1 = \theta_2 = \theta$

$$Y^M_L(\theta, \psi) = \frac{1}{a_L} \sum \langle l_1 l_2; m_1 m_2 | LM \rangle Y^m_{l_1}(\theta, \psi) Y^{m_2}_{l_2}(\theta, \psi)$$

This is known as the spherical harmonic addition relation.

Take the inner product:

$$\int d^2 \Omega Y^m_{l_1}(\Omega) Y^{m_2}_{l_2}(\Omega) Y^{m_3}_{l_3}(\Omega) =$$

$$\sum \langle l_1 l_2; m_1 m_2 | LM \rangle a_L \int Y^m_{l_3}(\Omega) Y^M_L d\Omega$$

Using the relation: $Y^m_L = (-1)^m Y^{-m}_L$. We see that only one term survives:

$$\int d^2 \Omega Y^m_{l_1}(\Omega) Y^{m_2}_{l_2}(\Omega) Y^{m_3}_{l_3}(\Omega) = (-1)^m a_L \langle l_1 l_2; m_1 m_2 | l_3 - m_3 \rangle$$
Angular momentum

\[ m_1 + m_2 = -m_3 \text{ and } |l_1 - l_2| \leq l_3 \leq l_1 + l_2 \]

This relates to last semester when we studied the rotational eigenstates of the ammonia molecule.

\[ \langle l_1 m_1 \left| Y_{l_2}^{m_2} \right| l_3 m_3 \rangle \]  \hspace{1cm} (163)

Where the spherical harmonic represents the electric dipole moment and the \( m \) values represent the energy levels.

**Wigner-Eckart Theorem**

**Scalars**

What is the definition of an scalar? It is invariant under rotations or in other word it is simply one number. An example of common scalar operators are \( \vec{r}^2 \) and \( \vec{p}^2 \).

\[ \vec{r} \cdot \vec{p} + \vec{r} \cdot \vec{p} \]  \hspace{1cm} (164)

Total angular momentum always generates rotation and we have which consists of Angular Momentum and Spin:

\[ \vec{J} = \vec{L} + \vec{S} \]  \hspace{1cm} (165)

To check to see if it is invariant under rotation we just need to check for an infinitesimal roation. We have a convenient check at hand, by definition of a QM scalar, if the commutator with \( J \) vanishes then we have a scalar. If the commutator is zero then no matter how complicated the matrix is it means that the quantity is scalar.

\[ [A, \vec{J}] = 0 \]  \hspace{1cm} (166)

**Vectors**

To see what is the definition of vector lets do the infinitesimal rotation again.

\[ \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \vec{p}, \vec{L}, \vec{S}, \vec{J} \]  \hspace{1cm} (167)

If we do a very small roation around the Z-axis, then we would have for rotations with \( \theta << 1 \):

\[ x' = x - \theta y \]
\[ y' = y + \theta x \]
\[ z' = z \]  \hspace{1cm} (168)

\[ \vec{r}' = \vec{r} + \frac{1}{i\hbar} \theta [L_z, \vec{r}] \]

\[ [L_z, x] = [xp_y - yp_x, x] = i\hbar y \]  \hspace{1cm} (169)
\[ [L_z, y] = [xp_y - yp_x, y] = -i\hbar x \] (170)
\[ [L_z, z] = [xp_y - yp_x, z] = 0 \] (171)

So in other words, the transformation of \( \vec{r} \) is going to be the \([\vec{r}, L_z] \):
\[ [L_z, \vec{r}] = [xp_y - yp_x, \vec{r}] = -i\hbar \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \] (172)
\[ \vec{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \] (173)

So if it is a vector the results of commuting with \( J \) should be like this.
\[ [J_i, V_j] = i\hbar \varepsilon_{ijk} V_k \] (174)

with
\[ \varepsilon_{ijk} = \begin{pmatrix} 1 : cyclic \\ -1 : noncyclic \\ 0 : repeated \end{pmatrix} \] (175)

Through the cyclic permutations of the above commutator we are given 9 relations which define a vector. We know three and the other 6 can be found with them. Let’s switch back to scalars, saying \( A \) is a scalar and we want to know the matrix elements of it. Eigenstates of \( A \) are given in the basis:
\[ | jm \rangle \] (176)
\[ \langle jm | A | jm \rangle = a_j m \langle jm | jm \rangle \] (177)

We know that: \([A, J_z] = 0 \) and \([A, J^2] = 0 \) and thus we see that \( A, J_z, J^2 \) form a C.S.C.O. We know that if two operators commute so they share the same common eigenstates.
\[ A | jm \rangle = a_j | jm \rangle \] (178)

The selection rules tell us a transition is only possible when \( \langle jmt | A | jm \rangle \neq 0 \) only if \( jt = j \) and \( mt = m \).
\[ \langle jmt | J_+, A | jm \rangle = 0 \] (179)
\[ \langle jmt | J_+ A | jm \rangle = \langle jmt | AJ+ | jm \rangle = \\
= \hbar \sqrt{j(j+1) - mt(mt-1)} a_{jm} \langle jmt-1 | jm \rangle = \\
= \hbar \sqrt{j(j+1) - m(m-1)} a_{jm-1} \langle jmt | jm+1 \rangle = \] (180)
Angular momentum

\[ m = m' - 1 \]
\[ a_{jm+1} = a_{jm} = a_{jm-1} = a_{jm-2} = a_j \]
\[ j' = j \]

When \( j \) is fixed then \( A \) is a matrix and we have \( 2j + 1 \) states and we can think about \( A \) as \( (2j + 1) \times (2j + 1) \) matrix, \( |jm\rangle \), where \( m = -j \cdots j \). This matrix is symmetric and it is identity.

\[ A = a_j II_{(2j+1) \times (2j+1)} \] (181)

\( B \) is also a scalar:

\[ B |jm\rangle = b_j |jm\rangle \] (182)

\( B = b_j II \)

So we can see \( A \propto B \) and it can be rewritten in the form of \( \Rightarrow A = \lambda B \)

where \( \lambda = \frac{a_j}{b_j} \).

Vectors

By using our definition of vectors. Instead of \( A \) here we have \( V \) which has three separate components:

\[ [V_z, J_z] = 0 \] (183)

\[ \langle jmt | [V_z, J_z] | jm\rangle = 0 \] (184)

\[ \langle jmt | V_z J_z | jm\rangle = \langle jmt | J_z V_z | jm\rangle \] (185)

\[ \hbar m \langle jmt | V_z | jm\rangle = \hbar m \langle jmt | V_z | jm\rangle \] (186)

So \( \langle jmt | V_z | jm\rangle = 0 \) or \( m = m' \) Lets try for other components.

\[ V_\pm = V_x \pm iV_y \] (187)

\[ [J_z, V_\pm] = \hbar V_\mp \] (188)

\[ [J_z, V_-] = -\hbar V_- \] (189)

Lets see what we can do for \( V_z \):

\[ \langle jmt | [J_z, V_\pm] | jm\rangle = \pm \hbar \langle jmt | V_\pm | jm\rangle \]
\[ = \hbar \langle jmt | V_\pm | jm\rangle - \hbar \langle jmt | V_\pm | jm\rangle \] (190)

So we can conclude that \( \langle jmt | V_\pm | jm\rangle = 0 \) unless \( mt = m + 1 \). For \( V_- \) we have the same. So we have selection Rules which are:
\[ V_z : \Delta m = 0 \]
\[ V_\pm : \Delta m = \pm 1 \]  

(191)

Now if we fix \( j \), what we can say about the matrices?

\[ [J_+, V_+] = [J_x + iJ_y, V_x + iV_y] = i(\hbar)\ V_z + i(-\hbar)\ V_z = 0 \]  

(192)

Now let’s consider the matrix elements, we specialize it for simplicity of calculations:

\[ \langle j\text{m}' | J_+ V_+ | j\text{m} \rangle = \langle j\text{m}' | V_+ J_+ | j\text{m} \rangle \]  

(193)

\[ \langle j\text{m} + 2 | J_+ V_+ | j\text{m} \rangle = \langle j\text{m} + 2 | V_+ J_+ | j\text{m} \rangle \]  

(194)

with:

\[ I = \sum | jm \rangle \langle jm | \]  

(195)

So we only can get something from that summation when \( m' = m + 1 \) contributes.

\[ \langle j\text{m} + 2 | J_+ | j\text{m} + 1 \rangle \langle j\text{m} + 1 | V_+ | j\text{m} \rangle = \langle j\text{m} + 2 | J_+ | j\text{m} + 1 \rangle \langle j\text{m} + 1 | V_+ | j\text{m} \rangle \]  

(196)

\[ \frac{\langle j\text{m} + 1 | V_+ | j\text{m} \rangle}{\langle j\text{m} + 1 | J_+ | j\text{m} \rangle} = \frac{\langle j\text{m} + 2 | V_+ | j\text{m} + 1 \rangle}{\langle j\text{m} + 2 | J_+ | j\text{m} + 1 \rangle} = \alpha_+ \]  

(197)

Notice, that this is independent of \( m \); thus the ratio is a constant and that is a very important conclusion. So we can say:

\[ V_+ = \alpha_+ J_+ \]
\[ V_- = \alpha_- J_- \]
\[ V_z = \alpha J_z \]  

(198)

We have to figure out all the three components.

\[ [J_-, V_+] = i(\hbar)V_z - i(-\hbar)V_z = -2\hbar V_z \]  

(199)

So:

\[ \langle j\text{m} | (J_- V_+ - V_+ J_-) | j\text{m} \rangle = -2\hbar \langle j\text{m} | V_+ | j\text{m} \rangle \]
\[ = \hbar \sqrt{j(j + 1) - m(m + 1)} \langle j\text{m} + 1 | V_+ | j\text{m} \rangle \]
\[ -\hbar \sqrt{j(j + 1) - m(m - 1)} \langle j\text{m} | V_+ | j\text{m} + 1 \rangle \]  

(200)

We know everything except for \( \alpha_{pm} \) so let us calculate them where:
\begin{align*}
\langle jm + 1 | V_+ | jm \rangle &= \alpha_+ J_+ = \alpha_+ \hbar \sqrt{j(j+1) - m(m+1)} \quad (201) \\
\text{and} \\
\langle jm | V_+ | jm + 1 \rangle &= \alpha_+ J_+ = \alpha_+ \hbar \sqrt{j(j+1) - (m-1)m} \quad (202)
\end{align*}

This gives
\begin{align*}
\alpha_+ h^2[j(j+1) - m(m+1) - j(j+1) + m(m-1)] \\
&= -2m\alpha_+ h^2 = -2hV_z 
\end{align*}

Now we can say:
\begin{align*}
\langle jm | V_z | jm \rangle &= m\hbar \alpha_+ = m\hbar \alpha_- \quad (204) \\
\alpha_+ = \alpha_- = \alpha
\end{align*}

\begin{align*}
[J_+, V_-] &= 2hV_z 
\langle jm | J_z | jm \rangle &= m\hbar \\
V_z &= \alpha J_z \quad (207)
\end{align*}

This gives us a generalized result and a very powerful statement:
\begin{align*}
\vec{V} &= \alpha \vec{J} 
\end{align*}

If we have: \( \vec{W} = \beta \vec{J} \) and \( \vec{V} \propto \vec{W} \), then when we fix \( j \):
\begin{align*}
V_\parallel &= \vec{V} \propto \vec{J} 
\end{align*}

So if we think of those as vectors we can write:
\begin{align*}
\vec{J} \cdot \vec{V} \propto \vec{J}^2 &= \alpha j(j+1)\hbar^2 
\end{align*}

and calculate coefficient \( \alpha \):
\begin{align*}
\langle \vec{J} \cdot \vec{V} \rangle &= \alpha j(j+1)\hbar^2 \\
\alpha &= \frac{\langle \vec{J} \cdot \vec{V} \rangle}{j(j+1)\hbar^2} = \frac{\langle \vec{J} \cdot \vec{V} \rangle}{\langle \vec{J}^2 \rangle} \\
V_\parallel &= V \cos(\theta) \\
\vec{J} \cdot \vec{V} &= JV \cos(\theta)
\end{align*}

So in this geometrical view we can conclude that:
\[ V_{\parallel} = \frac{\vec{J} \cdot \vec{V}}{J} \]  \hspace{1cm} (215)

\[ \alpha = \frac{V_{\parallel}}{J} = \frac{\vec{J} \cdot \vec{V}}{J^2} \]  \hspace{1cm} (216)

Recap from last class: Last time we spoke about vectors and found that for an arbitrary vector: \( \vec{V} \) is proportional to \( \vec{J} \) and more importantly for a fixed \( j \) value we have:

\[ \vec{V} = \alpha \vec{J} \]  \hspace{1cm} (217)

Today’s Notes:

The Hydrogen Atom

Remember that we found the energy levels for the hydrogen atom before

\[ E_n = -\frac{E_{\text{ion}}}{n^2} \]  \hspace{1cm} (218)

and we can denote the energy levels as \( |n, l, m) \).

Say you have a complicated atom with a Hamiltonian like:

\[ H_o = T_o + V_o + \ldots \]  \hspace{1cm} (219)

Also this complicated atom has total angular momentum \( \vec{L} \) and also \( \vec{S} \). For all the constituents, the total angular momentum, \( \vec{J} \), will be the sum of the spin and angular momentum components:

\[ \vec{J} = \vec{L} + \vec{S} \]  \hspace{1cm} (220)

Which is very important because this can generate rotation. And we know that \( [H_o, \vec{J}] = 0 \) even though we know nothing about the complicated atom. (\( H_o \) is a scalar and invariant under rotation)

We also know: \( [H_o, \vec{L}^2] = [H_o, \vec{S}^2] = 0 \) and \( \{H_o, J_z, L^2, S^2\} \)

In general our energy level can be written as: \( |E_o, L, S, J, M) \). If we fix \( J \) and \( E_o \), the energy level will have a fixed \( J \) and \( M = -J, \ldots, J \) with a degeneracy of at least \( 2J + 1 \) and in the case of hydrogen atom it is even more because it does not even depend on \( \vec{J} \). To break down this degeneracy we will add a magnetic field.
Angular momentum

Example 1:

Set $\vec{B} = B \hat{z}$.

$H = H_o + H_1$ \hspace{1cm} (221)

Where

$H_1 = -\frac{q}{2m} [ (\vec{B} \cdot \vec{L}) + 2 \vec{B} \cdot \vec{S} ]$ \hspace{1cm} (222)

Note: The coefficient 2 is due to relativistic effects.

$H_1 = \omega_L (L_z + 2S_z)$ \hspace{1cm} (223)

In which:

$\omega_L = -\frac{qB}{2m}$ \hspace{1cm} (224)

and as we know it is the Larmor frequency. By fixing $L, S, J$ we will try to figure out what will happen to the energy level $|E_o, L, S, J, M\rangle$

$\vec{L} = \frac{\langle \vec{L} \cdot \vec{J} \rangle}{J(J+1)\hbar^2} \vec{J}$ \hspace{1cm} (225)

$\vec{S} = \frac{\langle \vec{S} \cdot \vec{J} \rangle}{J(J+1)\hbar^2} \vec{J}$ \hspace{1cm} (226)

and these are two operators we write based on $\vec{J}$. Now we have to calculate the expectation values:

$\vec{L} \cdot \vec{J} = \vec{L}(\vec{L} + \vec{S}) = \vec{L}^2 + (\vec{L} \cdot \vec{S})$ \hspace{1cm} (227)

$\vec{L} \cdot \vec{J} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2) + \vec{L}^2 = \frac{1}{2}(\vec{J}^2 + \vec{L}^2 - \vec{S}^2)$ \hspace{1cm} (228)

Therefore,

$\vec{L} = \frac{[\frac{1}{2}(J(J+1) + L(L+1) - S(S+1)]}{J(J+1)} \vec{J}$ \hspace{1cm} (229)

and

$\vec{S} = \frac{[\frac{1}{2}(J(J+1) - L(L+1) + S(S+1)]}{J(J+1)} \vec{J}$ \hspace{1cm} (230)

$H_1 = \frac{\omega_L}{2J(J+1)} [3J(J+1) - L(L-1) + S(S+1)] J_z$ \hspace{1cm} (231)

By simplifying that we would have:

$H_1 = \omega_L \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} J_z$ \hspace{1cm} (232)
By acting this $H_1$ on our eigenstate we have:

$$H_1 |E_0, L, S, J, M⟩ = \omega_{L} g_J \hbar M |E_0, L, S, J, M⟩ \quad (233)$$

Here $g_J$ is the Lande factor.

**Spin-Spin Coupling: 2 magnetic moments**

Even if we do not have an external magnetic field; since each spin has a magnetic moment so one magnetic moment will produce magnetic field and the other spin can see that magnetic field so they can interact. For example in the simplest case like a hydrogen atom we have an electron with spin half and a proton with spin half. So electron can see the magnetic field due to proton spin and it generates very little splitting on electron’s energy levels which is very small.

$$H_0 = E_o I \quad (234)$$

$$H = H_o + a(\vec{S}_1 \cdot \vec{S}_2) \quad (235)$$

The question is do we know the energy levels of this Hamiltonian? we define the total spin as:

$$\vec{S}^2 = \vec{S}_{1^2} + \vec{S}_{2^2} + 2(\vec{S}_1 \cdot \vec{S}_2) \quad (236)$$

By acting it on our eigenstate we would have:

$$\tilde{S}^2 |SM⟩ = \left(\frac{3}{4} \hbar^2 + \frac{3}{4} \hbar^2 + 2(\vec{S}_1 \cdot \vec{S}_2)\right) |SM⟩ \quad (237)$$

$$\tilde{S}^2 |SM⟩ = S(S + 1)\hbar^2 |SM⟩ \quad (238)$$

$$\langle \vec{S}_1 \cdot \vec{S}_2 ⟩ |SM⟩ = \left(\frac{S(S + 1)}{2} - \frac{3}{4}\right)\hbar^2 |SM⟩ \quad (239)$$

$$H |SM⟩ = [E_o + \frac{\hbar^2}{2} - (S(S + 1) - \frac{3}{4})] |SM⟩ \quad (240)$$

and this is the new energy level, the new eigenvalue.

The system can transit between these two energy levels and when it transits, it emits a photon with frequency $\hbar \Omega$ which $\Omega$ is the Bohr frequency and as we can see from the above $\Omega = \hbar a$. Here we want to understand the spin of electron and just the first spin $\tilde{S}_1$ because usually one is going to do something and the other will not do anything. We want to see how its expectation value evolves in time:

$$\langle \vec{S}_1 ⟩_t \quad (241)$$
Angular momentum

That would tell us what kind of transition to expect and what kind of radiation we would observe. We have 4 states and we should have a 4 by 4 matrix:

\[ |SM\rangle = [ |1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle]\] (242)

\( S_{1z} \) is the easiest one to find so let's find it first:

\[ S_{1z} |SM\rangle = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \] (243)

We need two more components so:

\[ S_{1+} |SM\rangle = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \] (244)

To find the \( S_{1-} \) we don't have to calculate it because we do know that:

\[ S_{1-} = S_{1+}^T \] (245)

So we can conclude that:

\[ S_{1-} |SM\rangle = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \] (246)

As we can find out here:

\[ \vec{S} = \frac{1}{2} \vec{J} \] (247)

Which is the Wigner-Eckart theorem.

What information can we extract from:

\[ |\psi(0)\rangle = \alpha |0,0\rangle + \beta_1 |1,1\rangle \beta_0 |1,0\rangle + \beta_{-1} |1,-1\rangle \] (248)

Which is the wavefunction that the system can be determined with.

Apply the evolution operator.

\[ |\psi(t)\rangle = \alpha e^{-i(E_0 - \frac{3a\hbar^2}{\sigma^2}) \frac{t}{\tau}} |0,0\rangle + e^{-i(E_0 + \frac{3a\hbar^2}{\sigma^2}) \frac{t}{\tau}} [\beta_1 |1,1\rangle \beta_0 |1,0\rangle + \beta_{-1} |1,-1\rangle] \] (249)

This is the state of the system at time t and we can write it as:

\[ |\psi(t)\rangle = e^{-i(E_0 - \frac{3a\hbar^2}{\sigma^2}) \frac{t}{\tau}} [\alpha |0,0\rangle + e^{-i\Omega t} [\beta_1 |1,1\rangle \beta_0 |1,0\rangle + \beta_{-1} |1,-1\rangle] \] (250)
\[ \langle S_{1z} \rangle = \frac{\hbar}{\sqrt{2}} [ |\beta_1|^2 + e^{i\Omega t} \alpha \beta_0^* + e^{-i\Omega t} \alpha^* \beta_0 + |\beta_{-1}|^2 ] \] (251)

\[ \langle S_{1+} \rangle = \frac{\hbar}{\sqrt{2}} [ \beta_0^* \beta_0 + \beta_0^* \beta_1 - e^{i\Omega t} \beta_1^* \alpha + e^{i\Omega t} \beta_{-11} \alpha^* ] \] (252)

\[ \langle S_{1+} \rangle = \langle S_{1-} \rangle^* \] (253)

Example: Consider the states \(|1, 0\rangle \rightarrow |0, 0\rangle\) and here. Set \(\beta_{-1} = \beta_1 = 0\) and suppose \(\alpha\) is real so:

\[ \langle S_{1z} \rangle = \frac{\hbar}{\sqrt{2}} [ e^{i\Omega t} \alpha |\beta_0^*| e^{i\psi_0} + e^{-i\Omega t} \alpha^* |\beta_0| ] \] (254)

Where:

\[ \beta_0 = |\beta_0| e^{i\psi_0} \] (255)

\[ \langle S_{1z} \rangle = \frac{\hbar}{2}\alpha |\beta_0| (e^{i(\Omega t - \psi_0)} + e^{-i(\Omega t - \psi_0)}) \] (256)

\[ \langle S_{1z} \rangle = \hbar \alpha |\beta_0| \cos(\Omega t - \psi_0) \] (257)

\[ \langle S_{1+} \rangle = \langle S_{1-} \rangle = 0, \langle S_{1x} \rangle = \langle S_{1y} \rangle = 0 \] (258)

So we can think of \(S\) as a vector and we have oscillation along the \(z\)-axis and get linearly polarized radiation in the \(z\)-direction because of the oscillation.

Example B (for other transition): \(|1, 1\rangle \rightarrow |0, 0\rangle\)

\[ \beta_{-1} = \beta_1 = 0 \] (259)

\[ \langle S_{1z} \rangle = \frac{\hbar^2}{2} |\beta_1|^2 \] (260)

\[ \langle S_{1+} \rangle = -\frac{\hbar^2}{2} \alpha |\beta_1| e^{i\Omega t} \] (261)

\[ \beta_1 = |\beta_1| e^{i\psi_0} \] (262)

\[ \langle S_{1x} \rangle = -\frac{\hbar^2}{2} \alpha |\beta_1| \cos(\Omega t - \varphi_1) \] (263)

\[ \langle S_{1y} \rangle = -\frac{\hbar^2}{2} \alpha |\beta_1| \sin(\Omega t - \varphi_1) \] (264)

Therefore it precesses in the \(x\)-\(y\) plane around the \(z\)-axis and radiate circularly polarized light which is right handed (counterclockwise).