String Theory II

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UNIT 7

Superstrings

7.1 Bosons and fermions

Bosonic strings have the action

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \partial X^\mu \bar{\partial} X_\mu.$$

We wish to build a theory that has supersymmetry (SUSY). Why? It turns out that this is the only (known) way of obtaining a consistent theory.
For SUSY, each boson (commuting field), must have a fermionic (anticommuting) counterpart. We have already seen anticommuting fields. We called them b, c. Recall the b, c action

$$S_{bc} = \frac{1}{2\pi} \int d^2 z b \bar{\partial} c.$$

and their OPEs are

$$b(z)c(0) \sim \frac{1}{z}.$$

The wave equation was given by $\bar{\partial} b = \bar{\partial} c = 0$, i.e., b and c are purely holomorphic. The energy-momentum tensor is

$$T =: (\partial b) c : -\lambda \partial (: bc :)$$

where we assume the weights $h_b = \lambda,\ h_c = 1 - \lambda$. The OPE for the energy-momentum tensor is

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2} T(0) + \frac{1}{z} \partial T(0)$$

where $c = -3(2\lambda - 1)^2 + 1$. Earlier we required $\lambda = 2$, so $c = -26$ (hence $D = 26$ for the bosonic string) in order to do BRST quantization properly

($Q_{BRST}^2 = 0$). A more symmetric choice is $\lambda = \frac{1}{2}$. Then $h_b = h_c = \frac{1}{2}$ and $c = 1$. Define

$$b = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \qquad c = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2).$$

Then the action is

$$S = \frac{1}{2\pi}\int d^2 z b\bar{\partial}c = \frac{1}{4\pi}\int d^2 z(\psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2).$$

The stress-energy tensor written in terms if the new fields may be expressed as

$$T(z) = -\frac{1}{2}\psi_1\partial\psi_1 - \frac{1}{2}\psi_2\partial\psi_2.$$

The system splits into two identical copies. Since $c = 1$, for the two together, each system has $c = \frac{1}{2}$.
Pick one such system, $\psi = \psi_1$, say. Make D copies of it, $\psi \to \psi^\mu$ ($\mu = 0, 1, ..., D-1$) and let us try ψ^μ as a SUSY partner of X^μ.
The stress-energ tensor is given by

$$T = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu.$$

The TT OPE becomes

$$T(z)T(0) \sim \frac{(3D/2)}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0)$$

where we used

$$X^\mu(z,\bar{z})X^\nu(0,0) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z|^2, \qquad \psi^\mu(z)\psi^\nu(0) \sim \frac{1}{z}\eta^{\mu\nu}.$$

$T(z)$ is a conserved current that generates conformal transformations which are symmetries of the theory (in fact $v(z)T(z)$ is conserved for arbitrary $v(z)$, leading to an infinite number of symmetries). The new theory (X^μ, ψ^μ) has even more symmetries! Let us define a supercurrent as

$$T_F = i\sqrt{\frac{\alpha'}{2}}\psi^\mu(z)\partial X_\mu(z)$$

Any $\eta(z)T_F(z)$ is conserved and generates a symmetry mixing X^μ and ψ^μ (superconformal transformation) - η must be anticommuting so that ηT_F is commuting. To see this consider

$$\begin{aligned} T_F(z)X^\mu(0,0) &\sim -i\sqrt{\frac{\alpha'}{2}}\frac{\alpha'}{2}\frac{1}{z}\psi^\mu(0) = -i\sqrt{\frac{\alpha'}{2}}\frac{1}{z}\psi^\mu(0) \\ T_F(z)\psi^\mu(0) &\sim i\sqrt{\frac{2}{\alpha'}}\frac{1}{z}\partial X^\mu(0,0) \end{aligned}$$

So

$$\begin{aligned}
\delta X^\mu &= -i\epsilon \oint \frac{dz}{2\pi i}\eta(z)T_F(z)X^\mu(0,0) = -\sqrt{\frac{\alpha'}{2}}\epsilon\eta\psi^\mu(0) \\
\delta\psi^\mu &= -i\epsilon \oint \frac{dz}{2\pi i}\eta(z)T_F(z)\psi^\mu(0) = -\sqrt{\frac{2}{\alpha'}}\epsilon\eta\partial X^\mu(0,0)
\end{aligned}$$

The other OPEs are given by

$$\begin{aligned}
T(z)T_F(0) &\sim 2\left(-\frac{1}{a'}\right)\left(i\sqrt{\frac{\alpha'}{2}}\right)\left(\frac{\alpha'}{2}\partial^2 \ln|z|\right)\partial X^\mu(z)\psi_\mu(0) + \left(-\frac{1}{2}\right)\left(i\sqrt{\frac{\alpha'}{2}}\right)\left(\partial\frac{1}{z}\right)\psi^\mu(z)\partial X_\mu(0) \\
&\quad + \left(-\frac{1}{2}\right)\left(i\sqrt{\frac{\alpha'}{2}}\right)\frac{1}{z}\partial\psi_\mu(z)\partial X^\mu(0) \\
T(z)T_F(0) &\sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \\
T_F(z)T_F(0) &\sim \left(i\sqrt{\frac{\alpha'}{2}}\right)^2 \frac{1}{z}\left(\frac{\alpha'}{2}\partial^2 \ln|z|\right)D + \left(i\sqrt{\frac{\alpha'}{2}}\right)^2 \frac{1}{z}\partial X^\mu \partial X_\mu \\
&\quad + \left(i\sqrt{\frac{\alpha'}{2}}\right)^2 \left(\frac{\alpha'}{2}\partial^2 \ln|z|\right)\psi^\mu(z)\psi_\mu(0) \\
&\sim \frac{D}{z^3} + \frac{2}{z}T(0).
\end{aligned}$$

The first OPE shows that T_F has weight $h = 3/2$. There is a corresponding construction for the anti-holomorphic operators. Since ψ^μ is holomorphic, we need to add a *new* anti-holomorphic fermionic field $\tilde{\psi}^\mu(\bar{z})$ with the action

$$\tilde{S} = \frac{1}{4\pi}\int d^2z \tilde{\psi}^\mu \partial \tilde{\psi}_\mu.$$

The wave equation is given by

$$\partial\tilde{\psi}_\mu = 0,$$

so, indeed $\tilde{\psi}^\mu$ is anti-holomorphic. They OPE is

$$\tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu(0) \sim \frac{1}{\bar{z}}\eta^{\mu\nu}.$$

The stress-energy tensors are

$$\tilde{T} = -\frac{1}{2}\tilde{\psi}^\mu\bar{\partial}\tilde{\psi}_\mu, \qquad \tilde{T}_F = i\sqrt{\frac{\alpha'}{2}}\tilde{\psi}^\mu\bar{\partial}X_\mu.$$

The OPEs are similar to the OPEs of their holomorphic counterparts. Notice that the central charge for this theory is $c = 3D/2$. This is now a superconformal theory ($N = 1,\ \tilde{N} = 1$ where $N, \tilde{N}$ counts the number of $T_F, \tilde{T}_F$'s). Other examples

7.2 The ghosts

Recall,

$$S_{bc} = \frac{1}{2\pi}\int d^2z b\bar{\partial}c, \qquad T = (\partial b)c - \lambda\partial(bc), \qquad b(z)c(0) \sim \frac{1}{z}.$$

The weights and central charge for the bc system are

$$h_b = \lambda, \quad h_c = 1-\lambda, \quad c_{bc} = -3(2\lambda-1)^2 + 1.$$

Since (b, c) are anti-commuting fields, their partners will have to be commuting. We have already met them. They are the (β, γ) fields with action

$$S_{\beta\gamma} = \frac{1}{2\pi}\int d^2z \beta\bar{\partial}\gamma,$$

which is the same action as the bc action. Let $h_\beta = \lambda'$, $h_\gamma = 1-\lambda'$. The combined system will have SUSY if we can find a T_F that mixes b, c with β, γ. Such a T_F will most likely contain a $(\partial\beta)c$ (c.f. $(\partial b)c$ in T_{bc} and $(\partial\beta)\gamma$ in $T_{\beta\gamma}$,

$$T_{\beta\gamma} = (\partial\beta)\gamma - \lambda'\partial(\beta\gamma).$$

Since $h = 3/2$ for T_F, we need $1 + \lambda' + 1 - \lambda = 3/2$, i.e., $\lambda' = \lambda - 1/2$. The central charge is

$$c_{\beta\gamma} = 3(2\lambda'-1)^2 - 1 = 3(2\lambda-2)^2 - 1.$$

The central charge for the combination of the two systems becomes

$$c_{\text{total}} = c_{bc} + c_{\beta\gamma} = -3(2\lambda-1)^2 + 3(2\lambda'-2)^2 = 3(3-4\lambda).$$

For the special (interesting) case $\lambda = 2$, in which $c_{bc} = -26$ (hence $d = 26$ for bosonic strings), we have $c_{\text{total}} = 3(3-4\times2) = -15$. If we combine this system with the $(X^\mu,\ \psi^\mu,\ \tilde{\psi}^\mu)$, for which $c = 3D/2$ and demand $c_{\text{total}} = 0$, we need $3D/2 - 15 = 0 \Rightarrow D = 10$. Therefore superstrings must live in 10-dimensions.

Linear Dilaton

Recall

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu + V_\mu\partial^2X^\mu,$$

where V_μ is a fixed vector (breaking translational invariance). The central charge for this theory is

$$c = D + 6\alpha' V^\mu V_\mu.$$

By adding the fermion ψ^μ, with $T = -\frac{1}{2}\psi^\mu\partial\psi_\mu$ and $c = D/2$, we obtain

$$c = \frac{3D}{2} + 6\alpha' V^\mu V_\mu,$$

and

$$T_F = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X_\mu - i\sqrt{2\alpha'}V_\mu\partial\psi^\mu.$$

7.3 Mode Expansions

Let us do closed strings first. Recall the expansion

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu z^{-m-1},$$

where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$ and $[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$. X^μ obeys periodic boundary conditions. We could have imposed anti-periodic boundary conditions on X^μ, and we did so with U (the compactified coordinate) and got an orbifold, but this breaks translational invariance. That is ok for dimensions we cannot see (e.g., compactified), but not for the four dimensions that describe our space-time. ψ^μ and $\tilde{\psi}^\mu$ on the other hand have no such concerns (also note the absence of a spin-statistics theorem in two-dimensions), so we have two possibilities.

- anti-periodic boundary conditions (Neveu-Schwarz (NS)): $\psi^\mu(\sigma+2\pi) = -\psi^\mu(\sigma)$.
- periodic boundary conditions (Ramond (R)): $\psi^\mu(\sigma+2\pi) = \psi^\mu(\sigma)$.

These have two distinct Hilbert spaces (sectors). There are also two Hilbert spaces for $\tilde{\psi}^\mu$, so in all there are four Hilbert spaces (sectors): NS-NS, R-NS, NS-R, R-R.
Let us first describe $\psi\mu$ in NS. ψ^μ is a function of $\sigma+\tau$. When expanding in Fourier modes, because of anti-periodicity, only the terms $e^{-i(2m+1)(\sigma+\tau)/2}$ contribute (since $\sigma \to \sigma + 2\pi \Rightarrow e^{-i(2m+1)(\sigma+\tau)/2} \to e^{-\pi i(2m+1)}e^{-i(2m+1)(\sigma+\tau)/2}$)
Define $r = m + 1/2 \in Z + 1/2$, then

$$\psi^\mu(\sigma+\tau) = \sqrt{i}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu e^{-ir(\sigma+\tau)}$$

where the factor of $\sqrt{i}$ was introduced for convenience. Transforming to the z-picture, $z = e^{i(\sigma+\tau)}$, we have

$$\begin{aligned}\psi^\mu(z) &= \left(\frac{\partial w}{\partial z}\right)^h \psi^\mu(\sigma+\tau) \\ &= \frac{1}{\sqrt{iz}}\psi^\mu(\sigma+\tau) \\ &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu z^{-r-\frac{1}{2}}\end{aligned}$$

which is a Laurent expansion. We saw the same in terms of the X^μ field. We obtain anti-commutation relations of the ψ^μ fields by analyzing the OPE

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{1}{z}\eta^{\mu\nu}.$$

The anti-commutation relations are

$$\{\psi_r^\mu, \psi_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}.$$

We find similar results for the right-moving sector

$$\tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}_r^\mu \bar{z}^{-r-\frac{1}{2}}, \qquad \bar{\partial}X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}} \tilde{\alpha}_m^\mu \bar{z}^{-m-1},$$

and the anti-commutation relations are

$$\{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}.$$

The stress-energy tensor is

$$T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}, \qquad h = 2.$$

The OPE gives the Virasoro algebra with central extension

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m-1)(m+1)\delta_{m+n,0}.$$

In terms of the OPEs we find

$$T_F(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0).$$

We may expand $T_F(z)$ in terms of modes

$$T_F(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r z^{-r-\frac{3}{2}}.$$

Recall

$$[L_m, G_r] = ((h-1)m - r)G_{r+m} = (\frac{1}{2}m - r)G_{r+m}.$$

Finally

$$T_F(z)T_F(z') \sim \frac{D}{(z-z')^3} + \frac{2}{z-z'}T(z'), \quad \frac{3D}{2} = c, \text{ so } D = \frac{2c}{3}.$$

Find the anit-commutator $\{G_r, G_s\}$ in two steps. First

$$G_r = \oint \frac{dz}{2\pi i} r^{r+\frac{1}{2}} T_F(z),$$

and

$$\oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z)T_F(z') = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} \frac{D}{(z-z')^3} + 2z'^{r+\frac{1}{2}}T(z')$$

$$f(z) = z^{r+\frac{1}{2}}, \quad f'(z) = \left(r+\frac{1}{2}\right) r^{r-\frac{1}{2}}, \quad f''(z) = \left(r^2 - \frac{1}{4}\right) z^{r-\frac{3}{2}},$$

so

$$\oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} \frac{D}{(z-z')^3} = \frac{D}{2}\left(r^2 - \frac{1}{4}\right) z'^{r-\frac{3}{2}}$$

Second step: apply $\oint \frac{dz'}{2\pi i} z'^{s+\frac{i}{2}}$ to isolate G_s:

$$\begin{aligned} \{G_r, G_s\} &= \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\oint \frac{dz'}{2\pi i} z'^{r+s-1} + 2\oint \frac{dz'}{2\pi i} z'^{r+s+1} T(z') \\ &= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \\ &= 2L_{r+s} + \frac{c}{12}\left(4r^2 - 1\right)\delta_{r+s,0} \end{aligned}$$

The algebra of $(L_m,\, G_r)$ closes, as expected: NS algebra. Next, let us study the mode expansion: using

$$\partial X^\mu = -i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \alpha^\mu_m z^{-m-1}, \quad \psi^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi^\mu_r z^{-r-\frac{1}{2}},$$

and

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial\psi_\mu = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{4}(\psi^\mu\partial\psi_\mu - (\partial\psi^\mu)\psi_\mu)$$

we have

$$\begin{aligned} L_m &= \oint \frac{dz}{2\pi i} z^{m+1} T(z) = \frac{1}{2}\sum_{n,n'} \oint \frac{dz}{2\pi i} \alpha^\mu_n \alpha_{n'\mu} z^{-n-n'-m-1} + \frac{1}{4}\sum_{r,r'} \oint \frac{dz}{2\pi i} \psi^\mu_r \psi_{r'\mu} (r-r') z^{-r-r'+m-1} \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^\mu_{m-n}\alpha_{n\mu} + \frac{1}{4}\sum_{r\in\mathbb{Z}+\frac{1}{2}} (2r-m)\psi^\mu_{m-r}\psi_{r\mu}. \end{aligned}$$

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu \partial X_\mu \Rightarrow G_r = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z) = \sum_{n,r'} \oint \frac{dz}{2\pi i} \alpha^\mu_n \alpha_{r'\mu} z^{-n+r+r'-1} = \sum_{n\in\mathbb{Z}} \alpha^\mu_n \psi_{r-n\ \mu}.$$

Normal ordering: No question in G_r, $\forall\, r$ and L_m, $\forall\, m \neq 0$. Potential problem with L_0. After normal ordering, we get $L_0 + a$ where a is a constant to be determined. To determine a, look at $[L_+, L_{-1}] = 2L_0$. We have $L_1|0\rangle = 0$, so $\langle 0|[L_+, L_{-1}]|0\rangle = \langle 0[L_{+1}L_{-1}]|0\rangle = ||L_{-1}|0\rangle||^2$, because $L^+_{-1} = L_1$.
Now $L_{-1}|0\rangle = \frac{1}{2}\sum \alpha^\mu_{-1-n}\alpha_{n\mu}|0\rangle + \frac{1}{4}\sum(2r+1)\psi^\mu_{-1-r}\psi_{r\mu}|0\rangle$ There are non-vanishing terms only if $-n-1 < 0\; n < 0$, i.e., $0 < n < -1$ which is impossible! Also $0 < r < -1$, which implies $r = 1/2$, but then $2r+1 = 0$, so it also vanishes. Therefore

$$L_{-1}|0\rangle = 0, \qquad ||L_{-1}|0\rangle||^2 = 0,$$

so

$$\langle 0|2L_0|0\rangle = 2a = 0 \;\Rightarrow a = 0.$$

In the above, we used $\alpha^\mu_n|0\rangle = \psi^\mu_r|0\rangle,\ n, r > 0$, and the hermicity property, $(\alpha^\mu_{-n})^\dagger = \alpha^\mu_n,\ (\psi^\mu_{-r})^\dagger = \psi^\mu_r$.

The ghosts

The ghost system $(b, c; \beta, \gamma)$ is a superconformal system on its own right. It is opposite to (X^μ, ψ^μ) in that the role of X^μ is played by the fermionic (b, c). So $b, c,$ obey periodic boundary conditions (necessary due to definition of Q_{BRST}). Then (β, γ) may obey periodic (R) or anti-periodic (NS) boundary conditions. Let us do NS first. Recall

$$h_b = \lambda, \quad h_c = 1 - \lambda, \quad h_\beta = \lambda', \quad h_\gamma = 1 - \lambda', \quad \lambda' = \lambda - \frac{1}{2}.$$

We are interested in the $\lambda = 2$ case, in order to couple this system to the (X^μ, ψ^μ) system. Then

$$h_b = 2, \quad h_c = 1 - 1, \quad h_\beta = \frac{3}{2}, \quad h_\gamma = -\frac{1}{2},$$

and the expansions are

$$b = \sum_{m \in \mathbb{Z}} b_m z^{-m-2}, \; c = \sum_{m \in \mathbb{Z}} c_m z^{-m+1}, \; \beta = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \beta_r z^{-r-\frac{3}{2}}, \; \gamma = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \gamma_r z^{-r+\frac{1}{2}}.$$

From the operator product expansions, we get standard (anti) commutators

$$\{b_m, c_n\} = \delta_{m+n,0}, \qquad [\gamma_r, \beta_s] = \delta_{r+s,0}.$$

$b_m, \; c_m, \; \beta_r, \; \gamma_r$ are all annihilation operators for $r, m > 0$. Recall the subtlety with the zero modes b_0, c_0, satisfying $\{b_0, c_0\} = 1$. We have two choices for the vacuum. Choose $c_0|0\rangle = 0$. The conformal generators are

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T(z),$$

$$\begin{aligned}
T(z) &= (\partial b)c - \lambda \partial(bc) + (\partial \beta)\gamma - \lambda' \partial(\beta\gamma) \\
&= (\partial b)c - 2\partial(bc) + (\partial\beta)\gamma - \frac{3}{2}\partial(\beta\gamma) \\
&= \sum_{n,n'} (-n' - 2) b_{n'} z^{-n'-3} c_n z^{-n+1} - 2(-n - n' - 1) b_{n'} c_n z^{-n-n'-2} \\
&\quad + \sum_{r,r'} \left(-r' - \frac{3}{2}\right) \beta_{r'} z^{-r'-\frac{5}{2}} \gamma_r z^{-r+\frac{1}{2}} - \frac{3}{2}(-r - r' - 1)\beta_{r'}\gamma_r z^{-r-r'-2} \\
&= \sum_{n,n'} (n' + 2n) b_{n'} c_c z^{-n-n'-2} + \frac{1}{2} \sum_{r,r'} (3r + r') \beta_{r'} \gamma_r z^{-r-r'-2}
\end{aligned}$$

So

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T(z) = \sum_n (m + n) b_{m-n} c_n + \frac{1}{2} \sum_r (m + 2r) \beta_{m-r} \gamma_r$$

The SUSY generators are

$$G_r = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z), \qquad T_F(z) = -\frac{1}{2}(\partial\beta)c + \lambda' \partial(\beta c) - 2b\gamma$$

where

$$\begin{aligned} T_F(z) &= \sum_{s,n} -\frac{1}{2}\left(-s-\frac{3}{2}\right)\beta_s z^{-s-\frac{5}{2}} c_n z^{-n+1} + \frac{3}{2}\left(-s-n-\frac{1}{2}\right)\beta_s c_n z^{-s-n-\frac{3}{2}} - 2b_n\gamma_s z^{-n-s-\frac{3}{2}} \\ &= \sum_{s,n} -\frac{1}{2}(2s+3n)\beta_s c_n z^{-n-s-\frac{3}{2}} - 3b_n\gamma_s z^{-n-s-\frac{3}{2}}. \end{aligned}$$

So

$$G_r = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z) = -\sum_n \frac{1}{2}(2r+n)\beta_{r-n}c_n + 2b_n\gamma_{r-n}.$$

Normal ordering: again, only L_0 has a problem; should be $L_0 + a$. To find a, consider $[L_1, L_{-1}] = 2L_0$.

$$L_{-1} = \sum_n (n-1)b_{-1-n}c_n + \frac{1}{2}\sum_r (2r-1)\beta_{-1-r}\gamma_r.$$

When applied to the ground state, $|0\rangle$, only the terms $n = -1$, $r = -1/2$ contribute (recall $c_0|0\rangle = 0$), so $L_{-1}|0\rangle = -2b_0c_{-1}|0\rangle - \beta_{-1/2}\gamma_{-1/2}|0\rangle$. Similarly, we obtain $\langle 0|c_{-1} = \langle 0|(2b_1c_0 + \beta_{1/2}\gamma_{1/2})$. So

$$\langle 0|[L_1, L_{-1}]|0\rangle = \langle 0|L_1L_{-1}|0\rangle = -2\langle 0|(b_1c_0b_0c_{-1} - \beta_{\frac{1}{2}}\gamma_{\frac{1}{2}}\beta_{-\frac{1}{2}}\gamma_{-\frac{1}{2}}|0\rangle = -2+1 = -1.$$

So $2a = \langle 0|2L_0|0\rangle = -1$, so $a = -1/2$ (-1 from the bc and $1/2$ from the $\beta\gamma$).

7.4 Open Strings

Open strings do not have independent ocsillators α_n^μ, $\tilde{\alpha}_n^\mu$. Instead, $\alpha_n^\mu = \tilde{\alpha}_n^\mu$. Thus,

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu z^{-m-1}, \qquad \bar{\partial} X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu \bar{z}^{-m-1}.$$

where $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ (c.f. $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$ for closed strings). Similarly for the ψ^μ's:

$$\psi^\mu(z) = \sum_r \psi_r^\mu z^{-r-\frac{1}{2}}, \qquad \tilde{\psi}^\mu(\bar{z}) = \sum_r \psi_r^\mu \bar{z}^{-r-\frac{1}{2}}.$$

The spectrum

For a physical state, $|\psi\rangle$, we demand

$$L_n|\psi\rangle = G_r|\psi\rangle = 0, \quad \text{for } r, n > 0.$$

Also, $L_{-n}|\psi\rangle,\ G_r|\psi\rangle$ are orthogonal to all physical states $|\psi'\rangle$: $\langle\psi'|L_{-n}|\psi\rangle = \langle\psi|L_n|\psi'\rangle = 0$, and similarly for $G_r|\psi\rangle$. They are in the equivalence class of zero. Check also $L_{-n}|\psi\rangle$ is null: $||L_{-n}|\psi\rangle||^2 = 0$. Physical states also obey the constraint

$$\left(L_0 - \frac{1}{2}\right)|\psi\rangle = 0$$

i.e., the Hamiltonian $H = L_0 - 1/2 = 0$ (vanishes).
We build the Hilbert space by applying $\alpha^\mu_{-n},\ \psi^\mu_{-r}$ oscillators only (no ghost modes- the lead to states in the same equivalence classes as above) to the ground state.

$$H = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha^\mu_{-n}\alpha_{n\mu}: + \frac{1}{2}\sum_r r : \psi^\mu_{-r}\psi_{r\mu} : -\frac{1}{2}$$

plus the ghost oscillators, but they do not contribute. Since $\alpha_0 = \sqrt{2\alpha'}p^\mu$ for open strings, we have

$$H = \alpha' p^2 + N - \frac{1}{2}, \qquad N = \sum_{n=1}^\infty \alpha^\mu_{-n}\alpha_{n\mu} + \sum_{r=\frac{1}{2}}^\infty r\psi^\mu_{-r}\psi^{r\mu}.$$

The lowest state: $|0;k\rangle$ for which $N = 0$, so $\alpha' k^2 - 1/2 = 0$, so $m^2 = -k^2 = -1/2\alpha'$, a tachyon!
So we still have a tachyon. This was to be expected, because we took the bosonic theory and enlarged it therefore we should expect the new SUSY theory to contain all the states of the bosonic theory and more.
The next state: $|1\rangle$

$$A_\mu(k)\psi^\mu_{-\frac{1}{2}}|0;k\rangle$$

has $N = \frac{1}{2}$. We see that this is a massless state since

$$\alpha' k^2 + \frac{1}{2} - \frac{1}{2} = 0, \quad \Rightarrow \quad m^2 = -k^2 = 0.$$

Also, $G_r = \sum_n \alpha^\mu_n \psi_{r-n\mu}$, so when $G_{1/2}$ acts on our state, only the $n = 0$ term contributes. So

$$G_{\frac{1}{2}}|1\rangle = \alpha^\mu_0 A_\mu(k)|0;k\rangle = \sqrt{2\alpha'}k\cdot A|0;k\rangle = 0, \ \Rightarrow \ k\cdot A = 0,$$

i.e., transverse polarization. Also note that this is a null state:

$$G_{-\frac{1}{2}}|0;k\rangle = \alpha^\mu_0\psi_{-\frac{1}{2}\mu}|0;k\rangle = \sqrt{2\alpha'}k\cdot\psi_{-\frac{1}{2}}|0;k\rangle$$

i.e., the state with longitudinal polarization is null (and orthogonal to all physical states).
Thus the massless state is a $D-2=8$ dimensional vector. It transforms under the group $SO(8)$. For **closed strings**, the situation is similar.
The $H=0$ constraint translates into $L_0=\tilde{L}_0=0$, i.e.,

$$\frac{\alpha'}{4}p^2+N-\frac{1}{2}=\frac{\alpha'}{4}p^2+\tilde{N}-\frac{1}{2}=0$$

Notice the difference in $\alpha' p^2 \to \frac{\alpha'}{4}p^2$, which is due to the different definitions of α_0^μ between closed and open string.
The lowest state: $|0;k\rangle$ with $\frac{\alpha'}{4}k^2=\frac{1}{2}$, so $m^2=-k^2=-\frac{2}{\alpha'}$ which is a tachyon!
The next level: $A_{\mu\nu}\psi^\mu_{-1/2}\psi^\nu_{-1/2}|0;k\rangle$, with $\frac{\alpha'}{4}k^2=0$, i.e., $m^2=0$. This de-omposes into a scalar, an antisymmetric tensor, and a traceless symmetric tensor:

$$A_{\mu\nu}=\frac{1}{D-2}A^\rho_\rho\eta_{\mu\nu}+\frac{1}{2}(A_{\mu\nu}-A_{\nu\mu})+\frac{1}{2}(A_{\mu\nu}+A_{\nu\mu}+\frac{2}{D-2}A^\rho_\rho\eta_{\mu\nu}.$$

SectionGetting rid of the tachyon Comparing the tachyon with the massless states, there is a clear difference: the tachyon has one less fermionic excitation then the massless states. If we select the states with as *odd* number of fermionic excitations, that will get rid of the tachyon. This is similar to the harmonic oscillator, where we could select, e.g., all the odd states and still have a perfectly well defined physical system.
The operator that did the trick there was P (parity) which commuted with the Hamiltonian and could therefore be simultaneously diagonalized with it. Here we need to find an operator that has two eigenvalues and commutes with all generators of space-time symmetries (not just the Hamiltonian). The space-time symmetries from the Lorentz group (Poincare group rather, but Lorentz suffices). Let us review briefly. The angular momentum $\vec{L}=\vec{r}\times\vec{p}$. In terms of components we have

$$L_x=yp_z-zp_y,\quad L_y=zp_x-xp_z,\quad L_z=xp_y-yp_x.$$

where x and p obey the commutation relations $[x_i,p_j]=\delta_{ij}$.
Define the antisymmetric tensor $L_{ij}=x_ip_j-x_jp_i$, then $L_i=\frac{1}{2}\epsilon_{ijk}L_{jk}$. An antisymmetric tensor is a vector in three-dimensions. **Not** so in four-dimensions. So generalize $L_{ij}\to L_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu$, $[x_\mu,p_\nu]=i\eta_{\mu\nu}$ which includes *time*. $\vec{L}$ generates rotations:

$$\delta x_i=-\frac{i}{2}\omega^{kl}[L_{kl},x_i]=\omega_{ij}x_j$$

where ω_{ij} is an anti-symmetric tensor. In terms of the vector $\vec{\omega}$ we have $\delta\vec{x}=\vec{\omega}\times\vec{x}$. This generalizes to $L_{\mu\nu}\ :\ \delta x_\mu=\omega_{\mu\nu}x^\nu$. For e.g., L_{01}, we have $\delta t=\omega_{01}x,\ \delta x=-\omega_{01}t$, a boost! L_{0i} is a boost in the x_i-direction. The algebra of these Lorentz generators is

$$\begin{aligned}[L_{\mu\nu}, L_{\rho\sigma}] &= [x_\mu p_\nu - x_\nu p_\mu, x_\rho p_\sigma - x_\sigma p_\rho] \\ &= i(\eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho})\end{aligned}$$

Lie algebra of $SO(3,1)$, or in D-dimensions, $SO(D-1,1)$. Introduce spinors: we need to add a piece to $L_{\mu\nu}$ that will rotate the spinor (or boost it). Call this piece $\Sigma_{\mu\nu}$. It needs to satisfy the same $SO(D-1,1)$ algebra and will commute with $L_{\mu\nu}$ by constuction (since $L_{\mu\nu}$ involves space-time and $\Sigma_{\mu\nu}$ involves fermionic operators).
Guess:

$$\Sigma^{\mu\nu} = -i \sum_r \psi_r^\mu \psi_{-r}^\nu = -\frac{i}{2} \sum_r [\psi_r^\mu, \psi_{-r}^\nu].$$

Then the algebra is

$$\begin{aligned}[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] &= -\frac{1}{4} \left(\sum_r [\psi_r^\mu, \psi_{-r}^\nu], \sum_s [\psi_s^\rho, \psi_{-s}^\sigma] \right) \\ &= -\left(\sum_r \psi_r^\mu \psi_{-r}^\nu, \sum_s \psi_s^\rho, \psi_{-s}^\sigma \right) \\ &= i(\eta^{\nu\rho} \Sigma^{\mu\sigma} - \eta^{\mu\rho} \Sigma^{\nu\sigma} - \eta^{\nu\sigma} \Sigma^{\mu\rho} + \eta^{\mu\sigma} \Sigma^{\nu\rho})\end{aligned}$$

where we used $\{\psi_r^\mu, \psi_s^\nu\} = \eta^{\mu\nu} \delta_{r+s,0}$.
$\Sigma^{\mu\nu}$ generates Lorentz transformations on the fermionic fields $\psi^\mu(z)$. Notice that in $D = 10$, there are five operators that commute with each other: Σ^{01}, Σ^{23}, Σ^{45}, Σ^{67}, Σ^{89} (trivial - they contain different ψ_r^μ modes). They can be simultaneously diagonalized. How do they act? Let us be specific and consider Σ^{23}. It acts on ψ_r^2, ψ_r^3 as follows:

$$\begin{aligned}\left[\Sigma^{23}, \psi_r^2\right] &= -i \sum_s [\psi_s^2 \psi_{-s}^3, \psi_r^2] = -\sum_s \{\psi_s^2, \psi_r^2\} \psi_{-s}^3 = i\psi_r^3 \\ \left[\Sigma^{23}, \psi_r^3\right] &= -[\Sigma^{32}, \psi_r^3] = -i\psi_r^2\end{aligned}$$

Eigenstates: $\psi_r^2 + i\psi_r^3$, $\psi_r^2 - i\psi_r^3$.

$$\begin{aligned}\left[\Sigma^{23}, \psi_r^2 + i\psi_r^3\right] &= \psi_r^2 + i\psi_r^3 \quad \text{eigenvalue} : +1 \\ \left[\Sigma^{23}, \psi_r^3 - i\psi_r^3\right] &= -\psi_r^2 + i\psi_r^3 \quad \text{eigenvalue} : -1\end{aligned}$$

Consider a finite transformation (rotation) $U(\theta) = e^{i\theta\Sigma^{23}}$. Then $U(\theta)(\psi_r^2 + i\psi_r^3)U^\dagger(\theta) = e^{i\theta}(\psi_r^2 + i\psi_r^3)$.
Proof:

$$\Sigma^{23}(\psi_r^2 + i\psi_r^3) = (\psi_r^2 + i\psi_r^3)(1 + \Sigma^{23}) \Rightarrow (\Sigma^{23})^n(\psi_r^2 + i\psi_r^3 = (\psi_r^2 + i\psi_r^3)(1 + \Sigma^{23})^n$$

$$\Rightarrow U(\theta)(\psi_r^2 + i\psi_r^3)U^\dagger(\theta) = (\psi_r^2 + i\psi_r^3)e^{i\theta(1+\Sigma^{23})} = e^{i\theta}(\psi_r^2 + i\psi_r^3)U(\theta).$$

Similarly, $U(\theta)(\psi_r^2 + i\psi_r^3)U^\dagger(\theta) = e^{-i\theta}(\psi_r^2 + i\psi_r^3)$. In particular, for $\theta = \pi$, the action of $U(\pi)$ on **both** $\psi_r^2 \pm i\psi_r^3$ is the same. Therefore $U(\pi)\psi_r^{2,3}U^\dagger(\pi) = e^{i\pi}\psi_r^{2,3} = -\psi_r^{2,3}$, i.e., $U(\pi)$ and $\psi_r^{2,3}$ anti-commute!
On the other hand $U(\pi)$ commutes with all other ψ_r^μ, $\mu \neq 2,3$. Thus $U(\pi)$ only has two eigenvalues, ± 1, like parity! If a state has an even number of $\psi_{-r}^2, 3$'s $(r > 0)$, then it belongs to eigenvalue $+1$ - with an odd number of $\psi_{-r}^{2,3}$'s, it has $U(\pi) = -1$. E.g.:

$$\psi_{-r}^2|0\rangle \;:\; U(\pi)\psi_{-r}^2|0\rangle = -\psi_{-r}^2 U(\pi)|0\rangle = -\psi_{-r}^2|0\rangle : \;(-1)$$
$$U(\pi)\psi_{-r_1}^2\psi_{-r_2}^3|0\rangle = -\psi_{-r_1}^2 U(\pi)\psi_{-r_2}^3|0\rangle = \psi_{-r_1}^2\psi_{-r_3}^3|0\rangle \;(+1)$$

etc.
We can do the same with all other Σ's. Thus we have

$$U_1(\pi) = e^{\pi\Sigma^{12}},\; U_2(\pi) = e^{i\pi\Sigma^{23}},\; U_3(\pi) = e^{i\pi\Sigma^{45}},\; U_4(\pi) = e^{i\pi\Sigma^{67}},\; U_5(\pi) = e^{i\pi\Sigma^{89}}.$$

Notice that $U_1(\pi)$ has no i in the exponential. This is because $\{\psi_r^0, \psi_s^0\} = -\delta_{r+s,0}$. The product

$$U_1(\pi)U_2(\pi)...U_5(\pi) = e^{i\pi(-i\Sigma^{01}+\Sigma^{23}+\Sigma^{45}+\Sigma^{67}+\Sigma^{89})} = e^{i\pi F}.$$

This anti-commutes with **all** ψ_r^μ. F is a fermion number operator. e^{iF} will play the role of parity in the harmonic oscillator case. Correction: $e^{i\pi F}V_{\text{gh}}$ will. V_{gh} is the ghost contribution. Since there are no ghost oscillators, all it does is act on the vacuum: $V_{\text{gh}}|0\rangle = -|0\rangle$. Thus restrict Hilbert space to eigenstates of $e^{i\pi F}V_{\text{gh}}$ of eigenvalue $+1$ (invariant states). This gets rid of the tachyon, for $e^{i\pi F}V_{\text{gh}}|0;k\rangle = -|0;k\rangle$ but keeps all massless states $\psi_{-1/2}^\mu|0;k\rangle$.

Consistent truncation

Since $e^{i\pi F}$ is made of Lorentz generators it is guaranteed to be conserved by the OPEs of vertex operators. So even states will produce even states when they interact with other even states.
Thus, we now have a consistent string theory without a tachyon! Or do we? We still need to check modular invariance. The X^μ part of the partition function is modular invariant by itself,

$$Z_X(\tau) = \left(\frac{1}{2\pi\sqrt{\alpha'\tau_2}}|\eta(q)|^{-2}\right)^D, \quad \eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad q = e^{2\pi i\tau}.$$

the fermionic part of the partition function is similarly calculated. The result is a Jacobi-theta function. But, alas, it is **not** modular invariant. This can be seen without doing any calculation as follows.
Before we demanded $\psi^\mu(\sigma + 2\pi) = -\psi^\mu(\sigma)$ (anti-periodic boundary conditions). On a torus, we demand $\psi^\mu(z + 2\pi) = -\psi^\mu(z)$ and also $\psi^\mu(z + 2\pi\tau) = -\psi^\mu(z)$. But then,

$$\psi^\mu(z + 2\pi(\tau + 1)) = -\psi^\mu(z + 2\pi\tau) = +\psi^\mu(z).$$

Therefore the transformation $\tau \to \tau + 1$ changes the boundary conditions to periodic! Therefore $\tau \to \tau + 1$ is not a symmetry of the theory. Our theory is **not** modular invariant. The above argument also shows how to fix the theory. We need to include (somehow) the sector in which ψ^μ obeys periodic boundary conditions. That is the Ramond sector and we study it next.

7.5 The Ramond (R) sector

The R-sector can only exist in two-dimensions, because there is no spin-statistics theorem there. The mode expansion is

$$\psi^\mu(z) = \sum_{n\in\mathbb{Z}} \psi^\mu_n z^{-n-\frac{1}{2}}$$

indices are integers, since $\psi^\mu(z)$ is periodic. The expansion has a factor of $z^{-1/2}$, because the weight of ψ^μ is $h = 1/2$. Therefore this is **not** a Laurent expansion and has a branch cut. We still have

$$\{\psi^\mu_m, \psi^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}$$

as in the NS-sector. We also have the same algebra for L_m, G_r (note it is now $G_m,\ m \in \mathbb{Z}$).

Normal ordering

We only have a problem with L_0. Using $[L_1, L_{-1}] = 2L_0$, we have

$$2\langle 0|L_0|0\rangle = \langle 0|L_{+1}L_{-1}|0\rangle,$$

$$L_{-1}|0\rangle = \left(\frac{1}{2}\sum_n \alpha^\mu_{-1-n}\alpha_{n\mu} + \frac{1}{4}\sum_n (2n+1)\psi^\mu_{-1-n}\psi_{n\mu}\right)|0\rangle$$

For the α's we need $-1-n,\ n<0$, so $-1<n<0$, which is impossible. For the ψ's, we need $-1-n,\ n\le 0$, so $-1\le n\le 0$, so $n=0$, or $n=-1$. Therefore

$$L_{-1}|0\rangle = \frac{1}{4}\left(-\psi^\mu_0\psi_{-1\mu} + \psi^\mu_{-1}\psi_{0\mu}\right)|0\rangle = \frac{1}{2}\psi^\mu_{-1}\psi_{0\mu}|0\rangle.$$

Therefore

$$\begin{aligned}
\langle 0|L_1L_{-1}|0\rangle &= \frac{1}{4}\langle 0|\psi_{0\nu}\psi^\nu_1\psi^\mu_{-1}\psi_{0\mu}|0\rangle, \\
&= \frac{1}{4}\langle 0|\psi^\mu_0\psi_{0\mu}|0\rangle \\
&= \frac{1}{8}\langle 0|\{\psi^\mu_0, \psi_{0\mu}\}|0\rangle \\
&= \frac{D}{8}
\end{aligned}$$

Therefore $\langle 0|L_0|0\rangle = D/16 = a$ (i.e., $L_0 =: L_0 : -D/16$).

The ghosts

$$b = \sum_{m\in\mathbb{Z}} b_m z^{-m-2},\ c = \sum_{m\in\mathbb{Z}} c_m z^{-m+1},\ \beta = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \beta_r z^{-r-\frac{3}{2}},\ \gamma = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \gamma_r z^{-r+\frac{1}{2}}.$$

where $\beta,\ \gamma$ are **not** Laurent expansions. The algebras are

$$\{b_m, c_n\} = \delta_{m+n,0}, \qquad [\gamma_m, \beta_n] = \delta_{m+n,0},$$

which are the same as before, but in addition, the zero modes: $[\gamma_0, \beta_0] = 1$, i.e., $\gamma_0,\ \beta_0$ are creation and annihilation operators respectively. This define $|0\rangle$ by $b_m|0\rangle = 0,\ \ m > 0,\ \ \beta_m|0\rangle = 0,\ \ m \geq 0$ and $c_m|0\rangle == g_m|0\rangle = 0$ for $m > 0$.

Normal ordering

L_0 again has a problem. We can solve as we did before.

$$\begin{aligned} L_{-1}|0\rangle &= \left(\sum_n (n-1)b_{-1n-1}c_n + \frac{1}{2}\sum_n (2n-1)\beta_{-n-1}\gamma_n\right)|0\rangle \\ &= \left(-b_{-1}c_0 - \frac{1}{2}\beta_{-1}\gamma_0\right)|0\rangle \end{aligned}$$

There is only one possibility since $-1 < n \leq 0$, so

$$\begin{aligned} \langle 0|L_1 L_{-1}|0\rangle &= -\langle 0|b_0 c_1 b_{-1} c_0|0\rangle - \frac{1}{4}\langle 0|\beta_0\gamma_1\beta_{-1}\gamma_0|0\rangle \\ &= -1 - \frac{1}{4} = -\frac{5}{4} \end{aligned}$$

and

$$\langle 0|L_0|0\rangle = \frac{1}{2}\langle 0|L_1 L_{-1}|0\rangle = -\frac{5}{8} = a.$$

The spectrum

First observe that the defintion $|0\rangle$ is ambiguous. Indeed $|0\rangle$ is defined by $\psi_m^\mu|0\rangle,\ m > 0$. But then $\psi_0^\mu|0\rangle$ is as good as $|0\rangle$, for $\psi_m^\nu\psi_0^\nu|0\rangle = -\psi_0^\nu\psi_m^\mu|0\rangle = 0,\ m > 0$. the ground state is then a representation of the algebra of the zero modes, $\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}$ (Clifford - Dirac algebra). $|0\rangle$ therefore is a spinor. Instead of one spin, here we have five, because we are in ten-dimensions. The spin operators are $\Sigma^{01}, \Sigma^{23}, \Sigma^{45}, \Sigma^{67}, \Sigma^{89}$. They commute with each other so they can be simultaneously diagonalized. We can then define a basis of ground states $|s_1, s_2, s_3, s_4, s_5\rangle$ where $s_i = \pm 1/2 (i = 1, 2, 3, 4, 5)$. We will use the notation $\vec{s} = (s_1, s_2, s_3, s_4, s_5)$. There are $2^5 = 32$ such states (c.f. $2^2 = 4$ states in the four-dimensional Dirac spinor). All states built from $|\vec{s}\rangle$ have

integer $+1/2$ spin, because ψ^{μ}_{-m} has spin one (eigenstate of S^{XXXX+1} with eigenvalue $+1$). to be contrasted with NS-sector where all states have integer spin. Thus, the inclusion of the R-sector is important, because we need all spins to describe Nature.
The Hamiltonian (L_0) has const. $D/16 - 5/8 = 10/16 - 5/8 = 0$, so $H = \alpha' p^2 + N$ (c.f. $H = \alpha' p^2 + N - 1/2$ in the NS-sector)
The lowest level: $N = 0$, so $H = 0$ and $m^2 = -p^2 = 0$. There is no tachyon! The lowest states, $|0;k\rangle$ are massless! Non-trivial constraint: $G_0|\vec{s};k\rangle = 0$. Relevant piece: $G_0 = \sqrt{2\alpha'} p_\mu \psi^\mu_0$, so $k_\mu \psi^\mu_0 |\vec{s};k\rangle = 0$ which is the Dirac equation ($\gamma^\mu = \frac{1}{\sqrt{2}}\psi^\mu_0$, then $k_\mu \psi^\mu |\vec{s};k\rangle = 0$). Notice also that the algebra $\{G_0, G_0\} = 2L_0$, i.e., $G_0^2 = L_0$. G_0 is the square root of the Hamiltonian!
This is just like in the Dirac case. It is also a generic feature of a SUSY theory: the Hamiltonian can be written as the square of a SUSY charge.
Notice that this also implies that the ground state has zero eigenvalue, because $G_0|0\rangle = 0$, which makes it very hard to have a finite cosmological constant in a SUSY theory. In terms of the fields, the contribution of the boson always exactly cancels the contribution of the fermions (due to SUSY boson $\leftrightarrow$ fermion) and we get zero vacuum expectation energy (cosmological constant). The R-sector can also be split into two eigenspaces of $e^{i\pi F}$ with eigenvalues ± 1. The ground states belongs to $+1$.

7.6 Superstring Theories

We may now combine the NS and R-sectors to form a consistent superstring theory. We need to have analycity in the OPEs (which is not guaranteed in the R-sector, due to branch cuts in the expansions of the fields). This severely constrains the possibilities (we also do not want a tachyon) to ...

$$
\begin{array}{lllll}
IIA: & (NS+,NS+) & (R+,NS+) & (NS+,R-) & (R+,R-) \\
IIB: & (NS+,NS+) & (R+,NS+) & (NS+,R+) & (R+,R+) \\
IIA': & (NS+,NS+) & (R-,NS+) & (NS+,R+) & (R-,R+) \\
IIB': & (NS+,NS+) & (R-,NS+) & (NS+,R-) & (R-,R-)
\end{array}
$$

It can be shown that IIA' is the same as IIA (also, similarly, IIB' is the same as IIB)
Proof: Transform $X^9 \to -X^9,\ \psi^9 \to -\psi^9,\ \tilde{\psi}^9 \to -\tilde{\psi}^9$. Then $e^{i\pi S^{89}} \to e^{-i\pi S^{89}}$ (same eigenvalue), but $S^{89}|0\rangle \to -S^{89}|0\rangle$, so the sign is reversed in the R-sector (S^{89} annihilates the NS vacuum (no zero modes), so no change there). Therefore this transformation maps $R+ \to R-$ and vice versa. *QED*

Open Strings

Only one possibilty: type I: NS+,R+. The projection of eigenspaces of $e^{i\pi F}$ and $e^{i\pi\tilde{F}}$ is known as the Gliozzi-Scherk-Olive (GSO) projection.

The resulting theories turn out to have *space-time* SUSY and obey the spin-statisics theorem (which has to be obeyed for $D > 2$). The fact that space-time SUSY and the spin-statistics theorem emerge is rather unexpected. One would expect that these two should be evident from the start - built in formalism. This fact remains elusive.

Modular Invariance

We have already seen that modular invariance for the NS-NS sector alone cannot possibly work. Now we have a multitude of sectors and a hope that modular transformations will map one onto others and somehow the combination will be invariant. Let us start with the NS-sector. Only the NS+ subsector appears. The partition function for the X^μ's is the same as before and we have already established it is modular invariance, so we will concentrate on the ψ^μ's.
The partition function is as always

$$Z_{NS+} = \text{Tr }(q^H), \quad q = e^{2\pi i\tau}.$$

If $|\psi\rangle$ is in NS+, then $e^{i\pi F}|\psi\rangle = |\psi\rangle$. To find such a $|\psi\rangle$, we can start with an arbitrary state $|\psi'\rangle$ and project onto the eigenspace of $e^{\pi iF}$ of eigenvalue $+1$. The projection operator is

$$P = \frac{1}{2}(1 + e^{i\pi F}), \quad P^2 = P.$$

Also, $e^{i\pi F}P|\psi'\rangle = P|\psi'\rangle$, so eigenvalue $+1$. Thus, to compute the $\text{Tr }_{NS}(PA) = \frac{1}{2}\text{Tr }_{NS}A + \frac{1}{2}\text{Tr }_{NS}(e^{i\pi F}A)$. First trace: for each μ, we have the creation operators $\psi^\mu_{-r},\ r > 0$ where of course $r \in \mathbb{Z} + \frac{1}{2}$. A state can have 0 or 1 ψ^μ_{-r}, since $(\psi^\mu_{-r})^2 = 0$ (fermionic mode). So for fixed r, μ we get a factor $q^0 + q^r = 1 + q^r$ (since $N = 0, r$) the rest of H has already been considered in the X^μ part).
Varying r, we get a product

$$\prod_{r>0}(1 + q^r) = \prod_{m=1}^{\infty}(1 + q^{m-1/2}).$$

Varying μ, we get eight copies of this product (because only the transverse μ's contribute and there are $10 - 2 = 8$ of them). Thus

$$\text{Tr }_{NS}q^H = \left(q^{-1/48}\prod_{m=1}^{\infty}(1 - q^{m-\frac{1}{2}})\right)^8.$$

NB the factor of $q^{-1/48}$ which comes from the new tensor transformation of T (stress-energy “tensor”) as we go from z to $\sigma + \tau$ ($z = e^{i(\sigma+\tau)}$) c.f. in the bosonic case we got $q^{-1/24}$, double becuase for a boson $c = 1$ whereas for a fermion $c = 1/2$. We can write this partiion function in terms of the Jacobi ϑ-function. Recall ... $(z = e^{2\pi i\nu},\ \ q = e^{2\pi i\tau})$

$$\begin{aligned}
\vartheta_{00}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2}) \\
\vartheta_{01}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-1/2})(1-z^{-1}q^{m-1/2}) \\
\vartheta_{10}(\nu,\tau) &= 2e^{\pi i\tau/4}\cos\pi\nu\prod_{m=1}(1-q^m)(1+zq^m)(1+z^{-1}q^m) \\
\vartheta_{11}(\nu,\tau) &= -2e^{\pi i\tau/4}\sin\pi\nu\prod_{m=1}(1-q^m)(1-zq^m)(1-z^{-1}q^m)
\end{aligned}$$

For $\nu = 0,\ z = 1$, so

$$\begin{aligned}
\vartheta_{00}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1+q^{m-1/2})(1+q^{m-1/2}) \\
\vartheta_{01}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1-q^{m-1/2})(1-q^{m-1/2}) \\
\vartheta_{10}(\nu,\tau) &= 2q^{1/8}\prod_{m=1}(1-q^m)(1+q^m)(1+q^m) \\
\vartheta_{11}(\nu,\tau) &= -2q^{1/8}\sin\pi 0\prod_{m=1}(1-q^m)(1-q^m)(1-q^m) = 0!
\end{aligned}$$

Also $\eta(\tau) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m)$. Thus,

$$\vartheta_{00}(0,\tau) = q^{-1/24}\eta(\tau)\left(\prod_{m=1}^{\infty}(1+q^{m-1/2})\right)^2.$$

So

$$\mathrm{Tr}_{\ NS}q^H = \left(\frac{\vartheta_{00}(0,\tau)}{\eta(\tau)}\right).$$

Next, let us do $\mathrm{Tr}_{\ NS}e^{\pi iF}q^H$. Let us fix μ and r again. If the state has zero ψ^μ_{-r}'s then $F = 0$ and $N = 0$. So we get $q^0 = 1$. If the state has one ψ^μ_{-r}, then $e^{i\pi F} = -1$ (recall $e^{i\pi F}\psi^\mu_{-r}e^{-\pi iF} = -\psi^\mu_{-r}$) and $N = r$, so we get $-q^r$.
So this case differs from the previous one by a mere sign change which implies that $\vartheta_{00} \to \vartheta_{01}$. Moreover, the ground state $|0\rangle$ has eigenvalue -1 $(e^{i\pi F}|0\rangle_{NS} = -|0\rangle_{NS})$, so we get an overall "-" sign. Thus

$$\mathrm{Tr}_{\ NS}e^{i\pi F}q^H = -\left(\frac{\vartheta_{01}(0,\tau)}{\eta(\tau)}\right)^4.$$

Similarly, in the Ramond sector,

$$Z_{R+} = \mathrm{Tr}_{\ R+}Pq^H = \frac{1}{2}\left(\mathrm{Tr}_{\ R}q^H + \mathrm{Tr}_{\ R}e^{\pi iF}q^H\right).$$

The creation modes are ψ^{μ}_{-m} , $m > 0$ and we need to take special care of the zero modes ψ^{μ}_{0}.
Fix μ and $m > 0$. Then, for zero ψ^{μ}_{-m}'s, we obtain $q^0 = 1$ and for one $\psi^{m}_{-m}u$, we obtain q^m, so overall, $1 + q^m$. The ground state has energy $H = 1/16$ (normal ordering constant we obtained earlier, so $1 + q^m \to q^{1/16}(1 + q^m)$. Varying $\mu,\ m$ we obtain the product

$$\text{Tr }_R q^H = \left(q^{\frac{1}{16}} q^{-\frac{1}{48}} \prod_{m=1}^{\infty} (1 + q^m) \right)^8 \times \text{“0”}$$

where “0” is the contribution of the zero-modes. Recall the ground state $|\vec{s}\rangle$. Consider the Σ_{23} spin, for example. There are two states $|\uparrow\rangle,\ |\downarrow\rangle$, with spin $\pm\frac{1}{2}$ respectively. Each contributes 1, so overall $1 + 1 = 2$. we have four independent such states (since we have eight transverse dimensions Σ^{01} does not produce independent physical states). Therefore the overall factor “0” $= 2^4$ c.f. with

$$\vartheta_{01}(0, \tau) = 2q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + q^m)^2 = 2q^{1/8} q^{-1/24} \eta(\tau) \left(\prod_{m=1}^{\infty} (1 + q^m) \right)^2$$

$$\text{Tr }_R q^H = \left(2q^{1/8} q^{-1/24} \left(\prod_{m=1}^{\infty} (1 + q^m) \right)^2 \right)^4 = -\left(\frac{\vartheta_{10}(0, \tau)}{\eta(\tau)} \right)^4$$

where the minus sign comes from space-time spin-statistics (ghosts). Finally, $\text{Tr }_R e^{i\pi F} q^H$ gives a similar product, but with two changes

- $1 + q^m \to 1 - q^m$ (- form $e^{i\pi F}$, as in NS-sector
- $|\uparrow\rangle$ and $|\downarrow\rangle$ have opposite eigenvalues, contributing $1 - 1 = 0$!

Therefore

$$\text{Tr }_R = q^H = -\left(\frac{\vartheta_{10}(0, \tau)}{\eta(\tau)} \right)^4 = 0.$$

Putting everything together, the partition function for ψ^μ in NS+ and R+ sector is

$$\begin{aligned} Z_\psi(\tau) &= \text{Tr }_{NS+} q^H + \text{Tr }_{R+} q^H \\ &= \frac{1}{2}\text{Tr }_{NS} q^H + \text{Tr }_{NS} \frac{1}{2} e^{\pi i F} q^H + \frac{1}{2}\text{Tr }_R q^H + \frac{1}{2}\text{Tr }_R e^{\pi i F} q^H \\ &= \frac{1}{2(\eta(\tau)^4} \left(\vartheta_{00}(0, \tau)^4 - \vartheta_{01}(0, \tau)^4 - \vartheta_{10}(0, \tau)^4 + \vartheta_{11}(0, \tau)^4 \right). \end{aligned}$$

This is complicated combination of Jacobi-theta functions, yet not only is it modular invariant, but it vanishes identically! This should not be too surprising, since we have space-time SUSY and so the cosmological constant should vanish. This fact was known to Jacobi himself, for he proved the "abstruse identity"

$$\vartheta_{00}(0,\tau)^4 - \vartheta_{01}(0,\tau)^4 - \vartheta_{10}(0,\tau)^4 = 0,$$

and we have already seen $\vartheta_{11}(0,\tau)^4 = 0$.
Of course the total Z is a product of Z_ψ and $Z_\to = Z_\psi^*$ in the case of IIB and

$$(Z'_\psi)^* = \frac{1}{2\eta(\tau)^4}\left(\vartheta_{00}(0,\tau)^4 - \vartheta_{01}(0,\tau)^4 - \vartheta_{10}(0,\tau)^4 - \vartheta_{11}(0,\tau)^4\right)$$

which of course $Z'_\psi = Z_\psi$.

Modular invariance of type-I

Type-I is an open string theory. Instead of a torus, we have a cylinder.
The cylinder can easily be deduced from the torus. Recall for the torus

$$Z = \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} Z(\tau), \qquad Z(\tau) = \mathrm{Tr}\; q^H, \;\; q = e^{2\pi i\tau}$$

where F_0 is the fundamental region and $\frac{d\tau d\bar{\tau}}{4\tau_2}$ is a modular invariant measure on the torus $(\tau_2(2\pi)^2$ is the volume of the torus = volume of the group of translations). The cylinder defines a more honest partition function, because $\tau \to t \in \mathbb{R}$ and $Z(t) = \mathrm{Tr}\; q^H, \;\; q = e^{-2\pi t}$, i.e., $\tau = i\tau_2, \; \tau_2 = t$, and

$$Z = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}\; e^{-2\pi t L_0}.$$

Notice that thee is no fundamental region, so we have potential divergences from both limits $t \to \infty$ and $t \to 0$. $t \to \infty$ is usually associated with the IR region (long-distance, low energy). $t \to 0$ is associated with UV divergences (short-distances - high energies). In closed strings, there is no $t \to 0$ limit, for it is cut by the restriction to the fundamental region F_0. In the open string case, it is there. But does open string theory have UV divergences? That would make it as bad as field (particle) theory. To answer this, concentrate on $X^\mu, \; \mu = 0, 1, .., D-1$. The partition function (easily deduced from the torus) is

$$Z(t) = \mathrm{Tr}\; e^{-2\pi t L_0} = iV\left(\sqrt{8\pi^2\alpha' t}\right)^{-D} (\eta(it))^{-(D-2)}$$

c.f. on torus:

$$Z(t) = \mathrm{Tr}\; e^{-2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\tilde{L}_0} = iV\left(\sqrt{4\pi^2\alpha'\tau_2}\right)^{-D} |\eta(it)|^{-2(D-2)}$$

where

$$\eta(it) = e^{-\pi t/12} \prod_{m=1}^{\infty} (1 - e^{-2\pi m t}).$$

Let $D = 26$. In the $t \to \infty$ limit, we may expand

$$\begin{aligned}(\eta(it))^{-24} &= e^{2\pi t}\prod_{m=1}^{\infty}(1-e^{-2\pi m t})^{-24} = e^{2\pi t}(1+24e^{-2\pi t}+...)\\ &= e^{2\pi t}+24+...\end{aligned}$$

Each term in the expansion comes from a certain mass level. The first term is from the tachyon, and diverges, because $m^2 < 0$. The second term is from the massless modes (24 transverse photons). Again, it diverges, but only logarithmically. This is expected and is similar to field theory. These divergences cancel in physical quantities.
Now look at $t \to 0$. This appears to be a high energy effect, but it is not! The cylinder becomes very thin and it looks like a closed string is being created, propogating and disappearing again (NB: t does **not** represent a *physical* distance). So $t \to 0$ is *still* an IR effect (long-distance). To show this, use the modular property, $\eta(-1/\tau) = \sqrt{i\tau}\eta(\tau)$. For $\tau = it$, we get $\eta(i/t) = \sqrt{t}\eta(it)$, so

$$\eta(it) = \frac{1}{\sqrt{t}}\eta\left(\frac{i}{t}\right).$$

Change variables to $s = \frac{\pi}{t}$. Then, apart from constants

$$Z \sim \int_0^\infty \frac{dt}{t}t^{-13}\eta(it)^{-24} = \int_0^\infty \frac{dt}{t^2}\eta\left(\frac{i}{t}\right)^{-24} \sim \int_0^\infty ds\eta\left(\frac{is}{\pi}\right)^{-24}.$$

$t \to 0$ is obtained by expanding in large s,

$$\eta\left(\frac{is}{\pi}\right)^{-24} = e^{2s}+24+...$$

(same expansion as before). The first term is from the tachyon (pathological). The second term is from the massless modes. The propagator for them is $1/k^2$ and since $k^2 = -m^2 = 0$, we have $1/0 = \infty$. The pole is due to the propagator for a long time (on-shell).
Let us return to type-I. In this case $d = 10$, so for the X^μ's, we have

$$Z_X(t) = iV(8\pi^2\alpha' t)^{-5}\eta(it)^{-8}.$$

Moreover, there is a subtlety: define the world-sheet parity Ω by

$$\Omega:\ \ X^\mu(\sigma) \to X^\mu(\pi-\sigma).$$

In terms of modes, $\Omega\alpha_n^m u\Omega^{-1} = (-1)^n\alpha_n^\mu$ (recall $X^\mu(\sigma) \sim \sum \alpha_n^\mu e^{-in\sigma}$). Obviously, $\Omega^2 = 1$, sp Ω has two eigenvalues, ± 1. We need to restrict to the $+1$ eigenspace for consistency of the theory. This is easily implemented: we need to keep the states with and **even** number of α_{-n}^μ modes. [NB: In bosonic theory, this would give garbage, for it would exclude the photon! Here, the photon is $\psi_{-1/2}^\mu|0;k\rangle$, so it has 0 (even!) α_{-n}^μ's.

The various partition functions are not affected by the presence of Ω, but we get an extra factor of $1/2$ from the projection $\frac{1}{2}(1+\Omega)$. Thus

$$Z = \int_0^\infty \frac{dt}{2t}\frac{1}{2}\frac{1}{2}Z_X(t)Z_\psi(t)$$

where

$$Z_\psi(t) = \vartheta_{00}(0,it)^4 - \vartheta_{01}(0,it)^4 - \vartheta_{10}(0,it)^4 - \vartheta_{11}(0,it)^4$$

and the two factors of $1/2$ come from Ω and the GSO projections respectively. To study the divergences (even though $Z_\psi = 0$! - we still need to study the, otherwise $Z_\psi = 0$ is a $\infty - \infty = 0$ statement; also these divergences appear (and did not cancel) in other amplitudes) Define $s = \pi/t$. Then

$$Z_X(t) = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds\,\eta\left(\frac{is}{\pi}\right)^{-8} Z_\psi\left(\frac{\pi}{s}\right).$$

Modular properties

$$\eta(it) = \frac{1}{\sqrt{t}}\eta(i/t), \qquad \vartheta_{00}(0,it) = \frac{1}{\sqrt{t}}\vartheta_{00}(0,i/t).$$

Separate NS and R. Then in the NS-sector

$$Z_{NS}(t) = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds\,\eta\left(\frac{is}{\pi}\right)^{-12}\left(\vartheta_{00}(0,is/\pi)^4 - \vartheta_{10}(0,is/\pi)^4\right).$$

To leading order, $\eta\left(\frac{is}{\pi}\right)^{-12} = q^{-1/2} = e^s$ and

$$\vartheta_{10}(0,is/\pi)^4 = 2^4q^{1/2} = 2^4e^s, \qquad \vartheta_{00}(0,is/\pi)^4 = 1 + ...$$

So

$$Z_{NS} = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16 + o(e^{-2s})).$$

Notice that the tachyon has disappeared, but of course, we still have the divergence from the sixteen massless modes, as expected. What can we do? Well, the cylinder is not the only possibility. We also have the Möbius strip and the Klein bottle.

The Möbius Strip

Same as the cylinder, but we twist before we identify the ends. In other words, $X^\mu(\omega, 2\pi t) = X^\mu(\pi-\sigma,0) = \Omega X^\mu(\sigma,0)\Omega^{-1}$. The partition function is very similar to the cylinder. The only difference is the insertion of the parity operator, Ω. Thus $Z_{\text{Mobius}} = \text{Tr}\,(q^{L_0}\Omega)$. The action of Ω is simple. If a state has an even (odd) number of α^μ_{-n}'s, $\Omega = +1(-1)$. Thus, $(1-q^m)^{-1}$ is replaced by $(1-(-)^mq^m)^{-1}, \;\; q = e^{=2\pi t}$ and so

$$\eta(it) = e^{-\pi t/12}\prod_{m=1}^\infty(1-e^{-2\pi mt}) \Rightarrow e^{-\pi t/12}\prod_{m=1}^\infty(1-(-)^m e^{-2\pi mt})$$

which can be written in terms of

$$\vartheta_{00}(0,\tau)=\prod_{m=1}^{\infty}(1-q^m)(1-q^{m-1/2})^2.$$

As follows: let $\tau=2it$,

$$\begin{aligned}\vartheta_{00}(0,2it) &= \prod_{m=1}^{\infty}(1-q^m)(\prod(1+e^{-2\pi t(2m-1)}))^2\\ &= \prod_{m=1}^{\infty}(1-q^m)^{-1}\left[\prod(1+e^{-2\pi t(2m)})(1+e^{-2\pi t(2m-1)})\right]^2\\ &= e^{-\pi t/16}\frac{1}{\eta(2it)}\left[\prod(1-(-)^m e^{-2\pi tm})\right]^2\end{aligned}$$

so

$$e^{-\pi t/12}\prod(1-(-)^m e^{-2\pi tm})=\sqrt{\vartheta_{00}(0,2it)\eta(2it)}$$

replaces $\eta(it)$. zero modes are still the same , so... Recall the cylinder

$$Z_X=iV\left(\frac{1}{\sqrt{8\pi^2\alpha' t}}\right)^D\eta(it)^{-(D-2)}$$

The partition function for the Möbius strip is

$$Z_X=iV\left(\frac{1}{\sqrt{8\pi^2\alpha' t}}\right)^D(\vartheta_{00}(0,2it)\eta(it))^{-(D-2)/2}.$$

Next, do the ψ's. Easier to work in the R-sector (only one contribution)

$$Z_{\psi,R}=\mathrm{Tr}\;\Omega q^{N_\psi}=-2^4\left[q^{1/16}q^{-1/48}\prod_{m=1}^{\infty}(1+(-)^m q^m)\right]^8$$

which can be written in terms of Jacobi-theta functions as follows:

$$\begin{aligned}\vartheta_{01}(0,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1-q^{m-1/2})\\ \vartheta_{10}(0,\tau) &= 2q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+q^m)^2\end{aligned}$$

so

$$\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)=2q^{1/8}\prod_{m=1}^{\infty}(1-q^m)\prod_m(1+(-)(\sqrt{q})^m)^2,$$

so let $q=e^{-4\pi t}$.

$$\frac{\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)}{\eta(0,2it)^2}=2e^{\pi t/3}e^{-\pi t/2}\prod_m(1+(-)(\sqrt{q})^m)^2$$

so

$$Z_{\psi,R} = -\left(\frac{\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)}{\eta(0,2it)^2}\right)^4.$$

At $D = 10$

$$Z_R = \int_0^{\int} \frac{dt}{2t}\frac{1}{2}\frac{1}{2}Z_X Z_{\psi,R} = iV\int_0^{\int} \frac{dt}{8t}(8\pi^2\alpha' t)^{-5}\left(\frac{\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)}{\eta(0,2it)^2}\right)^4.$$

To study the $t \to 0$ limit, switch the variable to $s = \pi/t$. Then

$$Z_R = iV\frac{8}{(8\pi^2\alpha')^5}\int_0^\infty ds\left(\frac{\vartheta_{01}(0,2is/\pi)\vartheta_{10}(0,2is/\pi)}{\eta^3(2is/\pi)\vartheta_{00}(2is/\pi)}\right)^4$$

for small s, we have

$$\begin{aligned}
\vartheta_{01}(0,2is/\pi) &\simeq 1+...,\\
\vartheta_{01}(0,2is/\pi) &\simeq 2q^{1/8}+...\\
\vartheta_{00}(0,2is/\pi) &\simeq 1+...,\\
\eta(2is/\pi) &\simeq q^{1/24}+...
\end{aligned}$$

so

$$\frac{\vartheta_{01}\vartheta_{10}}{\eta^3\vartheta_{00}} = \frac{2q^{1/8}}{q^{1/8}} = 2+... \Rightarrow \left(\frac{\vartheta_{01}\vartheta_{10}}{\eta^3\vartheta_{00}}\right)^4 = 2^4+... = 16+...$$

no tachyon, and sixteen massless modes contributing, as expected. c.f. for the cylinder,

$$Z_R = -Z_{NS} = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...).$$

of opposite sign, but they do not cancel!

The Klein Bottle

Even though a bottle looks more appropriate for closed strings, and amplitudes are defined in terms of closed string modes, the Klein bottle contributes to open strings.

DEFINITION: Consider a torus with $\tau = it$. we identify the sides $\sigma = 0$ and $\sigma = 2\pi$ and obtain a cylinder, but just like with the Möbius strip, we identify the sides $\tau = 0$ and $\tau = 2\pi\tau$ by twisting them first

$$X^\mu(\sigma,0) = X^\mu(-\sigma,2\pi t) = \Omega X^\mu(\sigma,2\pi t)\Omega^{-1}$$

The partition function is given by

$$Z_X = \text{Tr }\Omega e^{-2\pi t L_0}e^{-2\pi t\tilde{L}_0}$$

In this case, $\Omega\alpha_n^\mu\Omega^{-1} = -\tilde{\alpha}_n^\mu$ (unlike for open strings, where $\alpha_n^\mu \to -\alpha_n^\mu$) Therefore, the diagonal elements if Ω have exactly the same α_n^μ's as $\tilde{\alpha}_n^\mu$'s.
So the trace is effectively over the α_n^μ's only, which explains why this is an open string amplitude.
For the diagonal elements of Ω we have $\Omega = +1$ (even total # of $\alpha_n^\mu, \tilde{\alpha}_n^\mu$.) and Ł$_0 = \tilde{L}_0$, so

$$Z_X = \text{ Tr } e^{-4\pi t L_o}\Omega\big|_{\Omega=1}$$

which is the same as open string partition function, but with q^2 instead of q (or $2t$ instead of t)

$$Z_X = iV(4\pi\alpha' t)^{-D/2}(\eta(2it))^{-(D-2)}$$

Note the first factor has a 4 rather than an 8 due to the closed string.
The partition function for the ψ^μ's is obtained similarly. The result is the same as the open string (cylinder) again, but with $t \to 2t$.

$$Z_\psi^{NS} = \frac{1}{(\eta(2it))^4}\left[\vartheta_{00}^4(2it) - \vartheta_{10}^4(2it)\right]$$

and $Z_\psi^{NS} = -Z_\psi^R$. Overall

$$Z_{NS} = \int_0^\infty \frac{dt}{2t}\frac{1}{2}\frac{1}{2}Z_X Z_\psi^{NS} = iV\int_0^\infty \frac{dt}{8t}(4\pi^2\alpha' t)^- s(\eta(2it))^{-12}\left[\vartheta_{00}^4(2it) - \vartheta_{10}^4(2it)\right]$$

and $Z_R = -Z_{NS}$. The study of the $t \to -$ limit can be copied from the cylinder with an extra 2^10 factor

$$Z_{NS} = i\frac{2^10V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16 + ...)$$

Again there is no tachyon, but alas, $Z_{cylinder} + Z_{mobius} + Z_{klein}$ still has a non-vanishing divergence. What do we do? We need to introduce **Chan-Paton** factors!
Chan-Paton factors were first introduced in QCD, where the string was made of glue. They attached quarks at the ends of the string which carried indices labeling *color*.
In the present setting, we will introduce them because we can. They do not spoil Lorentz invariance, because they live at the ends of the string. They are useful because they give us extra degrees of freedom, which are needed to describe gauge interactions.
e.g. E&M: Kaluza-Klein added an extra index throughout the string (didn't know about strings, but that is what they effectively did.) $(X^0, ..., X^3, X^4)$: X^4 was the extra-dimension. This spoiled Lorentz invariance, but that was ok, because we only care about Lorentz invariance in four dimensions. The extra dimension gave us a gauge group (U(1)) corresponding to a photon. More dimensions give us more complicated gauge groups and extra degrees of freedom.

With Chan-Paton factors, the gauge group does not come from extra dimensions, but from extra degrees of freedom at the ends of the string (open of course). Yet another innovation of string theory!
So all states now carry two more indices $|0\rangle \to |0, ij\rangle$, so, e.g. we now have n^2 tachyons or photons, if $i, j = 1, 2, ..., n$. Thus, the photon can be the weak boson multiplet $(W^{\pm}, Z^0)$, or the gluon.
How does this effect the partition function? For the cylinder, all n^2 states contribute equally, so Z is multiplied by n^2. For the Möbius strip ,because of the twist, i needs to be identified with j, and there are n possibilities, $Z_{\text{Mobius}} \to nZ_{\text{Mobius}}$.
For the Klein bottle, we have no indices, because we have closed strings, so $Z_{\text{Klein}} \to Z_{\text{Klein}}$.
Overall, the partition function is now

$$Z = n^2 Z_{\text{cylinder}} + nZ_{\text{Mobius}} + Z_{\text{Klein}}.$$

Recall for the R-sector

$$\begin{aligned}
Z_{\text{cylinder}} &= -i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...) \\
Z_{\text{Mobius}} &= i\frac{2^6 V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...) \\
Z_{\text{Klein}} &= -i\frac{2^{10} V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...)
\end{aligned}$$

$$Z = -i(n-2^5)^2\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...)$$

We obtain a **finite** answer if and only if $n = 2^5 = 32$. This implies that out of all possible gauge groups, type I string theory makes a unique choice: SO(32). This was a crucial discovery that led to the explosion of interest in string theory.