String Theory II

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Last update: 2006

Contents

UNIT 10

D-Branes

10.1 T-duality (again)

Consider type-II theories. We have

$$\begin{array}{lllll} IIA: & (NS+,NS+) & (R+,NS+) & (NS+,R-) & (R+,R-) \\ IIB: & (NS+,NS+) & (R+,NS+) & (NS+,R+) & (R+,R+) \end{array}$$

Compactify the 10th dimension on a circle of radius R in IIA, say. As we showed in the bosonic theory (argument is the same) the theory at R is identical to the theory at $R' = \frac{\alpha'}{R}$ (T-duality).
To show this, we started with the coordinate $X^9 = U = U_L(z) + U_R(\bar{z})$ and introduced the coordinate $z = U_L(z) - U_R(\bar{z})$. The resulting theory is at $R' = \frac{\alpha'}{R}$. In other words, the parity transformation on the right-moving part (only!).

$$X_R^9(\bar{z}) \to -X_R^9(\bar{z})$$

relates the theory at R with the theory at $R' = \frac{\alpha'}{R}$. Because of the superconformal invariance, this parity transformation is also applied tp the superpartner, $\tilde{\psi}^9(\bar{z})$

$$\tilde{\psi}^9(\bar{z}) \to -\tilde{\psi}^9(\bar{z}).$$

This, in particular, reverses the chirality of the states in the antiholomorphic part, so R- $\leftrightarrow$ R+. Therefore IIA $\leftrightarrow$ IIB, because that is the only difference between the two theories. Therefore IIA at R is equivalent to IIV at $R' = \frac{\alpha'}{R}$.
In particular, the IIA R-R fields, $C_\mu,\ C_{\mu\nu\lambda}$ are mapped onto the IIB R-R fields, $C,\ C_{\mu\nu},\ C_{\mu\nu\rho\sigma}$ as follows:

$$C_9 \to C,\ \ C_\mu \to C_{\mu 9},\ \ C_{\mu\nu 9} \to C_{\mu\nu},\ \ C_{\mu\nu\lambda} \to C_{\mu\nu\lambda 9}.$$

Of course, e.g., $C_{\mu\nu\rho\sigma}$ is obtained from $C_{\mu\nu\rho\sigma 9}$ in IIA, but $C_{\mu\nu\rho\sigma 9}$ is <u>not</u> an independent field (can be expressed in terms of $C_\mu,\ C_{\mu\nu\lambda}$) $8 + 56 = 64$.

Type-I Strings If we compactify the 10th dimension, $X^9(z,\bar{z})$, then the theory in the $R \to 0$ limit is mapped onto a T-dual theory at $R = \frac{\alpha'}{R} \to \infty$ which contains a D-brane.
Recall the argument, in the R' theory, the 10th dimension is

$$Z(z,\bar{z}) = X^9_L(z) - X^9_R(\bar{z}), \;\; \partial_\sigma Z = \partial_\tau X^9$$

so

$$\begin{aligned} Z(\sigma=\pi) - Z(\sigma=0) &= \int_0^\pi d\sigma \partial_\sigma Z = \int_0^\pi d\sigma \partial_\tau X^9 = \int_0^\pi \partial_\tau (2\alpha' p \tau) \\ &= 2\alpha' p \pi = 2\alpha' \frac{n}{R}\pi = 2\pi n R' = 0. \end{aligned}$$

Translation invariance is broken in the T-dual theory. Massless modes (same as in uncompactified theory)

$$NS: \;\; A_\mu \psi^\mu_{-1/2}|k\rangle, \;\; A\psi^9_{-1/2}|k\rangle, \quad R: \;\; |\vec{s};k\rangle$$

where A_μ represents a photon tangent to the brane. The second state shifts the position of the brane making it a dynamical object. (A is a function of k $\to$ its F.T. is a function of $X^\mu, \; \mu = 0,1,...,8$).
Even though the translation invariance is broken, the original theory has 32 supersymmetries! Of these, only half are broken. Thus the brane is a supersymmetric object with 16 supersymmetries! This large amount of symmetry implies the existence of conserved charges. What are they?
Our brane has 8+1 dimensions, so its volume element couples to the R-R potential, $C_{\mu_1\mu_2,...,\mu_9}$ $(dV \sim \epsilon_{\mu_1\mu_2...\mu_9} dx^{\mu_1}...dx^{\mu_9})$.
Recall familiar examples:

- A point charge q moving along a trajectory $x^\mu(\tau)$ has the action $q\int d\tau v^\mu A_\mu = \int d\tau j^\mu A_\mu = q\int dx^\mu A_\mu$. The charge q is conserved.
- The magnetic flux: $\Phi = \int \vec{B}\cdot d\vec{s}$, $\vec{B} = \nabla\times\vec{A}$ Define a field strength: $F_{ij} = \partial_i A_j - \partial_j A_i$. Then $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$, so $\Phi = \int F_{jk} dS^{jk}$ where $dS^{jk} = \frac{1}{2}\epsilon^{ijk} dS_i$ is the surface element. This is the magnetic charge, i.e., 0. Similarly, for the electric charge field, $\Phi_E = \int F_{0i} d\Sigma^{0i} \propto q$.

For the R-R charge on the D8 brane, we have

$$Q \propto \int dx^{\mu_1}...dx^{\mu_9} C_{\mu_1...\mu_9}$$

If we dualize two more dimension, the brane becomes a 6+1 dimensional object, (D6-brane). Two more gives D4, two more gives D2 and two more gives a D0 brane which represents a point particle. The charges are $\int dx^\mu C_\mu, \; \int dx^{\mu_1} dx^{\mu_2} ... C_{\mu_1\mu_2...}$ which are the R-R fields in type IIA theory! On the other hand, the D(2p+1)-branes couple to $C_{\mu_1\mu_2}, \; C_{\mu_1\mu_2\mu_3\mu_4}$, etc., which are the potentials in the type-IIB theory!

Not all these potentials are independent. Consider, e.g., D0-brane coupled to C_μ. The D0-brane is a point particle (with strings attached- hairy) with charge q which is the source of C_μ and field strength $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. q is an electric charge. The flux $\int F_{\mu\nu} d\Sigma^{\mu\nu} \propto q$ (Gauss' Law).
In four dimensional electromagnetism we may define the dual of $F_{\mu\nu}$ as $\tilde{F}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ which interchanges $\vec{E} \leftrightarrow \vec{B}$. Then the electric charges become magnetic charges. One may define a vector potential $\tilde{A}_\mu$ corresponding to $\tilde{F}_{\mu\nu}$ and describe electromagnetics in terms of $\tilde{A}_\mu$ instead of A_μ. $\tilde{A}_\mu$ can not be defined globally, since the magnetic flux around a charge is no longer zero, but it can be defined in patches, or almost everywhere apart from the string (Dirac string). If we include both electric and magnetic charges, then **no action** can be defined, yet the theory still makes sense. The existence of a monopole leads to quantization of the electric charge (Dirac).
Proof: Consider a point particle moving from $\vec{x_1} \to \vec{x_2}$. Its wavefunction changes $\psi(\vec{x}_1) \to \psi(\vec{x}_2)$. If I want to compare $\psi(\vec{x}_1)$ and $\psi(\vec{x}_2)$, then I will define the quantity $\psi(\vec{x}_2) * \psi(\vec{x}_1)$. In the limit $\vec{x}_2 \to \vec{x}_1$ (closed path) we obtain $|\psi(\vec{x}_1)|^2$. Gauge invariance: $\psi(\vec{x}) \to e^{iq\lambda(x)}\psi(\vec{x})$, so $\psi^*(\vec{x}_2)\psi(\vec{x}_1) \to e^{iq(\lambda(x_2)-\lambda(x_1))}\psi^*(\vec{x}_2)\psi(\vec{x}_1)$ This is not a gauge-invariant oblect. To make it gauge-invariant, multiply by $e^{iq\int \vec{A}\cdot d\vec{\ell}}$, $\vec{A} \to \vec{A} - \nabla\lambda$, so $\delta e^{iq\int \vec{A}\cdot d\vec{\ell}} = e^{-iq(\lambda(\vec{x}_1)-\lambda(\vec{x}_2))}$, so $\psi^*(\vec{x}_2)e^{iq\int \vec{A}\cdot d\vec{\ell}}\psi(\vec{x}_1)$ is gauge-invariant (physical)!
Go around a loop: we have $e^{iq\oint \vec{A}\cdot d\vec{\ell}}|\psi(\vec{x}_1)|^2$. By Stoke's theorem, $\oint_{\mathcal{C}} \vec{A}\cdot d\vec{\ell} = \int_{\mathcal{S}} \vec{B}\cdot d\vec{s}$ (flux through S).
If the path shrinks to zero, then $\oint_{\mathcal{C}} \vec{A}\cdot d\vec{\ell} = \int_{\mathcal{S}} \vec{B}\cdot d\vec{s} = 0$.
In the presence of a magnetic monopole, $\int_{\mathcal{S}} \vec{B}\cdot d\vec{s} = m$, the magnetic charge, so $e^{iq\int_{\mathcal{S}} \vec{B}\cdot d\vec{s}} = e^{iqm}$. We must have $e^{iqm} = 1$, therefore $qm = 2\pi n$, i.e., q is quantized even if only one magnetic monopole exists in the entire Universe.
Returning to D-branes, the C_μ potential on the D0-brane has field strength $F_{\mu\nu}$ whose dual is $\epsilon^{\mu_1\mu_2\cdots\mu_{10}}F_{\mu_9\mu_{10}}$ (8 indices). It corresponds tp a potential with seven indices, $C_{\mu_1\mu_2\ldots\mu_7}$ which resides on a D6-brane.
Thus the D0 electric charge is a source for the same field for which the D6-branes magnetic charge is a source. More generally, the electric Dp-brane charge and the magnetic D(6-p)-brane charge are sources for the same field.
Action for D0-branes electromagnetism:

$$S = -\frac{1}{2}\int d^{10}x\,\sqrt{-g}F_{\mu\nu}F^{\mu\nu} + q\int dx^\mu A_\mu$$

The potential between two points (D0-branes) is a Coulomb potential (in 10D)

$$V(y) \propto \frac{q^2}{y^7}$$

In momentum space, this is obtained from the propagator $-\frac{i}{k^2}$ where k^μ is the momentum of the exchanged boson (photon). Then

$$V(y) = -i\int d^{10}k e^{i\vec{k}\cdot\vec{y}}\frac{q^2}{k^2} = -i\frac{15V}{32\pi^4}\frac{q^2}{y^7}$$

where $V = \int d\omega$.
With D-branes, the potential comes from the exchange of closed strings. This may also be viewed as an open string with ends at $y = 0$ and $y = y$ moving around a loop. We already know how to calculate it.
The answer is

$$Z = \int_0^\infty \frac{dt}{2t} Z(t), \;\; Z(t) = \text{Tr}\, e^{-2\pi t L_0}.$$

Recall our result earlier

$$Z_{NS} = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16 + o(e^{-2s})), \;\; s = \frac{\pi}{t}.$$

Now we have 9 dimensions (16 compactified), so $s = \frac{9}{2}$. Also there is no integral over spatial momenta, only the energy D0-branes have world-lines, so the contribution from 0-modes $\left(8\pi^2\alpha' t\right)^{-D/2} \rightarrow \left(8\pi^2\alpha' t\right)^{-1/2}$, therefore, there is an additional factor $\left(8\pi^2\alpha' t\right)^{-(1-D)/2} \rightarrow \left(8\pi^2\alpha' t\right)^{9/2}$.
An extra factor of $4 = 2 \times 2$ (2 from **XXXXX** and no need to average over orientations). Finally, since $Z(\sigma = \pi) - Z(\sigma = 0) = y$, the expansion contains an extra term $Z = y\frac{\sigma}{\pi} + ...$ which gives an extra contribution to $L_0 = \frac{y^2}{4\pi^2\alpha'} + ...$
Therefore the extra factor is given by $e^{-2\pi t \frac{y^2}{4\pi^2\alpha'}} = e^{-ty^2/2\pi\alpha'}$. The partition function becomes

$$\begin{aligned} Z &\rightarrow \frac{iV(4\times 16)}{8\pi(8\pi^2\alpha')^5}\int_0^\infty \frac{\pi dt}{t^2}(8\pi^2\alpha' t)^{9/2} e^{-ty^2/2\pi\alpha'} \\ &= iV(2\pi)(4\pi^2\alpha')^3\frac{15}{32\pi^4}\frac{1}{y^7} \end{aligned}$$

This is compared to the potential $V(y) = -i\frac{15V}{32\pi^4}\frac{q^2}{y^7}$. In fact it is the R-sector $Z_R = V(y)$, but $Z_R = -Z_{NS}$, so $q^2 = 2\pi(4\pi^2\alpha')^3$.
This generalizes to Dp-branes: $(8\pi^2\alpha' t)^{9/2} \rightarrow (8\pi^2\alpha' t)^{(9-p)/2}$. The potential generalizes to $V_p(y) \sim \frac{1}{y^{7-p}}$ and the charges become $q_p^2 = 2\pi(4\pi^2\alpha')^{3-p}$.
For the D6-brane, $q_6^2 = \frac{2\pi}{(4\pi^2\alpha')^3}$, so $q_6 q_0 = 2\pi$, the Dirac quantization condition with $n = 1$! In general, $q_p q_{6-p} = 2\pi$, confirming that the Dp-brane and D(6-p)-brane act as electric and magnetic sources for the same field.

10.2 D-branes at angles

So far we have considered similar D-branes separated by a distance y. These are parallel D-branes. More generally, we can have a Dp-brane and a Dp'-brane along different subspaces and they may even intersect e.g., a D8-brane obtained by dualizing X^9 and a D8-brane from X^8 dualization. These two branes have the space $(X^1, X^2, ..., X^7)$ common. One may be obtained by rotating the other by $90°$ in the (X^8, X^9) plane. An open string may stretch between these two branes. Then its X^9 coordinate will obey Dirichlet boundary conditions at one end and Neumann boundary conditions at the other. The X^8 coordinate is reversed: Neumann boundary conditions at one end and Dirichlet at the other. Thus the modes expansions will be different. Recall for Neumann boundary conditions on both ends (set $\tau = 0$ for simplicity)

$$X^\mu_{NN}(\sigma) = x^\mu + i\sqrt{2\alpha'}\sum \frac{1}{n}\alpha^\mu_n \cos(n\sigma).$$

Check $\partial_\sigma X_{NN} 6\mu = 0$ at $\sigma = 0, \pi$.
For Dirichlet boundary conditions on both ends,

$$X^\mu_{DD}(\sigma) = \frac{y\sigma}{\pi} - i\sqrt{2\alpha'}\sum \frac{1}{n}\alpha^\mu_n \sin(n\sigma)$$

where y is the separation of the two (parallel in the μ-direction) branes. Check $X^\mu_{DD}(0) = 0,\ X^\mu_{DD}(\pi) = y$. X^μ_{NN} is split into holomorphic and antiholomorphic pieces as such

$$\begin{aligned} X^\mu_L &= \frac{1}{2}x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum \frac{1}{n}\alpha^\mu_n e^{in\sigma} \\ X^\mu_R &= \frac{1}{2}x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum \frac{1}{n}\alpha^\mu_n e^{-in\sigma} \end{aligned}$$

X_{DD} is split as $X^\mu_{DD} = X^\mu_L - X^\mu_R$ (dual!) For DN-b.c., i.e., $X^\mu_{DN}(\sigma = 0) = 0\ ,\ \partial_\sigma X^\mu_{DN}(\sigma = \pi) = 0$, we obtain

$$X^\mu_{DN}(\sigma) = -\sqrt{2\alpha'}\sum_{r\in\mathbb{Z}+1/2} \frac{\alpha^\mu_r}{r}\sin(r\sigma).$$

For ND-boundary conditions, we have $X^\mu_{ND}(\sigma) = i\sqrt{2\alpha'}\sum_{r\int \mathbb{Z}+1/2} \frac{\alpha^\mu_r}{r}\cos(r\sigma)$.
The superpartners ψ^μ and $\tilde{\psi}^\mu$ are similar.
Generalize to general angles. Suppose that there is an angle ϕ between the branes and consider strings stretched between the two. Define $Z = X^8 + iX^9$ (the brane at $X^9 = 0$ is not rotated-no loss of generality). At $\sigma = 0,\ X^9 = 0$ and $\partial_\sigma X^8 = 0$, so Im$(Z) = 0$, Re$(Z) = 0$. At $\sigma = \pi$, the brane is rotated by ϕ, so $Z \to e^{i\phi}Z$, so Im$(Ze^{-i\phi} = \partial_\sigma$ Re$(Ze^{-i\phi}) = 0$.
We may expand in terms of the modes

$$Z = \sqrt{\frac{2}{\alpha'}}\sum_{r\in\mathbb{Z}+\frac{\phi}{\pi}} \frac{\alpha_r}{r}e^{ir\sigma} + \sqrt{\frac{2}{\alpha'}}\sum_{r\in\mathbb{Z}-\frac{\phi}{\pi}} \frac{\alpha^+_r}{r}e^{-ir\sigma}$$

α_r and α^+_r are independent, because they involve α^8_r and α^9_r ($\alpha_r = \alpha^8_r + \alpha^9_r$).
At $\sigma = 0$: $Z \sim O(\alpha_r + \alpha^+_r)$, so, Im$(Z) = 0$, and $\partial_\sigma Z \sim i(\alpha_r - \alpha^+_r)$, so, Re$(\partial_\sigma Z) = 0$.
At $\sigma = \pi$: $Z \sim (\alpha_r + \alpha^+_r)e^{i\phi}$, so, Im$(Ze^{-i\phi}) = 0$, and $\partial_\sigma Z \sim i(\alpha_r - \alpha^+_r)e^{i\phi}$, so, Re$(\partial_\sigma Ze^{-i\phi}) = 0$.

10.3 Partition Function

The partition function has contributions from both the α_r's and α_r^+'s. It is easy to see that for $q = e^{-2\pi t}$

$$\begin{aligned} Z &= q^a \prod_{r\in Z+\frac{\phi}{\pi}} (1-q^r)^{-1} \prod_{r\in Z-\frac{\phi}{\pi}} (1-q^r)^{-1}, \\ &= q^a \prod_{m=0}^{\infty} (1-q^{m+\frac{\phi}{\pi}})^{-1} \prod_{m=1}^{\infty} (1-q^{m-\frac{\phi}{\pi}})^{-1}, \end{aligned}$$

where a is the Casimir energy (normal ordering constant in $L_0 =: L_0 : -a$). Recall $a = -1/24$ for a boson, because $a = \frac{1}{2}\sum_{n=1}^{\infty} n = \frac{1}{2}\zeta(1) = -1/24$. Here the sum becomes

$$\frac{1}{2}\sum_{r\in Z-\frac{\phi}{\pi}} r = \frac{1}{2}\sum_{m=1}^{\infty}\left(m-\frac{\phi}{\pi}\right) = \frac{1}{2}\left[\frac{1}{24}-\frac{1}{8}\left(2\frac{\phi}{\pi}-1\right)^2\right].$$

To prove this, look at the twisted sum problem (Polchinski 2.9.19) done last semester.
Also,

$$\frac{1}{2}\sum_{\substack{r\in Z+\frac{\phi}{\pi}\\ r>0}} r = \frac{1}{2}\sum_{m=0}^{\infty}\left(m+\frac{\phi}{\pi}\right) = \frac{1}{2}\left[\frac{1}{24}-\frac{1}{8}\left(2\left(1-\frac{\phi}{\pi}\right)-1\right)^2\right] = \frac{1}{2}\left[\frac{1}{24}-\frac{1}{8}\left(1-2\frac{\phi}{\pi}\right)^2\right],$$

which is the same as before. So, $a = \frac{1}{24}-\frac{1}{8}\left(1-2\frac{\phi}{\pi}\right)^2$. Therefore,

$$Z = q^a(1-z)^{-1}\left[\prod_{m=1}^{\infty}(1-zq^m)(1-z^{-1}q^m)\right]^{-1}, \quad z = q^{\phi/\pi} = e^{-2\phi t} = e^{2\pi i\nu}$$

This can be expressed in terms of

$$\vartheta_{11}(\nu, it) = -2q^{1/8}\sin\pi\nu \prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m),$$

and

$$\eta(it) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m).$$

Indeed,

$$\begin{aligned} \frac{\eta(it)}{\vartheta_{11}(\nu,it)} &= -\frac{1}{2}q^{-1/24-1/8-a}\frac{1}{\sin\pi\nu}\left[\prod(1-zq^m)(1-z^{-1}q^m)\right]^{-1} \\ &= -\frac{1}{2}q^{1/24-1/8-a}\frac{1-z}{\sin\pi\nu}Z \\ &= iq^{\phi^2/2\pi^2}Z \end{aligned}$$

Therefore

$$Z = -1\frac{e^{\phi^2 t/\pi\eta(it)}}{\vartheta_{11}(\nu, it)}.$$

Similarly for the fermion, we obtain

$$Z = \frac{\vartheta_{ab}(\nu, it)}{e^{\phi^2 t/\pi}\eta(it)},$$

for $a, b = 0, 1$ (NS-NS, NS-R, etc.)
Notice that the bosonic Z diverges as $\nu \to 0$, i.e., $\phi \to 0$. In this limit the two branes become parallel to each other, and the string is free to move along them, i.e., it has an additional (continuous) momentum, whose trace gives $\mathrm{Tr}\, q^{L_0} \sim \frac{V}{\sqrt{8\pi\alpha' t}}$, where V is the volume of the dimension along the brane. Therefore,

$$\begin{aligned} Z &= q^a \frac{1}{\sqrt{8\pi^2\alpha' t}} \prod_{m=1}^{\infty}(1-q^m)^{-2}, \quad a = 1/24 - 1/8 = -1/12 \\ &= \frac{V}{\sqrt{8\pi^2\alpha' t}}(\eta(it))^{-2}. \end{aligned}$$

The fermionic partition functions Z_{ab} do not change. Suppose as $\phi \to 0$, both branes are in the X^8 direction. Now take the dual of X^8. Since we have Neumann boundary conditions in X^8 (Dirichlet in X^9), in the dual, we will have Dirichlet in X^8. So in the dual picture, the two branes will become distinct points separated by a distinct y.
If originally we had Dp-branes, we end up with D(p-1) branes in the dual space. Open strings are stretched between the two branes. Thus, instead of a continuous momentum, we now have a contribution $\frac{y^2}{4\pi^2\alpha'}$ in L_0, therefore $e^{-ty^2/2\pi\alpha'}, \quad , y^2 = y_8^2 + y_9^2$, in general. The partition function is

$$Z = q^a e^{-ty^2/2\pi\alpha'} \prod_{m=1}^{\infty}(1-q^m)^{-2} = e^{-ty^2/2\pi\alpha'}(\eta(it))^{-2}$$

Example: Consider two D4-branes at an angle ϕ_1 in the 23-plane, ϕ_2 in the 45-plane, ϕ_3 in the 67-plane, ϕ_4 in the 89-plane and separated by a ditance y in the 1-directioin. In each plane, we obtain a partition function for the fermions:

$$Z_{ab}(\phi_i, it) = \frac{\vartheta_{ab}(\nu_i, it)}{e^{\phi_6^2 t/\pi}\eta(it)}, \quad ,\nu_= i\phi_i t/pi, \;\; i = 1, 2, 3, 4.$$

Putting them together, the fermionic partition function is

$$Z_f = \frac{1}{2}\left[\prod_{i=1}^{4}\frac{\vartheta_{00}(\nu_i, it)}{e^{\phi_i^2 t/\pi}\eta(it)} - \prod_{i=1}^{4}\frac{\vartheta_{10}(\nu_i, it)}{e^{\phi_i^2 t/\pi}\eta(it)} - \prod_{i=1}^{4}\frac{\vartheta_{01}(\nu_i, it)}{e^{\phi_i^2 t/\pi}\eta(it)} - \prod_{i=1}^{4}\frac{\vartheta_{11}(\nu_i, it)}{e^{\phi_i^2 t/\pi}\eta(it)}\right].$$

Generalizing our earlier result, when $\phi_i = 0 = \nu_i$,

$$Z_\psi = \frac{1}{2\eta^4(it)}\left(\vartheta_{00}^4(0,i\tau) - \vartheta_{10}^4(0,i\tau) - \vartheta_{01}^4(0,i\tau) - \vartheta_{11}^4(0,i\tau)\right).$$

Earlier we used the abtruse identity to show $Z_\psi = 0$. Now, we shall use the generalization of the abstruse identity:

$$\prod_{m=1}^{\infty}\vartheta_{00}^4(0,i\tau) - \prod_{m=1}^{\infty}\vartheta_{10}^4(0,i\tau) - \prod_{m=1}^{\infty}\vartheta_{01}^4(0,i\tau) - \prod_{m=1}^{\infty}\vartheta_{11}^4(0,i\tau) = 2\prod_{m=1}^{\infty}\vartheta_{11}(\nu_i', it)$$

$$\nu_i' = i\phi_i' t/\pi, \quad \phi_1' = \tfrac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4), \quad \phi_2' = \tfrac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4)$$
$$\phi_3' = \tfrac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4), \quad \phi_4' = \tfrac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4)$$

Notice $\sum_{i=1}^4 \phi_i'^2 = \sum_{i=1}^4 \phi_i^2$, so $\prod_{i=1}^4 e^{\phi_i^2 t/\pi} = \prod_{i=1}^4 e^{\phi_i'^2 t\pi}$ and

$$Z_f = \frac{\prod_{i=1}^4 \vartheta_{11}(\nu_i', it)e^{-\phi_i'^2 t/\pi}}{\eta^4(it)}.$$

Bosons: Recall in the 89-plane

$$Z_{boson} = -i\frac{e^{\phi^2 t/\pi}\eta(it)}{\vartheta_{11}(\nu, it)}$$

so in the $234...9$ direction

$$Z_{boson} = \eta^4(it)\prod_{i=1}^4 \frac{e^{\phi_i^2 t/\pi}}{\vartheta_{11}(\nu_i, it)}.$$

In the 0(time)-direction, we have a continuous distribution, so $Z \sim \frac{1}{\sqrt{8\pi^2\alpha' t}}$.
In the 1-direction, we have branes separated by a distance y, so $L_0 = \frac{y^2}{4\pi^2\alpha'} + \ldots$, so $Z_1 \sim e^{-ty^2/2\pi\alpha'}$.
Multiplying everything, the partition function becomes (potential)

$$V = -\int_0^\infty \frac{dt}{t}\frac{1}{\sqrt{8\pi^2\alpha' t}}e^{-ty^2/2\pi\alpha'}\prod_{i=1}^4\frac{\vartheta_{11}(\nu_i', it)}{\vartheta_{11}(\nu_i, it)}$$

This is a complicated function of y. We will calculate it for large distances. If y is large, the dominant contribution to the integral comes from small t (due to the $e^{ty^2/2\pi\alpha'}$ factor). If we set $t = 0$ in the ϑ-function, we obtain a constant and the integral diverges. We will calculate the force, which is a physical quantity and define the potential on the integral, $V = -\int F dy$.

$$F = -\frac{dV}{dy} = -y\int_0^\infty \frac{dt}{\pi\alpha'}\frac{e^{-ty^2/2\pi\alpha'}}{\sqrt{8\pi^2\alpha' t}}\prod_{i=1}^4\frac{\vartheta_{11}(\nu_i', it)}{\vartheta_{11}(\nu_i, it)}$$

From

$$\vartheta_{11}(\nu, it) = -2q^{1/8} \sin \pi\nu \prod_{m=1}^{4} (1-q^m)(1-zq^m)(1-z^{-1}q^m),$$

and

$$\vartheta_{11}(-i\nu/t, i/t) = -i\sqrt{t}e^{\pi\nu^2/t}\vartheta_{11}(\nu, it),$$

we obtain

$$\prod_{i=1}^{4} \frac{\vartheta_{11}(\nu_i', it)}{\vartheta_{11}(\nu_i, it)} = \prod_{i=1}^{4} \frac{\vartheta_{11}(-i\nu_i'/t, i/t)}{\vartheta_{11}(-i\nu_i/t, i/t)}.$$

As $t \to 0$, $q = e^{-2\pi/t} \to 0$, so $\Pi \to \prod_{i=1}^{4} \frac{\sin i\pi\nu_i'/t}{\sin i\pi\nu_i/t}$

$$\nu_i = i\phi t/\pi \to i\pi\nu_i/t = -\phi_i, \quad \Pi = \prod_{i=1}^{4} \frac{\sin \phi_i'}{\sin \phi_i}.$$

So

$$F \sim Cy \int_0^\infty \frac{dt}{\sqrt{t}} e^{-ty^2/2\pi\alpha'}, \;\; y \to \infty \quad const.: \frac{1}{\pi\alpha'\sqrt{8\pi^2\alpha'}} \prod_{i=1}^{4} \frac{\sin \phi_i'}{\sin \phi_i}.$$

and the potential is $V \sim Cy$.

10.4 Scattering

How do you make a D-brane move? Simple. Motion in e.g., the 1-direction is motion in Minkowski space (X^0, X^1) just like a rotation in Euclidean space (X^8, X^9) we studied above.

$$\begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \to \begin{pmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix}, \qquad \begin{pmatrix} X^8 \\ X^9 \end{pmatrix} \to \begin{pmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} X^8 \\ X^9 \end{pmatrix}.$$

where$X^1 = vX^0$ and the speed (v) is defined by the rapidity (ζ) as $v = \tanh\zeta$. The rapidity is related to the velocity via

$$\cosh\zeta = \frac{1}{\sqrt{1-v^2}}, \qquad \sinh\zeta = \frac{v}{\sqrt{1-v^2}}.$$

Consider two parallel Dp-branes moving with relative velocity v in the X^1-direction and separated by a distance y in the z-direction (branes are perpendicular to bot X^1 and X^2). In the 01-plane (Minkowski), we may copy our earlier result with the substitution $\phi = -i\zeta$: The bosonic part of the partition function is

$$Z_{bosonic\ (01)} = -i\frac{e^{\phi^2 t/\pi}\eta(it)}{\vartheta_{11}(\nu, it)}, \qquad \phi = -i\zeta, \;\; \nu = i\phi t/\pi - \zeta t/\pi.$$

The fermionic part is

$$Z_{ab} = \frac{\vartheta_{ab}(\nu, it)}{e^{\phi^2 t/\pi}\eta(it)}.$$

In the rest of the direction, the D-branes are parallel, so all other angles are zero. Therefore, the fermionic piece is

$$Z_f = \frac{1}{2\eta^4(it)} e^{-\phi^2 t/\pi} \left[\vartheta_{00}(\nu, it)\vartheta_{00}^3(0, it) - \vartheta_{10}(\nu, it)\vartheta_{10}^3(0, it)\right.$$

$$\left. -\vartheta_{01}(\nu, it)\vartheta_{01}^3(0, it) - \vartheta_{11}(\nu, it)\vartheta_{11}^3(0, it)\right]$$

This may be computed by applying the generalized abstruse identity. We have

$$\phi_1 = \phi, \ \ \phi_2 = \phi_3 = \phi_4 = 0,$$

so

$$\phi_1' = \phi_2' = \phi_3' = \phi_4' = \frac{1}{2}\phi$$

and therefore

$$Z_f = \frac{1}{2\eta^4(it)} e^{-\phi^2 t/\pi} \vartheta_{11}^4(\frac{1}{2}\phi, it).$$

The bosonic piece in the other directions ($X^2, X^3, ..., X^9$ total of eight ... six of which are transverse) is

$$Z_{\substack{bosonic\\2,3,\ldots,9}} = V_p \left(\frac{1}{\sqrt{8\pi^2\alpha' t}}\right)^p e^{-ty^2/2\pi\alpha'}(\eta(it))^{-6}$$

Therefore the partition function is

$$Z = -iV_p \int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-p/2} e^{-ty^2/2\pi\alpha'} \frac{\vartheta_{11}^4(\nu/2, it)}{\vartheta_{11}(\nu, it)}(\eta(it))^{-9}, \quad \nu = \zeta t/\pi$$

As the branes move the distance changes to $r^2 = y^2 + v^2\tau^2$. The potential may be extracted from

$$Z = -1\int_{-\infty}^\infty d\tau V[r(\tau), v].$$

We easily obtain

$$V(r, v) = i\frac{2V_p v}{(\sqrt{8\pi^2\alpha'})^{p+1}} \int_0^\infty dt\ t^{(5-p)/2} e^{-tr^2/2\pi\alpha'} \frac{\vartheta_{11}^4(i\zeta/2\pi, i/t)}{\eta^9(i/t)\vartheta_{11}(i\zeta/\pi, i/t)},$$

where we used the modular properties of the ϑ and η functions.
Note: as $v \to 0,\ u \to 0$, so $V \to 0$.
Since

$$\vartheta_{11}(\nu, it) = -2q^{1/8}\sin\pi\nu \prod(1-q^m)(1-zq^m)(1-z^{-1}q^m), \quad \eta(it) = q^{1/24}\prod(1-q^m)$$

we have, as $v \to 0, \;\; \nu \to 0, \;\; Z \to 1.$

$$\frac{\vartheta_{11}^4(i\zeta/2\pi, i/t)}{\eta^9(i/t)\vartheta_{11}(i\zeta/\pi, i/t)} = \frac{8i\sinh^4(\zeta/2)}{\sinh(\zeta)} + ... = \frac{1}{2}v^3 + ..., \qquad \zeta \to v$$

So

$$\begin{aligned} V(r,v) &= -\frac{2V_p v^4}{(\sqrt{8\pi^2\alpha'})^{p+1}}\int_0^\infty dt\; t^{(5-p)/2}e^{-tr^2/2\pi\alpha'} + o(v^6) \\ &\sim -\frac{v^4}{r^{7-p}}\frac{V_p}{\alpha'^{p-3}} \end{aligned}$$

Problem: as $r \to 0, V \to \infty$! How can string theory claim finiteness at short distances (r is real distance - not bogus!)?
Answer: Let $r \to 0$ *before* expanding in v. r only appears in $e^{-tr^2/2\pi\alpha'}$. If we rescale $t \to t/r^2$, the $r \to 0$ corresponds to *large* t. If t is large in ϑ, η, then

$$\frac{\vartheta_{11}^4}{\eta^9\vartheta_{11}} \to \frac{\sinh^4\left(\frac{vt}{4}\right)}{\sinh(vt)}, \quad \zeta \sim v$$

From the exponential, $t \sim 2\pi\alpha'/r^2$ dominates. $ut \sim 2\pi\alpha' u/r^2$, so in the limit that $r \to 0$, ut becomes large and the integral oscillates rapidly. OScillation on a scale $ut \sim 1$, i.e., $2\pi\alpha' u \sim r^2$, i.e., $r \sim \sqrt{\alpha' v}$. This is the effective scale probed by the brane: $r_0 = \sqrt{\alpha' v}$. A slow brane $(v \to 0)$ probes scales smaller than the string scale! Moreover, we obtain an uncertainty in the position

$$\delta x \geq \sqrt{\alpha' v}.$$

The time it takes for this scattering process is

$$\delta t \sim \delta x/v$$

Therefore,

$$\delta x \delta t \geq \frac{\delta x}{v}\sqrt{\alpha' v} \simeq \frac{\alpha' v}{v} = \alpha'.$$

A new uncertainty principle! It implies that coordinates do **not** commute! It seems that Nature is described by noncommutative geometry. What can this possibly mean??
Consider two branes separated by a distance y. Strings ending on the same brane have a massless mode each, so we have two massless modes. A string stretched between the two branes has

$$L_0 = \frac{y^2}{4\pi^2\alpha'} + ...$$

This extra term makes $L_0 > 0$ i.e., there are no massless modes. At low energies, we only see two massless particles from the two branes. However, when $y \to 0$, the stretched string develops a massless mode, and there are two of

them. So when the two branes coincide, there are four massless modes. These four modes can be grouped into a matrix X_{ij} in an obvious notation.
Each X_{ij} is the position of the brane! When we develop a particle theory we need to treat the position of the brane as a 2×2 matrix. More generally, n distinct branes have n massless modes. The particle theory is just n copies of the same theory. When all n branes coincide, we have n^2 massless modes. Each massless mode corresponds to a symmetry of the theory (U(1)). With n^2 massless states the symmetry is enhanced to U(n) (n^2 generators).
Familiar Examples
Photon: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $U(1)$ symmetry ($A_\mu \to A_\mu + \partial_\mu \lambda$).
3 Photons: $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$, $U(1)^3$ symmetry.
Weak Bosons: Demand $SU(2)$ symmetry which has three generators, so $F^i_{\mu\nu} \neq \partial_\mu A^i_\nu = \partial_\nu A^i_\mu$. There is a correction, to obey the enhanced symmetry $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ (A_μ is a matrix ... $A_\mu = A^i_\mu \sigma_i$)
Gluons: Demand $SU(3)$ - eight gluons $(3^2 - 1)$.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad A_\mu = A^i_\mu \lambda_i$$

where λ_i represent the Gell-Mann matrices. The action is given by

$$S \sim \int d^4x \text{ Tr } F_{\mu\nu} F^{\mu\nu}.$$

If we only had eight copies of electromagnetism, we would have

$$S \sim \int d^4x \text{ Tr } F^i_{\mu\nu} F^{\mu\nu}_i.$$

Now we have interactions between gluons - enhanced symmetry (gluons and weak bosons, unlike photons have charge).
Potential: Set A_μ =constant, then

$$\text{Tr } F_{\mu\nu} F^{\mu\nu} \sim \text{Tr } [A_\mu, A_\nu]^2.$$

Back to D-branes: X^μ is like a^μ (that can be made precise - see Polchinski **8.6**). So the enhanced symmetry contains a potential

$$V \sim \text{Tr } [X^\mu, X^\nu]^2$$

where μ, ν run over that dimension transverse to the branes. Expand around $X^\mu = 0$ in a Taylor series. There are no linear or quadratic terms in X^μ, so there is no mass term (which would come from $V(\phi) = V(0) + V'(0)\phi + \frac{1}{2}V''(0)\phi^2/m^2 + ...$)
So we have kn^2 massless modes, where k is the number of transverse dimensions. Also, $V = 0$ if and only if all $[X_m, X_n] = 0$, i.e., all X_m commute. This can be accomplished if we make them all diagonal. There are n diagonal elements, each corresponding to one of the D-branes. Thus this potential correctly describes n coincident non-interacting free D-branes.