Useful constants

- \( \hbar = 1.05 \times 10^{-34} \text{ J} \cdot \text{s} \)
- \( c = 3 \times 10^8 \text{ m/s} \)
- 1 eV = \( 1.6 \times 10^{-19} \text{ J} \)

Problem 1

A high-resolution neutron interferometer narrows the energy spread of thermal neutrons of kinetic energy \( E = 0.02 \text{ eV} \) to a wavelength dispersion level of \( \Delta \lambda/\lambda \approx 10^{-9} \). A neutron has mass \( m_n = 1.67 \times 10^{-27} \text{ kg} \).

(a) Estimate the length of the wave packets in the direction of motion.

(b) Over what length of time will the wave packets spread appreciably (\( \Delta E \sim E \))?

Problem 2

For a wavefunction \( \psi(x,t) \) satisfying the one-dimensional Schrödinger equation

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = i\hbar \frac{\partial \psi}{\partial t}
\]

we define the probability charge and current densities, respectively, by

\[
\rho = |\psi|^2, \quad j = \frac{i\hbar}{2m} \left( \psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right)
\]

(a) Derive the continuity equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0
\]

By integrating, show that the total probability \( P = \int dx \rho \) is conserved.

(b) If we add a constant imaginary term to the potential, i.e., replace \( V(x) \) by \( V(x) - iV_0 \), where \( V_0 > 0 \), how does the continuity equation change?

Integrate the new continuity equation and find \( dP/dt \) in terms of \( V_0, \hbar \) and \( P \). Hence show that \( P \) decays exponentially with time.

Explain why \( V_0 \) represents absorption.
Problem 3

The Schrödinger equation for a rigid body that is constrained to rotate about a fixed axis and has a moment of inertia $I$ about this axis is

$$-rac{\hbar^2}{2I} \frac{\partial^2 \psi}{\partial \theta^2} = i\hbar \frac{\partial \psi}{\partial t}$$

where $\psi(\theta,t)$ is a periodic function of the angle $\theta \in (0,2\pi]$.

Find the eigenvalues and normalized eigenfunctions of the corresponding time-independent Schrödinger equation.

Is there any degeneracy?

Problem 4

The spin operator $\vec{S}$ acting on a spin-1/2 particle has components

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Find the eigenvalues and corresponding normalized eigenstates of the component $\hat{n} \cdot \vec{S}$ where

$$\hat{n} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}$$

is the unit vector in the $xy$-plane forming an angle $\theta$ with the $x$-axis.

(b) Suppose the spin in the $z$-direction is measured and is found to be $+\hbar/2$. Subsequently the spin along $\hat{n}$ is measured. What are the possible outcomes of the second measurement and with what probability will each outcome occur?