String Theory II

George Siopsis and Students

Department of Physics and Astronomy
The University of Tennessee
Knoxville, TN 37996-1200
U.S.A.
e-mail: `siopsis@tennessee.edu`

Last update: 2006

Contents

UNIT 8

Heterotic Strings

8.1 Introduction

The Heterotic string was introduced in 1985 by the "string quartet" (Gross Harvey, Martinec and Rohm).
Basic idea: left and right movers need not be in the same theory. not even in the same dimension!
Thus, take the holomorphic part as bosonic ($\partial X^{\mu}(z), \mu = 0, 1, ..., 25$) and the anti-holomorphic part as a superstring ($\bar{\partial} X^{\mu}(\bar{z}), \tilde{\psi}^{\mu}(\bar{z}), \mu = 0, 1, ..., 9$).
Then the holomorphic piece has $c = 26$, so we need to throw in the bc ghosts with $c = -26$ in order to have a vanishing central charge. The anti-holomorphic piece has a central charge $c = 15$, so we need couple it to the superconformal ghosts (b,c and γ, β) with central charge, $\tilde{c} = -15$.
It is convenient to split ∂X^{μ} into $\partial X^{\mu}, \mu = 0, 1, ..., 9$, which will then combine with the anti-holomorphic piece to $X^{\mu}(z, \bar{z})$ and the rest $\partial X^{\mu}, \mu = 10, 11, ..., 25$ have $c = 16$ and have no anti-holomorphic partners. We may replace them with $\psi^{A}(z), A = 1, 2, ..., 32$ which have the same central charge (c=32/2=16).
The two theories are the same even though it may not be obvious.
Now we can write the action

$$S = \int d^2 z \left(\frac{1}{2\pi\alpha'} \partial X^{\mu} \bar{\partial} X_{\mu} + \tilde{\psi}^{\mu} \partial \tilde{\psi}_{\mu} + \partial^{A} \bar{\partial} \psi_{A} \right), \; \mu = 0, 1, ..., 9, \; A = 1, 2, ..., 32.$$

so μ is a space-time index and A is an internal index which represents the gauge degrees of freedom. Think of ψ^{A} as a 32 dimensional vector rotated in an abstract space. The gauge group is then the group of rotations, SO(32), which is large enough (too large!) to accommodate all interactions we see in Nature.
Notice that SO(32) is also the gauge group in type-I theory, yet they are different, for in type-I, SO(32) is at the ends of the strings, whereas in the heterotic theory, SO(32) is along the entire string.

Modern wisdom holds that type-I and heterotic only look different. Deep inside they are different manifestations of the same theory. The operator product expansions are as the usual

$$\begin{aligned} X^\mu(z,\bar z)X^\nu(0,0) &\sim -\eta^{\mu\nu}\frac{\alpha'}{2}\ln|z|^2 \\ \tilde\psi^\mu(\bar z)\bar\partial^\nu(0) &\sim \eta^{\mu\nu}\frac{1}{z} \\ \psi^A(z)\psi^B(0) &\sim \delta^{AB}\frac{1}{z} \quad \leftarrow \text{Euclidean signature!} \end{aligned}$$

Energy momentum tensor:

$$T(z) = -\frac{1}{\alpha}\partial X^\mu\partial X_\mu - \frac{1}{2}\psi^A\partial\psi_A \quad , \quad \tilde T(z) = -\frac{1}{\alpha}\bar\partial X^\mu\bar\partial X_\mu - \frac{1}{2}\tilde\psi^A\bar\partial\tilde\psi_A.$$

SUSY currents:

$$\tilde T_F = i\sqrt{\frac{2}{\alpha'}}\tilde\psi^\mu\bar\partial X_\mu, \quad , \quad T_F = 0$$

so this theory has $N = 0,\ \tilde N = 1$ SUSY (world-sheet)

To build the Hilbert space, we need to specify the boundary conditions on ψ^A and $\tilde\psi^\mu$. $\tilde\psi^\mu$ is as before, leading to NS and R sectors - we may apply GSO projection to split the sectors in NS$\pm$ and R$\pm$.

ψ^A is tricky. It is not restricted by Lorentz invariance, because A is an internal index. We may only require that $T(z)$ be periodic, so $\psi^A(\sigma+2\pi,\tau) = O^{AB}\psi^A(\sigma,\tau)$ where O is a 32×32 orthogonal matrix. This leaves the quadratic form $\psi^A\partial\psi_A$ and $T(z)$ invariant. A host of possibilities, but only two work!

<u>Possibility 1</u>

$$\psi^A(\sigma+2\pi,\tau) = \pm\psi^A(\sigma,\tau)$$

<u>same</u> sign from all components. This is easy and is the same as before. Define the fermion number operator $F = \Sigma^{12} + \Sigma^{34} + ... + \Sigma^{31\ 32}$ (we now have 16 spins). The GSO projection is onto eigenspaces of $e^{i\pi F}$.

We will select $e^{i\pi F} = +1$, thus restricting to NS+, R+ for ψ^A. <u>Partition Function</u>

Recall for ψ^μ:

$$Z_\psi = \frac{1}{2}\left[\left(\frac{\vartheta_{00}(0,\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_{10}(0,\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_{01}(0,\tau)}{\eta(\tau)}\right)^4 \pm \left(\frac{\vartheta_{11}(0,\tau)}{\eta(\tau)}\right)^4\right] = 0$$

which vanished by the abstruse identity.

In our case, instead of 4=8/2, we have 16=32/2 (no time-like coordinate therefore all components contribute) Answer:

$$Z_\psi = \frac{1}{2}\left[\left(\frac{\vartheta_{00}(0,\tau)}{\eta(\tau)}\right)^{16} + \left(\frac{\vartheta_{10}(0,\tau)}{\eta(\tau)}\right)^{16} + \left(\frac{\vartheta_{01}(0,\tau)}{\eta(\tau)}\right)^{16} \pm \left(\frac{\vartheta_{11}(0,\tau)}{\eta(\tau)}\right)^{16}\right]$$

Where the first "+" is due to the ghosts (β,γ) from which are absent in this case. The second "+" sign is due to space-time statistics, ψ^A is a scalar.

This partition function is multiplied by $Z_{\tilde{\psi}}$, which vanishes, but it is still useful to demonstrate the modular invariance of $Z_\psi Z^*_{\tilde{\psi}}$.
Under $\tau \to -\frac{1}{\tau}$

$$\begin{aligned}\vartheta_{01}(0,-1/\tau) &= \sqrt{-i\tau}\vartheta_{00}(0,\tau)\\ \vartheta_{01}(0,-1/\tau) &= \sqrt{-i\tau}\vartheta_{10}(0,\tau)\\ \vartheta_{10}(0,-1/\tau) &= \sqrt{-i\tau}\vartheta_{01}(0,\tau)\\ \eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau)\end{aligned}$$

It is obvious that both Z_ψ and $Z_{\tilde{\psi}}$ are invariant since $\vartheta_{10} \leftrightarrow \vartheta_{01}$.
Under $\tau \to \tau + 1$

$$\begin{aligned}\vartheta_{00}(0,\tau+1) &= \vartheta_{01}(0,\tau)\\ \vartheta_{01}(0,\tau+1) &= \vartheta_{00}(0,\tau)\\ \vartheta_{10}(0,\tau+1) &= e^{i\pi/4}\vartheta_{10}(0,\tau)\\ \eta(\tau+1) &= e^{i\pi/12}\eta(\tau)\end{aligned}$$

$Z_\psi \to e^{-16\pi i/12}Z_\psi = e^{2\pi i/3}Z_\psi$ and $Z_{\tilde{\psi}} \to -e^{-4\pi i/12}Z_{\tilde{\psi}} = e^{2\pi i/3}Z_{\tilde{\psi}}$. Therefore the product $Z_\psi Z^*_{\tilde{\psi}}$ is invariant under modular transformations.

8.2 The spectrum

Constraints: $L_0 = \tilde{L}_0 = 0$. Do $\tilde{L}_0$ first (anti-holomorphic part). This is the same as before.

$$\tilde{NS}: \quad \tilde{L}_0 = \frac{\alpha'}{4}p^2 + \tilde{N} - \frac{1}{2} = 0.$$

Lowest state: $|0,k\rangle$ has no $\tilde{N} = 0$, so $m^2 = -k^2 = -2/\alpha'$, a tachyon. This has $e^{i\pi\tilde{F}}|0,k\rangle = -|0,k\rangle$, where the minus sign is due to the (β,γ) ghosts. Therefore it is not a $\tilde{NS}+$.
Next level: $\tilde{\psi}^\mu_{-\frac{1}{2}}|0,k\rangle$ has $\tilde{N} = \frac{1}{2}$, so $m^2 = -k^2 = 0$. It is a vector transforming under SO(8). It has $e^{i\pi\tilde{F}}\tilde{\psi}^\mu_{-\frac{1}{2}}|0,k\rangle = \tilde{\psi}^\mu_{-\frac{1}{2}}|0,k\rangle \in \tilde{NS}+$.
R-Sector $\tilde{L}_0 = \frac{\alpha'}{4}p^2 + N + a$
a can be deduced as follows: recall in bosonic theory $a = -1$. This is because $a = -(D-2)/24$. Here $D = 10$, so we get $a = -1/3$. In the NS sector, we get $a = -1/2$, because the fermions contributed $-1/6(1/2 = 1/3 + 1/6)$-each fermion contributes $-1/48$. Now we have 32 fermions, so they contribute $-32/48 = -2/3$. Overall $a = -1/3 - 2/3 = -1$ in NS.
In the R sector, we get $a = 0$, so fermions contribute $1/24$ each. Overall, $a = -1/3 + 32/24 = +1$ in the R sector which implies all modes are massive in the R sector.
In NS, the lowest state has $m^2 = -k^2 = -\frac{4}{a'}$, which represents a tachyon! It has $e^{i\pi F}|0,k\rangle = |0,k\rangle$, so we keep it.

The next level has $N = 1/2: \quad \psi^A_{-1/2}|0,k\rangle$, so $m^2 = k^2 = -\frac{2}{\alpha'}$, which is another tachyon, but $e^{i\pi F}\psi^A_{-1/2}|0,k\rangle = -\psi^A_{-1/2}|0,k\rangle$, so we must reject it.
The next level has $N = 1: \quad m^2 = k^2 = 0$, massless! There are two possibilities:

$$A_\mu(k)\alpha^\mu_{-1}|0;k\rangle \;, \quad B_{AB}\psi^A_{1/2}\psi^B_{-1/2}|0;k\rangle$$

Both possibilities have the correct GSO projection i.e., $e^{i\pi F} = +1$, so we keep them.
$A_\mu(k)$ represents a photon with 8 transverse polarizations. B_AB is an antisymmetric 32×32 matrix with $\frac{32\times 31}{2} = 496$ components.
Summary

m^2	$NS+$	$R+$	$\tilde{NS}+$	$\tilde{R}+$
$-4/\alpha'$	$\|0;k\rangle$	$-$	$-$	$-$
0	A_μ, B_{AB}	$-$	$\tilde{\psi}^\mu_{-1/2}\|0;k\rangle$	$\|\vec{s};k\rangle$

Closed strings states must have $L_0 = \tilde{L}_0$, so the tachyon in NS+ is rejected, because there is no tachyon in $\tilde{NS}+$ or $\tilde{R}+$. At the massless level we have

$$A_{\mu\nu}\alpha^\mu_{-1}\tilde{\psi}^\nu_{-1/2}|0;k\rangle, \;\; A_{\mu,\vec{s}}\alpha^\mu_{-1}|\vec{s};k\rangle$$

where $A_{\mu\nu}$ may be decomposed via a scalar, antisymmetric and traceless symmetric tensors as $8\times 8 = 64 = 1{+}28{+}35$. $A_{\mu\vec{s}}$ is the supersymmetric partner to $A_{\mu\nu}$ and may be decomposed into $8 + 56$ irreducible representations of SO(8). We also have

$$B^{(\vec{s})}_{AB}\psi^A_{-1/2}\psi^B_{-1/2}|\vec{s};k\rangle, \;\; B^\mu_{AB}\psi^A_{-1/2}\psi^B_{-1/2}\tilde{\psi}_{\mu\;-1/2}|0;k\rangle$$

so B^μ_{AB} represents a gauge boson with 496 components. c.f. gluon has A^μ_i, where $i = 1, 2, ..., 8$. So SO(32) is a gauge symmetry in space-time, just like SU(3) is a gauge symmetry for QCD.
Comparing with SO(32) type-I theory, we see big differences. In type-I theory SO(32) resides at the ends of the strings and the spectrum does not match that of the heterotic string (except at the massless level). Yet, these two theories are one and the same (to be proved)!
Possibility #2
Divide $\psi^A(z)$ into two groups, $\psi^A, \; A = 1, 2, ..., 16, \; \psi^B, \; B = 17, 18, ..., 32$. This is possible because

$$\begin{array}{ll} \psi^A(\sigma + 2\pi) = -\psi^A(\sigma) \;\; (NS_1) & \psi^A(\sigma + 2\pi) = +\psi^A(\sigma) \;\; (R_1) \\ \psi^B(\sigma + 2\pi) = -\psi^B(\sigma) \;\; (NS_2) & \psi^B(\sigma + 2\pi) = +\psi^B(\sigma) \;\; (R_2) \end{array}$$

Total of four possibilities: $NS_1 + NS_2, \; NS_1 + R_2, \; R_1 + NS_2, \; R_1 + R_2$. There are two GSO projections because we have two fermion number operators. We will restrict to $e^{i\pi F_1} = e^{i\pi F_2} = +1$.
Partition Function

$$Z_\psi = Z_{\psi^A} Z_{\psi^B} = Z^2_{\psi^A},$$

because $Z_{\psi^A} = Z_{\psi^B}$. Z_{ψ^A} is the same as Z_ψ we derived in possibility #1, except $16 \to 8$ (we have half as many ψ^As now). Therefore

$$Z_{\psi^A} = \frac{1}{2}\left[\left(\frac{\vartheta_{00}}{\eta}\right)^8 + \left(\frac{\vartheta_{10}}{\eta}\right)^8 + \left(\frac{\vartheta_{01}}{\eta}\right)^8 \mp \left(\frac{\vartheta_{11}}{\eta}\right)^8\right].$$

Modular invariance: $\tau \to -1/\tau$ leave the partition function invariant (trivial). $\tau \to \tau + 1$ takes $\vartheta_{00} \leftrightarrow \vartheta_{01},\ \vartheta_{10} \to e^{i\pi/4}\vartheta_{10}$, so $\vartheta_{10}^8 \to \vartheta_{10}^8$. So only $\eta \to e^{i\pi/12}\eta$, i.e., $\eta^{-8} \to e^{-2\pi i/3}\eta^{-8}$, and the partition transforms as $Z_{\psi^A} \to e^{-2\pi i/3}Z_{\psi^A} \Rightarrow Z_{\psi^A}^2 \to e^{-4\pi i/3}Z_{\psi^A}^2 = e^{2\pi i/3}Z_{\psi^A}^2$. The additional factor cancels when we multiple by $Z_{\tilde{\psi}}^* \to e^{-2\pi i/3}Z_{\tilde{\psi}}^*$.
Gauge Group: Obviously, this theory has $SO(16) \times SO(16)$ symmetry. However, this is only a subgroup os the full gauge group, which is $E_8 \times E_8$ (exceptional group).
To summarize: there exist only two possibilities for heterotic strings, with gauge groups SO(32) and $E_8 \times E_8$, respectively.