String Theory II

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UNIT 6

Compactification and Duality

6.1 The Kaluza-Klein Mechanism

Before we introduce the Kaluza-Klein mechanism, let us briefly review electromagnetism and gauge symmetry. The field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and the action

$$S = \int d^4x \, \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right).$$

leads to the Maxwell equations

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

Gauge invariance in the theory implies $A_\mu \to A_\mu - \partial_\mu \lambda$ provided the current is conserved ($\partial_\mu J^\mu = 0$). The conserved charge is given by

$$Q = \int d^3x J^0, \quad \frac{dQ}{dt} = 0.$$

Suppose J^μ is due to a scalar field Φ which has mass m. If we forget about the charge for the moment, let p^μ be the momentum of the particle Φ represents (Φ represents no particle that anybody has observed). Einstein tells us

$$p^\mu p_\mu = -m^2$$

Quantize the system: $p_\mu \to i\partial_\mu$, so $\partial_\mu \partial^\mu \Phi = m^2 \Phi$ (Klein-Gordon equation). This is obtained from the action

$$S = \frac{1}{2} \int d^4x \; \partial_\mu \Phi^* \partial^\mu \Phi + m^2 |\Phi|^2$$

The conserved current: $j^\mu = \Phi^*\partial^\mu\Phi-$ c.c. , $\partial_\mu J^\mu = 0$. If Φ has charge, we need to couple J^μ to A^μ. This is done by $p_\mu \to p_\mu - qA_\mu$, or $\partial_\mu \to \partial_\mu + iqA_\mu$, where q is the charge. Therefore the action for a charged scalar field, Φ is

$$S = \frac{1}{2}\int d^2x \ \left((\partial_\mu - iqA_\mu)\Phi^*(\partial^\mu + iqA^\mu)\Phi + m^2|\Phi|^2\right)$$

comparing with $\int d^4x \ A_\mu J^\mu$, we obtain

$$J_\mu = -\frac{iq}{2}(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*).$$

Gauge invariance: $\Phi \to e^{iq\lambda}\Phi, \ A_\mu \to A_\mu - \partial_\mu\lambda$. We have $(\partial_\mu + igA_\mu)\Phi\tilde{\psi}e^{ig\lambda}(\partial_\mu + igA_\mu)\Phi$ So the action and the currents are gauge invariant. Since λ is real, $|\Phi|$ is invariant, so Φ moves on a circle in the complex plane as λ changes. λ represents an angle in this picture.
Kaluza-Kleins suggestion was to take this picture literally and assume the the e&m is nothing but the effect of an extra (fifth) dimension. How? Let us see... Imagine a five-dimensional manifold in which the fifth dimension is a circle of radius R. Choose the coordinates

$$x^\mu = (t, x, y, z, u), \qquad u \equiv u + 2\pi R.$$

The line element is

$$ds^2 = G_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + 2G_{\mu u}dx^\mu du + G_{uu}du^2.$$

Suppose G_{MN} is independent of u (invariance under u translation). Parametrize G_{MN} as follows: $A_\mu = G_{\mu u}/G_{uu}$, $g_{\mu\nu} = G_{\mu\nu} - G_{uu}A_\mu A_\nu$. Then

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + G_{uu}(du + A_\mu dx^\mu)^2$$

Reparametrizaions $u \to u + \lambda(x^\mu)$ implies $A_\mu \to A_\mu - \partial_\mu\lambda$, i.e., gauge transformations!
Let p_μ be the momentum conjugate to u. Then $e^{iap_\mu}f(u) = f(u+a)$ (p_μ generates translations). Since $f(u+2\pi R) = f(u)$, we need $e^{2\pi i R p_\mu} = 1$, so $p_\mu = n/R$ (momentum is quantized). A general $f(u)$ may be expanded in momentum eigenstates ($e^{inu/pR}$).

$$f(u) = \sum_{n=-\infty}^{\infty} a_n e^{inu/R}.$$

For a field $\phi(x^\mu) = \phi(x^\mu, u)$, we have

$$\phi(x^\mu) = \sum_{n=-\infty}^{\infty} a_n(x^\mu)e^{inu/R}.$$

The wave equation $\partial_\mu\partial^\mu\phi = 0$ (massless ϕ) becomes

$$\partial_\mu\partial^\mu\Phi_n - \frac{n^2}{R^2}\Phi_n = 0$$

i.e., $a_n(x^\mu)$ represents a **massive** field of mass $m = n/R$ from a four-dimensional point of view. Einstein's equation correspondingly reads $p_\mu p^\mu = -n^2/R^2$.
At energies $E \ll 1/R$, we only see the $n = 0$ mode of $\alpha_0(x^\mu)$. At high energies (early universe), we see more modes.
What about charge? To isolate the effects of A_μ, let $G_{uu} = 1$ and $g_{\mu\nu} = \eta_{\mu\nu}$.
From $\phi = \sum a_n(x^\mu)e^{inu/R}$ and $u \to u + \lambda(x^\mu)$ we obtain $a_n(x^\mu) \to e^{in\lambda/R}a_n$.
Comparing with $\phi \to e^{iq\lambda}\phi$ for a field ϕ of charge q, we see an indication that $q = u/R$ for the mode a_n. So, $q = m$; the mass of a_n.
To see the gauge invariance in full swing, go back to the wave equation for a_n and put back in the curvature of space-time:

$$D^\mu \partial_\mu (a_n e^{inu/R}) = 0$$

where

$$D_M v_N = \partial_M v_N - \Gamma^L_{MN} v_L, \quad D^M v_M = \partial^M v_M - G^{MN}\Gamma^L_{MN} v_L,$$

and the Christophel symbol expressed in terms of the metric may be written as

$$\Gamma^L_{MN} = \frac{1}{2} G^{LP} \left(\partial_M G_{PN} + \partial_N G_{MP} - \partial_P G_{MN} \right).$$

A short calculation reveals

$$\begin{aligned} (\partial_\mu + A_\mu \partial_u)^2 (a_n e^{inu/R}) &= 0 \\ \partial_\mu + i\frac{n}{R} A_\mu) a_n &= \frac{n^2}{R^2} a_n \end{aligned}$$

which confirms that a_n has charge $q = n/R$. Moreover, the Maxwell equations come from the Einstein action

$$S \sim \int d^5x \, \sqrt{-G} R^{(5)},$$

where

$$G = det G_{MN} = -1 \qquad\qquad R^{(5)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

So the Einstein action reduces to electromagnetism! If we put back four-dimensional curvature ($g_{\mu\nu} \neq \eta_{\mu\nu}$), then we get $G = det g_{\mu\nu}$ and $R^{(5)} = R^{(4)} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ which is four-dimensional gravity and electromagnetism!

6.2 Strings

Consider a closed string moving along u (as well as $x, y, z, ...$), described by the function $V(\sigma, \tau)$. The action is (concentrate on U)

$$S = \frac{1}{2\pi\alpha'} \int d^2z \; \partial U \bar{\partial} U$$

and we still demand $U \equiv U + 2\pi R$. As before, center-of-mass momentum is quantized: $p = n/R,\ n \in \mathbb{Z}$. Recall the mode expansion without compactification:

$$X^\mu(\sigma,\tau) = x^\mu + 2\pi\alpha' p^\mu \frac{\tau}{\ell} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m}\left(\alpha^\mu_m e^{-2\pi i m(\sigma+\tau)/\ell} + \tilde{\alpha}^\mu_m e^{2\pi i m(\sigma-\tau)/R}\right),$$

which can also be written in terms of $z = e^{2\pi i(\sigma+\tau)/\ell},\ \bar{z} = e^{-2\pi i(\sigma-\tau)/\ell}$ as

$$X^\mu(z,\bar{z}) = x^\mu + 2\pi\alpha' p^\mu \frac{\tau}{\ell} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m}\left(\alpha^\mu_m z^{-m} + \tilde{\alpha}^\mu_m \bar{z}^{-m}\right),$$

with derivatives

$$\begin{aligned} \partial X^\mu(z,\bar{z}) &= -i\frac{\alpha'}{2} p^\mu z^{-1} - i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \alpha^\mu_m z^{-m-1}, \\ \bar{\partial} X^\mu(z,\bar{z}) &= -i\frac{\alpha'}{2} p^\mu z^{-1} - i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \tilde{\alpha}^\mu_m \bar{z}^{-m-1}. \end{aligned}$$

Notice that the momentum is $p = \frac{1}{2\pi\alpha'}\oint(dz\partial X - d\bar{z}\bar{\partial}X)$ and if we go around the string once, we obtain

$$\oint(dz\partial X + d\bar{z}\bar{\partial}X) = 0. \tag{6.2.1}$$

With $u \equiv u + 2\pi R$ (compactified), (6.2.1) is no longer necessarily true. When we go around the string, u can change by a multiple of $2\pi R$, i.e.,

$$U(\sigma+\ell) = U(\sigma) + 2\pi R w, \qquad w \in \mathbb{Z}.$$

This allows a solution of the form $U(\sigma,\tau) = 2\pi w R \frac{\sigma}{\ell}$. Since $z/\bar{z} = e^{4\pi i\sigma/\ell} \Rightarrow 2\pi\sigma/\ell = -i/2\ln(z/\bar{z})$, so

$$U(\sigma,\tau) = -\frac{i}{2} w R \ln\left(\frac{z}{\bar{z}}\right)$$

has to be added. We obtain

$$\begin{aligned} U(z,\bar{z}) &= u - i\frac{\alpha'}{2}\frac{n}{R}\ln|z|^2 - \frac{i}{2}wR\ln\left(\frac{z}{\bar{z}}\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}(\alpha_m z^{-m} + \tilde{\alpha}_m \bar{z}^{-m}), \\ &= u - i\frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z - i\frac{\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}(\alpha_m z^{-m} + \tilde{\alpha}_m \bar{z}^{-m}), \end{aligned}$$

with the derivatives

$$\begin{aligned} \partial U(z,\bar{z}) &= -i\frac{\alpha'}{2} p_L z^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\alpha_m z^{-m-1}, \\ \bar{\partial} U(z,\bar{z}) &= -i\frac{\alpha'}{2} p_R \bar{z}^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\tilde{\alpha}_m \bar{z}^{-m-1}, \end{aligned}$$

where $P_{\frac{L}{R}} = \frac{n}{R} \pm \frac{wR}{\alpha'}$. Notice that

$$\frac{1}{2\pi\alpha'}\oint (dz\partial X - d\bar{z}\bar{\partial}X) = \frac{1}{2}(p_L + p_R) = p = \frac{n}{R},$$

and

$$\frac{1}{2\pi\alpha'}\oint (dz\partial X + d\bar{z}\bar{\partial}X) = \frac{1}{2}(p_L - p_R) = \frac{wR}{\alpha'} \neq 0.$$

We may express the Virasoro generators in terms of the left and right-handed momenta.

$$L_0 = \frac{\alpha' p_L^2}{4} + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n, \qquad \tilde{L}_0 = \frac{\alpha' p_R^2}{4} + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n$$

These are the same as in the uncompactified case, except now $p_L \neq p_R$.

6.3 Partition Function

First, let us recall the uncompactified case

$$\begin{aligned}
Z &= \text{Tr } (q^{L_0 - 1/24}\bar{q}^{\tilde{L}_0 - 1/24}), \qquad q = e^{2\pi i\tau}, \\
&= \text{Tr } (q\bar{q})^{\alpha' p^2/4 - 1/24} \prod_n \left(\sum_{N_n = 0}^{\infty} q^{nN_n} \right) \left(\sum_{\tilde{N}_n = 0}^{\infty} \bar{q}^{n\tilde{N}_n} \right), \\
&= \left(\text{Tr } e^{-\pi\tau_2\alpha' p^2} \right) \left| \prod_n (1 - q^n)^{-1} \right|^2 (q\bar{q})^{-1/24}, \\
&= |\eta(\tau)|^{-2} V \int \frac{dp}{2\pi} e^{-\pi\tau_2\alpha' p^2}, \\
&= 6|\eta(\tau)|^{-2} \frac{V}{2\pi} \frac{1}{\sqrt{\alpha'\tau_2}},
\end{aligned}$$

which is invariant under modular transformations ($\tau \to \tau + 1,\ \tau \to -1/\tau$). Notice that for $X^\mu = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2$, which is a solution of the wave equation,we have $\partial X^\mu = -i\frac{\alpha'}{2}p^\mu z^{-1},\ \bar{\partial}X^\mu = -i\frac{\alpha'}{2}p^\mu \bar{z}^{-1}$ and the action is

$$\begin{aligned}
S &= \frac{1}{2\pi\alpha'} \int d^2\, z\partial X_\mu \bar{\partial} X^\mu = \frac{\alpha'}{8\pi} p^2 \int \frac{d^2 z}{|z|^2}, \quad z = e^{i(\sigma^1 + i\sigma^2)}, \\
&= \frac{\alpha'}{8\pi} p^2 2V_{\text{Torus}}, \qquad (V_{\text{Torus}} = 2\pi(2\pi\tau_2)) \\
&= \alpha' p^2 \tau_2 \pi.
\end{aligned}$$

Therefore

$$e^{-S} = e^{-\pi\tau_2\alpha' p^2},$$

which is the factor whose trace contributes to the partition function.

Partition Function for Compactified Space

$$\begin{aligned} Z &= \mathrm{Tr}\,(q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24}), \\ &= |\eta(\tau)|^{-2}\sum_{n,w} q^{\alpha' p_L^2/4}\bar{q}^{\alpha' p_R^2/4}, \qquad p_{\frac{L}{R}} = \frac{n}{R} \pm \frac{wR}{\alpha'} \\ &= |\eta(\tau)|^{-2}\sum_{n,w} \exp\left[-\pi\tau_2\alpha'\left(\frac{n^2}{R^2}+\frac{w^2R^2}{\alpha'^2}\right)+2\pi i\tau_1 nw\right]. \end{aligned}$$

Use the Poisson resummation formula:

$$\sum_{n=-\infty}^{\infty} e^{-\pi a n^2+2\pi i b n} = \frac{1}{\sqrt{a}}\sum_{m=-\infty}^{\infty} e^{-\pi(m-b)^2/a}.$$

Therefore the partition function for the compactified case is

$$\begin{aligned} Z &= |\eta(\tau)|^{-2}\frac{R}{\sqrt{\tau_2\alpha'}}\sum_{m,w}\exp\left[-\frac{\pi R^2}{\alpha'\tau_2}\left((m-\tau_1 w)^2+\tau_2^2 w^2\right)\right], \\ &= |\eta(\tau)|^{-2}\frac{V}{2\pi\sqrt{\alpha'\tau_2}}\sum_{m,w}\exp\left(-\frac{\pi R^2}{\alpha'\tau_2}|m-\tau w|^2\right), \end{aligned}$$

which is the same as the uncompactified case. Modular invariance implies

$$\begin{aligned} \tau\to\tau+1 \quad &\Leftrightarrow \quad m\to m+w, \\ \tau\to-\frac{1}{\tau} \quad &\Leftrightarrow \quad m\to -w,\ w\to m. \end{aligned}$$

Notice that the solution with the correct boundary conditions satisfying

$$\begin{aligned} U(\sigma^1+2\pi,\sigma^2) &= U(\sigma^1,\sigma^2)+2\pi wR, \\ U(\sigma^1+2\pi\tau_1,\sigma^2+2\pi\tau_2) &= U(\sigma^1,\sigma^2)+2\pi mR. \end{aligned}$$

can be written as

$$U(\sigma^1,\sigma^2) = \frac{wR}{\tau_2}(\tau_2\sigma^1-\tau_1\sigma^2)+\frac{mR}{\tau_2}\sigma^2$$

where

$$\partial_1 U = wR, \qquad \partial_2 U = \frac{R}{\tau_2}(m-w\tau_1).$$

The action is given by

$$S = \frac{1}{4\pi\alpha'}\int d\sigma_1 d\sigma_2\left[(\partial_1 U)^2+(\partial_2 U)^2\right] = \frac{1}{4\pi\alpha'}2\pi 2\pi\tau_2\frac{R^2}{\tau_2^2}|m-w\tau|^2+\ldots$$

Thermodynamics

From thermodynamics we know the partition function is

$$Z = \sum_E e^{-\beta E}, \qquad \beta = \frac{1}{T}$$

and T is the temperature.
Let $|E\rangle$ be the eigenstate of the Hammiltonian with eigenvalue E. Then

$$Z = \sum_E \langle E|e^{-\beta H}|E\rangle = \text{Tr } e^{-\beta H}.$$

This is a special case of the string partition function ($\beta \in \mathbb{R}$). To calculate this, insert the complete sets

$$Z = \sum_{E,x,y} \langle E|x\rangle\langle x|e^{-\beta H}|y\rangle\langle y|E\rangle$$

Let $\beta \to it\hbar$, then $\langle x|e^{-\beta H}|y\rangle \to \langle x|e^{-iHt/\hbar}|y\rangle = \langle x(t)|y(0)\rangle$ a correlator (Greens function) of the Schrodinger equation. Suppose t is small. Then insert $1 = \sum_p |p\rangle\langle p|$, where $|p\rangle$ is an eigenstate of the momentum ($H = H(p,q)$).

$$\begin{aligned}\langle x(t)|y(0)\rangle &= \sum_p \langle x|p\rangle e^{-iHt/\hbar}\langle p|y\rangle, \\ &= \int dp\ e^{-ipx}e^{-iHt/\hbar}e^{ipy}.\end{aligned}$$

For $x - y \simeq -\dot{q}t$, so

$$\langle x(t)|y(0)\rangle = \int dp\ e^{i(\dot{q}p-H)t} = \int dp\ e^{iS}, \quad S = \int_0^t dt'\ (\dot{q} - H)$$

The dominant contribution is from the stationary point, $\frac{\partial S}{\partial p} = 0$. This is at $\dot{q} = \frac{\partial H}{\partial p}$, which is the Hamilton-Jacobi equation. Then

$$\langle x(t)|y(0)\rangle = e^{iS}.$$

This integrates for finite t. Then

$$Z = \sum_{e,x,y} e^{iS(x,y)}\psi_E^*(x)\psi_E(y) = \sum_x e^{iS(x,x)}$$

(sum over all possible closed paths) and we used the orthogonality of $|x\rangle$ such that $\langle x|y\rangle = \delta(x-y)$ and assumed the energy eigenstates formed a complete set.

6.4 Vertex Operators

Recall

$$U(z,\bar z) = u - i\frac{\alpha'}{2}\left(\frac{n}{r}+\frac{wR}{\alpha'}\right)\ln z - i\frac{\alpha'}{2}\left(\frac{n}{r}-\frac{wR}{\alpha'}\right)\ln \bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\alpha_m z^{-m}+\tilde\alpha_m \bar z^{-1}\right)$$

Momenta $p_L,\ p_R$ are different. Their eigenvalues are

$$p_L = \frac{n}{r}+\frac{wR}{\alpha'}, \qquad p_R = \frac{n}{r}-\frac{wR}{\alpha'}.$$

In the uncompactified case, u commutes with $p=\frac{1}{2}(p_L+p_R),\ [u,p]=i$. Here $p_L,\ p_R$ are independent operators, so u should alsp consist of two independent operators, $u=u_L+u_R$, such that

$$[u_L,p_L]=[u_R,p_R]=i.$$

Thus U may be broken into holomorphic (U_L) and antiholomorphic (U_R) pieces as

$$\begin{aligned} U_L(z) &= u_L - i\frac{\alpha'}{2}p_L\ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha_m z^{-m}, \\ U_R(\bar z) &= u_R - i\frac{\alpha'}{2}p_R\ln \bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\tilde\alpha_m \bar z^{-m}. \end{aligned}$$

The operator product expansions are

$$U_L(z)U_L(0)\sim -\frac{\alpha'}{2}\ln z, \qquad U_R(\bar z)U_R(0)\sim -\frac{\alpha'}{2}\ln \bar z, \qquad U_L(z)U_R(0)\sim 0.$$

The vertex operator also splits into a holomorphic and antiholomorphic pieces

$$V_L(z) =: e^{i\left(\frac{n}{r}+\frac{wR}{\alpha'}\right)U_L(z)}:, \qquad V_R(\bar z) =: e^{i\left(\frac{n}{r}-\frac{wR}{\alpha'}\right)U_R(\bar z)}:$$

c.f., the uncompactified case,

$$V_L(z) =: e^{ik\cdot X_L(z)}:, \qquad V_R(\bar z) =: e^{ik\cdot X_R(\bar z)}:$$

The OPE for these vertex operators is

$$\begin{aligned} V_L^k(z)V_L^{k'}(0) &= :e^{ik\cdot X_L(z)}::e^{ik'\cdot X_L(0)}:, \\ &\sim e^{ik\cdot k'\frac{\alpha'}{2}\ln z}:e^{i(k+k')X_L(0)}:, \\ &= z^{\frac{\alpha'}{2}k\cdot k'}:V_L^{k+k'}(0):. \end{aligned}$$

This has a branch cut. If you let z go around the circle $\mathcal{C}$ once, it picks up a factor $e^{\pi i\alpha' k\cdot k'}$ $(z^{\frac{\alpha'}{2}k\cdot k'} = e^{\frac{\alpha'}{2}k\cdot k'\ln z}\to e^{\frac{\alpha'}{2}k\cdot k'(\ln|z|+iArg z)})$. On the other hand,

$z^{-\frac{\alpha'}{2}k\cdot k'}$ picks up a factor $e^{-\pi i\alpha' k\cdot k'}$. The two factors cancel each other, so the OPE of the *full* vertex operator is single valued: $(V = V_L V_R)$

$$V^k(z,\bar z)V^{k'}(0,0) \sim |z|^{\alpha' k\cdot k'/2}V^{k+k'}(0,0)$$

In the compactified case, let $k_L = \frac{n}{r} + \frac{wR}{\alpha'},\ k_R = \frac{n}{r} - \frac{wR}{\alpha'}$. Then, similar to the uncompactified case, we find

$$V^{k_L}(z)V^{k_L'}(0) \sim |z|^{\alpha' k_L\cdot k_L'/2}V_L^{k_L+k_L'}(0) \qquad V_R^k(\bar z)V^{k_R'}(0) \sim |\bar z|^{\alpha' k_R\cdot k_R'/2}V^{k_R+k_R'}(0)$$

As z goes around a circle $\mathcal{C}$, we obtain factors $e^{-\pi i\alpha' k_L\cdot k_L'},\ e^{-\pi i\alpha' k_R\cdot k_R'}$. The total factor for the full vertex is $e^{i\pi\alpha'(k_L\cdot k_L' - k_R\cdot k_R')} = e^{2\pi i(nw'+n'w)} = 1$, so the full vertex is ok.
Subtlety: If instead of going around, consider points $z_1,\ z_2$ and interchange them and let $(k_L,k_R) \leftrightarrow (k_L',k_R')$ i.e., consider the commutator of two vertices. This is equivalent to letting $z \leftrightarrow -z$, which introduces a factor $(-1)^{\frac{\alpha'}{2}k_L\cdot k_L'} = e^{i\pi\frac{\alpha'}{2}k_L\cdot k_L'}$ in the left part and $e^{i\pi\frac{\alpha'}{2}k_R\cdot k_R'}$ in the right part. Overall,

$$e^{i\pi\alpha'(k_L\cdot k_L' - k_R\cdot k_R')} = e^{2\pi i(nw'+n'w)} = \pm 1$$

So if $nw'+n'w$ is odd, the vertices anticommute! To remedy this, we will define the vertex as

$$V^{k_L,k_R} = C_{k_L,k_r}(p) : e^{i(k_L U_l + k_R U_R)} :$$

where C_{k_L,k_R} is known as a *cocyle*. One possible choice is all choices are equivalent.

$$C_{k_L,k_r}(p) = e^{i\pi\frac{\alpha'}{2}(k_L-k_R)r}, \qquad p = /2(p_L+p_R)$$

It satisfies

$$C_k(p)C_{k'}(p) = C_{k+k'}(p), \qquad k = (k_L,k_R).$$

When we commute two vertices, we pick up the factor

$$e^{i\pi\frac{\alpha'}{2}(k_L-k_R)\frac{k'}{2}}e^{-i\pi\frac{\alpha'}{2}(k_L'-k_R')\frac{k}{2}} = e^{\pi i(nw'-n'w)}$$

Thus the overall factor is now

$$e^{\pi i(nw'-n'w)}e^{\pi i(nw'+n'w)} = e^{2\pi i nw'} = 1.$$

Thus the overall factor is one, so the vertices always commute.

6.5 Amplitudes

Recall in the uncompactified case,

$$A = \langle V_1(z_1)V_2(z_2) + ... + V_N(z_N)\rangle \sim 2\pi\delta(k_1+k_2+...+k_N)\prod_{i<j}|z_i - z_j|^{\alpha' k_i\cdot k_j} \tag{6.5.1}$$

where the vertex operators are given by

$$V_i(z_i) =: e^{ik_i\cdot X(z_i)} : .$$

The delta function in (6.5.1) comes from the zero mode, $e^{ik_i\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}$. Explicitly

$$\begin{aligned}\langle e^{ik_1\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}...e^{ik_N\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}\rangle &= \langle e^{ik_1\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}...e^{ik_{N-1}\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}|k_N\rangle \\ &\sim \langle 0|k_1+k_2+...+k_N\rangle \\ &\sim \delta(k_1+k_2+...+k_N)\end{aligned}$$

Each $e^{ik_i\cdot x}$ shifts the state $|k\rangle \to |k+k_i\rangle$. In the compactified case, we get the same holomorphic and antiholomorphic except momenta are not different:

$$\prod_{i<j}|z_i-z_j|^{\alpha' k_i\cdot k_j} \to \prod_{i<j}(z_i-z_j)^{\alpha' k_{Li}k_{Lj}/2}(\bar{z}_i-\bar{z}_j)^{\alpha' k_{Ri}k_{Rj}/2}.$$

The zero modes contribute

$$\delta(k_{L1}+k_{L2}+...+k_{LN})\delta(k_{R1}+k_{R2}+...+k_{RN}) \sim \delta(n_1+n_2+...+n_N)\delta(w_1+w_2+...+w_N).$$

Cocycles give additional $\pm$ signs.

6.6 Spectrum

Recall in the uncompactified case:

$$L_0 = \frac{\alpha' p^2}{4} + N, \qquad N = \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n.$$

In the compactified case, $p \to p_L$ in L_0 and $p \to p_R$ in $\tilde{L}_0$. A string will travel in $(t, x, y, z, ..., u)$ i.e., the 26 dimensional space-time with one-dimension compactified. Then

$$L_0 = \frac{\alpha'(p^2+p_L^2)}{4} + N, \qquad p^2 = p_\mu p^\mu$$

and

$$N = \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha_{n\mu} + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n,$$

and similarly for $\tilde{L}_0$. The Mass-shell condition states

$$(L_0-1)|\text{phys}\rangle = (\tilde{L}_0-1)|\text{phys}\rangle = (L_0-\tilde{L}_0)|\text{phys}\rangle.$$

The Hamiltonian is a constraint. We define the mass by $m^2 = k_\mu k^\mu$, where k^μ is the eigenvalue of p^μ. Then

$$L_0 - 1 = 0 \Rightarrow \frac{\alpha'}{4}(-m^2+k_L^2) + N - 1 = 0 \Rightarrow m^2 = k_L^2 + \frac{4}{\alpha'}(N-1)$$

From the antiholomorphic piece we get

$$m^2 = k_R^2 + \frac{4}{\alpha'}(\tilde{N} - 1)$$

Subtract the two and we find

$$\tilde{N} - N = \frac{\alpha'}{4}(k_L^2 = k_R^2) = nw$$

If we add the two conditions we find

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2) + \frac{k_L^2 + k_R^2}{2} = \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

We now explicitly see the Kaluza-Klein mass term and the winding potential energy.

Massless States

$n = w = 0,\ N = \tilde{N} = 1$, same as in the noncompactified theory. They are:

$$\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0;k\rangle, \qquad \alpha_{-1}^{\mu}\tilde{\alpha}_{-1}|0;k\rangle, \qquad \alpha_{-1}\tilde{\alpha}_{-1}^{\mu}|0;k\rangle, \qquad \alpha_{-1}\alpha_{-1}|0;k\rangle.$$

The states are represented by the (graviton and antisymmetric tensor), vectors, and scalar particles respectively. Recall in Kaluza-Klein field theory, we had $g_{\mu\nu},\ A_\mu,\ G_{uu}$. Our gravition corresponds to $g_{\mu\nu}$, the scalar to G_{uu}, but we have **two** vectors instead of one! Which vector corresponds to A_μ? To answer this question consider three-point amplitudes of two tachyons and a vector. The tachyons have momenta $k_1,\ k_2$ and the vector has momentum k. A tachyon is described by the vertex

$$: e^{ik_L U_l + ik_R U_R + ik\cdot X} :\Leftrightarrow \text{state} \quad |0, k, k_L, k_R\rangle$$

The two vectors are

$$|B\rangle = B_\mu(k)\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}|0;k\rangle, \qquad |C\rangle = C_\mu(k)\tilde{\alpha}_{-1}^{\mu}\alpha_{-1}|0;k\rangle$$

The amplitude for $|B\rangle$ is

$$A \sim B_\mu\langle 0; k_1, k_{1L}, k_{1R}| : e^{ik_{2L}U_L + ik_{2R}U_R + ik_2\cdot X(1)} : \alpha_{-1}^{\mu}\tilde{\alpha}_{-1}|0;k\rangle$$

The relevent parts of $U,\ X^\mu$ are:

$$\begin{aligned} U_L &= u_L - i\frac{\alpha'}{2}p_L \ln z + \ldots \\ U_R &= r_R - i\frac{\alpha'}{2}p_R \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_1\bar{z}^{-1} + \ldots \\ X^\mu &= x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\alpha_1^\mu z^{-1} + \ldots \qquad \text{where } z = 1 \end{aligned}$$

So

$$\begin{aligned} A &\sim B_\mu \frac{\alpha'}{2} \langle 0; k_1+k_2, k_{1L}+k_{2L}, k_{1R}+k_{2R}|\tilde{\alpha}_1 k_{2R} \alpha_1^\nu k_{2\nu} \alpha_{-1}^\mu \tilde{\alpha}_{-1}|0;k\rangle \\ &\sim \alpha' k_2 \cdot B k_{2R} \delta(k_1+k_2+k)\delta(k_{1L}+k_{2L})\delta(k_{1R}+k_{2R}) \\ &\sim \alpha'(k_2-k_3)\cdot B k_{2R}\delta(k_1+k_2+k)\delta(n_1+n_2)\delta(w_1+w_2) \end{aligned}$$

where we used $k \cdot B = 0$ (gauge invariance coming from $Q|B\rangle = 0$ or observe that if $B \propto k$, $|B\rangle$ is a null state. Notice that the two tachyons have *opposite* quantum numbers n, w which makes sense because they annihilate each other. Alternatively, if k_2 is outgoing, the two tachyons have the same quantum numbers (scattering of a single tachyon). The diagram for the other vector C is

$$A' \sim (k_2-k_3)\cdot B k_{2L}\delta(k_1+k_2+k)\delta(n_1+n_2)\delta(w_1+w_2).$$

i.e., $k_{2R} \to k_{2L},\ B_\mu = C_\mu$. Notice that if the sum (corresponding to $|B\rangle + |C\rangle$) is

$$A + A' \sim \alpha'(k_2-k_3)\cdot B\frac{n}{R}\delta(k_1+k_2+k)\delta(n_1+n_2)\delta(w_1+w_2)$$

i.e., the strength of the interaction is proportional to the change. Therefore, the photon is

$$A_\mu(k)(\alpha_{-1}^\mu \tilde{\alpha}_{-1} + \tilde{\alpha}_{-1}^\mu \alpha_{-1})|0;k\rangle.$$

The other vector $A'_\mu(k)(\alpha_{-1}^\mu \tilde{\alpha}_{-1} - \tilde{\alpha}_{-1}^\mu \alpha_{-1})|0;k\rangle$ leads to the amplitude

$$A - A' \sim (k_2-k_3)\cdot B\frac{wR}{\alpha'}\delta(k_1+k_2+k)\delta(n_1+n_2)\delta(w_1+w_2)$$

so it couples to a different charge: the winding number (magnetic?) This is absent in particle theory and is only a string effect.

6.7 $R = \sqrt{\alpha'}$

When $R = \sqrt{\alpha'}$, $m^2 = 0$ implies $\frac{n^2}{\alpha'} + \frac{w^2}{\alpha'} + \frac{2}{a'}(N + \tilde{N} - 2) = 0,\ \tilde{N} - N = nw$. Apart from $n = w = 0,\ N = \tilde{N} = 1$, we have the following possibilities

- $n = w = \pm 1$, in which case we have $N + \tilde{N} = 1,\ \tilde{N} - N = +1$, so $N = 0,\ \tilde{N} = 1$.
- $n = -w = \pm 1$, in which case we have $N + \tilde{N} = 1,\ \tilde{N} - N = -1$, so $N = 1,\ \tilde{N} = 0$.
- $n = \pm 2,\ w = 0$, in which case we have $N + \tilde{N} = 0,\ \tilde{N} - N = 0$, so $N = \tilde{N} = 0$.
- $n = 0,\ w = \pm 2$, in which case we have $N + \tilde{N} = 0,\ \tilde{N} - N = 0$, so $N = \tilde{N} = 0$.

The first and second possibilities are new vectors and they are charged! This is reminiscent of the Weak interactions where we have $W^\pm$ (charged vectors). Together with a mixture of γ and Z^0, they form a triplet, which is a representation of SU(2). Similarly, matter comes in doublets $(e, \nu_e),\ (u, d)$ which also transform under SU(2), which is a gauge symmetry, much like electromagnetism, where the photon and matter transform under U(1). ($W^\pm$ also have mass, but that is only because they ate a Higgs). Recall in E&M (matter represented by a scalar, e.g., tachyon - both fiction)

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \Phi|^2 + m^2 |\Phi|^2 \right],$$

where

$$D_\mu = \partial_\mu + iqA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Gauge invariance: $\Phi \to e^{iq\lambda}\Phi,\ \ A_\mu \to A_\mu - \partial_\mu \lambda$ based on the gauge group U(1).

To extend this to weak interactions, where we have three vectors, $A^1_\mu,\ A^2_\mu,\ A^3_\mu$, we view them as a vector in an abstract space (three-dimensional). Rotations in this space should be independent of the physics. In other words, the action, S, must be invariant under such rotations. We might guess that we should have

$$-\frac{1}{4} \sum_{i=1}^{3} F^i_{\mu\nu} F^{i\mu\nu}$$

(a Weak field for each vector). We also need $\int d^4x \sum_i A^i_\mu J^{i\mu}$, where $J^{i\mu}$ is made of $A^{i\mu}$. These requirements **severly** restrict the form of the action. It turns out that the fields must be defined by

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \epsilon^{ijk} A^j_\mu A^k_\nu$$

and the action is

$$S = -\frac{1}{4} \int d^4x \sum_i F^i_{\mu\nu} F^{i\mu\nu}$$

and leads to nonlinear Maxwell equations given by

$$\partial_\mu F^{i\mu\nu} - \epsilon^{ijk} A^j_\mu F^{k\mu\nu} = 0.$$

Gauge transformations: $A^i_\mu \to A^i_\mu - \partial_\mu \lambda^i - \epsilon^{ijk} A^j_\mu \lambda^k$ (infinitesimal). Finite transformations: Introduce Pauli matrices, σ^i. Define the matrix field

$$A_\mu = \frac{1}{2} A^i_\mu \sigma^i$$

where σ^i represents the Pauli spins matrices which obey the algebra

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

The field strength is

$$F_{\mu\nu} = \frac{1}{2} F^i_{\mu\nu} \sigma^i$$

$$S = e^{\frac{i}{2}\lambda^i \sigma^i} \in SU(2), \qquad SS^\dagger = \mathbb{I}$$

For $\lambda \ll 1$ we may expand S

$$S \simeq 1 + \frac{i}{2}\lambda^j \sigma^j$$

Let us see how $A_\mu = \frac{1}{2}A^i_\mu \sigma^i$ transforms under a finite transformation

$$\begin{aligned} A_\mu &\to S\left(A_\mu - i\partial_\mu\right)S^\dagger \\ &\to (1 + \frac{i}{2}\lambda^i\sigma^i)(A_\mu - i\partial_\mu)(1 - \frac{i}{2}\lambda^j\sigma^j) \\ &\to A_\mu - \frac{1}{2}\sigma^j\partial_\mu\lambda^j + \frac{i}{4}\lambda^i[\sigma^i, \sigma^j]A^j_\mu + O(\lambda^2) \\ &\to A_\mu - \frac{1}{2}\sigma^j\partial_\mu\lambda^j - \frac{1}{2}\epsilon^{ijk}\sigma^i\lambda^j A^k_\mu \\ A^i_\mu &\to A^i_\mu - \partial_\mu\lambda^i - \epsilon^{ijk}\lambda^j A^k_\mu \end{aligned} \tag{6.7.1}$$

For the matter fields, we need to define $D_\mu\Phi$. Φ is a doublet, (u, d) or (e, ν_e), etc. (not quite, because of the spin), so define

$$D_\mu = \partial_\mu + iA_\mu.$$

where A_μ is a matrix. Notice that we have no degree of freedom in introducing a charge q, because

$$\Phi \to S\Phi, \qquad A_\mu \to S(A_\mu - i\partial_\mu)S^\dagger,$$

so

$$\begin{aligned} D_\mu\Phi &\to \partial_\mu(S\Phi) + iA'_\mu S\Phi, \\ &= S(\partial_\mu\Phi + S^\dagger\partial_\mu S\Phi + iS^\dagger A'_\mu S\Phi), \\ &= SD_\mu\Phi. \end{aligned} \tag{6.7.2}$$

This would have worked nicely had we chosen

$$\Phi \to S'\Phi, \qquad S' = e^{\frac{i}{2}q\lambda^i\sigma^i},$$

because $iS'^\dagger A'_\mu S' \neq A_\mu - S^\dagger\partial_\mu S$. This is only true in the *Abelian* case, $S = e^{iq\lambda}$, because all "matrices" commute.
We found two vectors that coupled to charges p_L and p_R (or n and w). The charge operator (which measures the charge of a state) is then the momentum, and there are two of them. The corresponding (conserved) currents generate gauge symmetries, both being $U(1)$, so the gauge group is $U(1) \times U(1)$.

The two currents are ∂U and $\bar{\partial} U$. To see their action, consider a state of charge (n, w) or (k_L, k_r), e.g., $V =: e^{ik_L U + ik_R U}$:. The OPEs are given by

$$\begin{aligned}
\partial U(z) V(0,0) &\sim \partial U_L(z) : e^{ik_L U_L} : \\
&\sim ik_L \partial \left(-\frac{\alpha'}{2} \ln z \right) : e^{ik_L U_L}(0) : \\
&\sim -ik_L \frac{\alpha'}{2} \cdot \frac{1}{z} V(0,0) \\
\bar{\partial} U(\bar{z}) V(0,0) &\sim -ik_R \frac{\alpha'}{2} \cdot \frac{1}{\bar{z}} V(0,0)
\end{aligned}$$

The charges are

$$Q_L = \frac{1}{2\pi i} \oint_C dz \partial U(z) \qquad Q_R = -\frac{1}{2\pi i} \oint_C d\bar{z} \partial U(\bar{z}).$$

so

$$[Q_L, V] = -i \frac{\alpha'}{2} k_L V \qquad [Q_R, V] = -i \frac{\alpha'}{2} k_R V$$

The electric charge is $Q \sim Q_L + Q_R$. These are generators of gauge transformations, indeed

$$\delta V = -\lambda [Q_L, V] = -i \frac{\alpha'}{2} k_L \lambda V$$

so $V \to (1 + i\frac{\alpha'}{2} k_L \lambda) V$ which is the infinitesimal of $V \to e^{i\frac{\alpha'}{2} k_L \lambda} V$, or for the state $|V\rangle = V(0)$, $|V\rangle \to e^{i\frac{\alpha'}{2} k_L \lambda} |V\rangle$ (similarly for Q_R).
At the special radius ($R = \sqrt{\alpha'}$), we have vectors which are charged. They **must** combine with the two photons, just like the $X^{\pm}$ combine with the neutral vector in weak interactions (**must** because we know of no other consistent theory that has charged vectors).
The vectors are

$$① = \tilde{\alpha}^{\mu}_{-1} |0; k; \pm 1, \pm 1\rangle, \qquad ② = \alpha^{\mu}_{-1} |0; k; \pm 1, \pm 1\rangle$$

① corresponds to the vertex : $\tilde{\alpha}^{\mu}_{-1} |0, k, -1, -1\rangle \quad \sim \; : \partial X^{\mu} e^{ik \cdot X} e^{-i \frac{2}{\sqrt{\alpha'}} U_L(z)} :$
(since $n = w$, $k_R = 0$)
② corresponds to the vertex : $\alpha^{\mu}_{-1} |0, k, -1, +1\rangle \quad \sim \; : \partial X^{\mu} e^{ik \cdot X} e^{-i \frac{2}{\sqrt{\alpha'}} U_R(\bar{z})} :$
(since $n = -w$, $k_L = 0$)
Just like with the two photons, they lead to conserved currents.

$$j^{\pm}(z) =: e^{ik_L U_L}(z), \qquad \tilde{j}^{\pm}(\bar{z}) =: e^{ik_R U_R}(\bar{z})$$

where

$$k_L = \frac{n}{R} + \frac{wR}{\alpha'} = \pm \frac{2}{\sqrt{\alpha'}}, \qquad k_R = \pm \frac{2}{\sqrt{\alpha'}}.$$

The OPEs are given by

$$\begin{aligned}
j^+(z)j^+(0) &\sim \; :e^{i\frac{2}{\sqrt{\alpha'}}U_L}(z)::e^{i\frac{2}{\sqrt{\alpha'}}U_L}(0): \\
&\sim e^{\frac{-4}{\alpha'}ln(z)\frac{-\alpha'}{2}} :e^{i\frac{4}{\sqrt{\alpha'}}U_L}(0) \\
&\sim z^2 :e^{i\frac{4}{\sqrt{\alpha'}}U_L}(0): \sim \; 0 \\
j^-(z)j^-(0) &\sim 0
\end{aligned}$$

$$\begin{aligned}
j^+(z)j^-(0) &\sim z^{-2} :e^{i\frac{2}{\sqrt{\alpha'}}U_L}(z)e^{-i\frac{2}{\sqrt{\alpha'}}U_L}(0): \\
&= z^{-2}\left[1 + z\, i\frac{2}{\sqrt{\alpha'}}\partial U_L + z^2(\;\;) + ...\right] \\
&\sim \frac{1}{z^2} + i\frac{2}{\sqrt{\alpha'}}\partial U_L \frac{1}{z}
\end{aligned}$$

Define $j^3 = \frac{i}{\sqrt{\alpha'}}\partial U$ (the photon!). The OPEs may be expressed in terms of the photon.

$$\begin{aligned}
j^+(z)j^-(0) &\sim \frac{1}{z^2} + \frac{2}{z}j^3(z) + ... \\
j^3(z)j^+(0) &\sim -\frac{2}{\alpha'}\left(-\frac{\alpha'}{2}\partial \ln z\right) j^+(0) \sim \frac{1}{z}j^+(0) \\
j^3(z)j^-(0) &\sim -\frac{1}{z}j^-(0)
\end{aligned}$$

Be defining $j^1 = \frac{1}{2}(j^+ + j^-),\; j^2 = \frac{1}{2}(j^+ - j^-)$, we may write all the OPEs in the form

$$j^a(z)j^b(0) \sim \frac{1}{2z^2}\delta_{ab} + i\epsilon^{abc}j^c(0).$$

The corresponding charges

$$Q^a = \oint \frac{dz}{2\pi i} j^a(z)$$

satisfy an SU(2) algebra

$$[Q^a, Q^b] = i\epsilon^{abc}Q^c$$

so the gauge group is $SU(2)\times SU(2)$ (which is enlarged from $U(1)\times U(1)$). You can think of this as an abstract three-dimensional space (in fact, two) in which particles are free to rotate. U(1) is then a subgroup of SU(2) corresponding to rotations around the z-axis. In fact, the symmetry of the theory has an infinite number of generators (much like $T(z)$ generated an infinite number of symmetries through its modes, L_n). Expand

$$j^a(z) = \sum j^a_m z^{-m-1}$$

so that the charges Q^a are the zero modes, $Q^a = j_0^a$. Then the current algebra is an affine Lie algebra (Kac-Moody)

$$[j_\mu^a, j_\nu^b] = \frac{m}{2}\delta_{m+n,0}\delta^{ab} + i\epsilon^{abc} j^c_{m+n}.$$

The constant tells us that j^a is **not** a tensor. It can be shown that a general algebra has $\frac{km}{2}\delta_{m+n,0}\delta^{ab}$ constant term, $k \in \mathbb{N}$ (level), so in our case $k = 1$.

6.8 Away from $R = \sqrt{\alpha'}$

Once we realize there is a symmetric at the special radius $R = \sqrt{\alpha'}$, you can not ignore it when you move away from $R = \sqrt{\alpha'}$. This is because R is dynamical and we have already seen that there is a string mode, $\Phi(k)\tilde{\alpha}_{-1}\alpha_{-1}|0;k\rangle$ which mixes with G_{uu} and changes R, much like the graviton $g_{\mu\nu}(k)(\tilde{\alpha}^\mu_{-1}\alpha^\nu_{-1} + \tilde{\alpha}^\nu_{-1}\alpha^\mu_{-1})|0;k\rangle$ changes the background metric (in the uncompactified case).
So what happens to the SU(2) symmetry as we move away from $R = \sqrt{\alpha'}$?
Recall weak interactions...
A scalar called the Higgs moves from the unstable symmetric point to a stable minimum of the potential (Mexican hat) So $|\Phi|$ goes from 0 to a value $\langle|\Phi|\rangle \sim \nu$. Φ can settle into any minimum and all positions are equivalent. However each position breaks the symmetry. It is similar to the SUN-EARTH system. The underlying physics (Newton's law) is rotationally invariant, but the orbit of the Earth is not (it is an ellipse, even as a circle there is an axis that breaks the symmetry).
Recall the action for a scalar.

$$S = \int d^4x \left(|D_\mu \Phi|^2 + V(\Phi)\right).$$

It contains a term quadratic in the vectors A^i_μ, which gives rise to a mass term after the shift $|\Phi| \to |\Phi| + \nu$.
Back to strings...
Massless scalars $\forall R$: $\eta_{\mu\nu}\tilde{\alpha}^\mu_{-1}\alpha^\nu_{-1}|0;k\rangle$ (dilaton) $\tilde{\alpha}_{-1}\alpha_{-1}|0;k\rangle$ ($G_{uu|}$). The latter changes R and corresponds to the vertex

$$:\partial U(z)\bar{\partial}U(\bar{z})e^{ik\cdot X}:,$$

or

$$:j^3(z)\tilde{j}^3(\bar{z})e^{ik\cdot X}:.$$

At $R = \sqrt{\alpha'}$, we have additional scalars

① : $\bar{\partial}U e^{ik_L U_L} e^{ik\cdot X}$: (c.f. vector $:\bar{\partial}X^\mu e^{ik\cdot X} e^{ik_L U_L}:$) or $:j^\pm \tilde{j}^3 e^{ik\cdot X}:$

② : $\bar{\partial}U e^{ik_R U_R} e^{ik\cdot X}$: (c.f. vector $:\partial X^\mu e^{ik\cdot X} e^{ik_R U_R}:$) or $:j^3 \tilde{j}^\pm e^{ik\cdot X}:$

③ : $n = \pm 2,\ w = 0 \Rightarrow k_L = k_R = \dfrac{n}{R} = \pm\dfrac{2}{\sqrt{\alpha'}}$, vertex : $e^{ik_l U_L} e^{ik_R U_R} e^{k\cdot X} = j^\pm \tilde{j}^\pm e^{ik\cdot X}$

④ : $n = 0,\ w = \pm 2 \Rightarrow k_L = -k_R = \dfrac{wR}{\alpha'} = \pm\dfrac{2}{\sqrt{\alpha'}}$, vertex : $e^{ik_l U_L} e^{ik_R U_R} e^{k\cdot X} = j^\pm \tilde{j}^\pm e^{ik\cdot X}$

Putting everything together, we have

$$: j^a(z)\tilde{j}^b(\bar{z})e^{ik\cdot X} :$$

transforming as a $(\mathbf{3},\mathbf{3})$ of $SU(2)\times SU(2)$. This is the Higgs (**not** a doublet, unlike weak interactions). At the symmetric point, all scalars are massless. Away from the symmetric point, all except $j^3\tilde{j}^3$ get masses

$$m^2 = \left(\frac{R^2-\alpha'}{R\alpha'}\right)^2,$$

using

$$m^2 = \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N+\tilde{N}-2)$$

breaking the $SU(2)\times SU(2)$ symmetry, down to $U(1)\times U(1)$ (with one (two) massless vectors).
Unlike with weak interactions, we can get arbitrarily close to the symmetric point by varying $R\to\sqrt{\alpha'}$. This shows that the potential contains a *flat* direction (it costs nothing to move along this direction) - not a Mexican hat.

Conclusion

There is a underlying gauge symmetry $SU(2)\times SU(2)$ in string theory which is not present in particle theory (KK). Strings see space-time in an unusual way-not yet understood.

6.9 T-duality

Recall

$$m^2 = \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N+\tilde{N}-2).$$

As $R\to\infty$, $n=0$ states becomes infinitely massive and decouple. $w=0$ states go to a continuum of small masses so we get ordinary particle theory at an uncompactified extra dimension. As $R\to 0,\ w=0$ states becomes infinitely massive and decouples. $n=0$ states so to a continuum, so this is similar to the$R\to\infty$ limit, i.e., we still have an *extra* uncompactified dimension, even though it has been shrunk to 0! This behavior generalizes to a symmetry under

$$R\to R' = \frac{\alpha'}{R}.$$

Spectra are identical if we just interchange $(n\leftrightarrow w)$. THM: The theories at R and $R'=\frac{\alpha'}{R}$ are identical.

Proof: Let us start with the R theory. Recall $U = U_L + U_R$. Define $Z = U_L - U_R$. Z has the *same* OPE's as U, because they all come in pairs, so the signs cancel (e.g., $Z_R Z_R = (-U_R)(-U(R))$. However,

$$\begin{aligned} U_L &= u_L - i\frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z \ldots \\ U_R &= u_R - i\frac{\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln \bar{z} \ldots \end{aligned}$$

so

$$Z = u_L - u_R - i\frac{\alpha'}{2}\left[\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z - \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln \bar{z}\right] + \ldots \tag{6.9.1}$$

and the R' theory has

$$Z = u_L + u_R - i\frac{\alpha'}{2}\left[\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z + \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln \bar{z}\right] + \ldots$$

so

$$U = u_L + u_R - i\frac{\alpha'}{2}\left[\left(\frac{w}{R} + \frac{nR}{\alpha'}\right)\ln z - \left(\frac{w}{R} - \frac{nR}{\alpha'}\right)\ln \bar{z}\right] + \ldots$$

which has the same momentum as Z if we interchange $(w \leftrightarrow n)$. QED
The self-dual point is $R = R'$, which is the special point we discussed before. The set of inequivalent theories lies in the internal $[\alpha', \infty)$, so there is a "min" $R = \sqrt{\alpha'}$ from the string point of view.
T-duality, $R \leftrightarrow \frac{\alpha'}{R}$ is a $\mathbb{Z}_2$ symmetry. It is part of $SU(2) \times SU(2)$. Indeed, note that, if $\delta R = R - R_{\min}$, then for small δR, $\delta R' = R' + \sqrt{\alpha'} = \frac{\alpha'}{R} - \sqrt{\alpha'} = \frac{\sqrt{\alpha'}}{R}(\sqrt{\alpha'} - R) \simeq -\delta R$ since $(R \simeq \sqrt{\alpha'})$. $\delta R = j^3 \tilde{j}^3$, so reversing its sign means, e.g., in terms of the first $SU(2)$, a reflection in the 12-plane.
However, to get $\delta R \to -\delta R$, we may also rotate (generated by j^1) around the 1-axis by π. This is an $SU(2)$ transformation. Thus the points $\sqrt{\alpha'} + \delta R$ and $\sqrt{\alpha'} - \delta R$ are *gauge* equivalent.

$$R = \frac{\sqrt{\alpha'}}{k} \cong R = k\sqrt{\alpha'}$$

Recall

$$m^2 = \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2).$$

For $R = k\sqrt{\alpha'}$, we have massless scalars with $N = \tilde{N} = 0$,

$$m^2 = \frac{n^2}{k^2\alpha'} + \left(\frac{wk^2}{\alpha'}\right)^2 - \frac{4}{\alpha'} = 0 \Rightarrow n = \pm 2k,\ w = 0.$$

In the T-dual theory, $R=\sqrt{\alpha'}/k$, these scalars have $n=0,\ w=\pm 2k\ (w\leftrightarrow n)$. Therefore, $k_L=\frac{wR}{\alpha'}=\pm\frac{2}{\alpha'}=-k_R$, and the vertex operators are

$$:e^{ik_LU_L}e^{ik_RU_R}e^{ik\cdot X}:=:e^{\pm i\frac{2}{\alpha'}U_L}e^{\mp i\frac{2}{\alpha'}U_R}e^{ik\cdot X}:=:j^{\pm}\tilde{j}^{\mp}e^{ik\cdot X}:$$

which is part of the set

$$j^a\tilde{j}^b e^{ik\cdot X}:$$

at the symmetric point $R=\sqrt{\alpha'}$! How come? (Total of three: $j^3\tilde{j}^3,\ j^+\tilde{j}^-,\ j^i\tilde{j}^+$) To see the connection you must think of twists. Why? Because you can! Let us start with $k=2$ for simplicity. We wish to compare $r=\sqrt{\alpha'}$ and $R=\sqrt{\alpha'}/2$ (equivalent to $R=2\sqrt{\alpha'}$). Consider the expansions

$$\begin{aligned} U_L &= u_L-i\frac{\alpha'}{2}\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)\ln z+i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha_m z^{-m},\\ U_R &= u_R-i\frac{\alpha'}{2}\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)\ln \bar{z}+i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\tilde{\alpha}_m \bar{z}^{-m}.\end{aligned}$$

For

$$\begin{aligned} R=\sqrt{\alpha'} &\quad:\quad U_L=u_L-i\sqrt{\frac{\alpha'}{2}}(n+w)\ln z+\ldots\\ R=\frac{\sqrt{\alpha'}}{2} &\quad:\quad U_L=u_L-i\sqrt{\frac{\alpha'}{2}}(n+\frac{w}{2})\ln z+\ldots\end{aligned}$$

To mimic $R=\sqrt{\alpha'}/2$ at $R=\sqrt{\alpha'}$, we need to (a) $n\to 2n$, (b) $w\to w/2$. (a) is easier so let us try it first. We need to restrict n to even numbers. This is a restriction on the Hilbert space.
c.f. Harmonic Oscillator: $H=p^2/2m+1/2\ m\omega^2x^2$. H has eigenvalues $(n+1/2)\hbar\omega$. We can restrict wavefunctions to evenfunctions. This is consistent, because H is even (H commutes with the parity operator). Then H has the eigenvalues $(2n+1/2)\hbar\omega$. Given $\Psi(x)$, we can construct the even function

$$\psi_{\text{even}}=\frac{1}{2}(1+P)\psi(x)=\frac{1}{2}(\psi(x)+\psi(-x)),$$

where $1/2(1+P)$ is a projection operator.
For strings, the restriction on the Hilbert space (even n) is consistent. Indeen, consider two even-n states,

$$V=:e^{ik_LU_L}e^{ik_RU_R}:,\qquad V'=:e^{ik'_LU_L}e^{ik'_RU_R}:$$

The OPE gives

$$V(z,\bar{z})V(z',\bar{z}')\sim z^{\frac{\alpha'}{2}k_Lk'_L}e^{\frac{\alpha'}{2}k_Rk'_R}V_{kL+kL',k_R+k'_R}.$$

The operator similar to parity is $(-1)^n$ where $n\sim$ charge $\sim$ momentum, so it acts just like parity, $U\to -U$.

Having completed (a), we turn to (b). This is harder. We need to allow half-integer winding numbers. How can a closed string wind half-way? **Answer**: Fold the circle $U \equiv U + 2\pi R$ by identifying point $U \equiv -U$. There are two fixed points: 0 and $\pi R = -\pi R$. This creates a singular "manifold" known as an *orbifold*. Now the string can wind half-way, say from $-\pi R/2$ to $\pi R/2$ (these two points are identified, so the string is closed).
In general, the ends of the string can be at opposite points, i.e., $U(\sigma + 2\pi) = -U(\sigma)$. We are allowed to impose anti-periodic boundary conditions! Does the theory make sense? There is no a priori guarantee that it will, but alas, it does (also note, $p = 0$, string cannot move away from the fixed point, so $n = 0$). Of course, we need to restrict the Hilbert space again to "even parity" states. Again, this is a consistent truncation and the resulting theory is **identical** to the theory on a $R = \sqrt{\alpha'}/2$ circle.
The truncated theory is called "twisted". Now let us compare the massless scalars. In the original $R = \sqrt{\alpha'}$ theory we had 9 (3×3) scalars, $: j^a \tilde{j}^b e^{i\kappa \cdot X} :$.
These are indeed the massless scalars at $R = \sqrt{\alpha'}/2$. The above generalizes to $\forall k \in \mathbb{N}$.

Open Strings

Just like closed strings, open strings have a quantized momentum in the compact dimensions, $p = \frac{n}{R}$. However, there is no *winding* for open strings, so $w = 0$ (just like KK particles). The mass formula is

$$m^2 = \frac{n^2}{R^2} + \frac{1}{\alpha'}(N-1).$$

As $R \to 0$, $n \neq 0$ states become infinitely massive and decouple. Thus the compact dimensions disappears. That would be considered normal behavior, were it not for the fact that open strings cannot help but create and interact with closed strings. The latter exhibit weird behavior (the compact dimension does **not** disappear as $R \to 0$). So how does one reconcile the two pictures? It is easier to think in terms of the $R \to \infty$ limit and we have already seen that this is possible with closed strings because of T-duality. Recall that the theory at $R' = \alpha'/R$ is equivalent to the theory at R if written in terms of $Z = U_L - U_R$ instead of $U_L + U_R$.
Recall the expansion

$$U(z,\bar{z}) = u - i\alpha' p \ln |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{1}{m} \alpha_m (z^{-m} + \bar{z}^{-m}).$$

For compact U, $p = n/R$.

$$z = e^{\pi i(\sigma+\tau)/\ell}, \qquad \bar{z} = e^{-\pi i(\sigma-\tau)/\ell}$$

We will set $\ell = \pi$ for simplicity. So

$$\begin{aligned} \partial_\sigma U_L = \partial_\sigma z \partial U = iz\partial U_L, \quad & \partial_\tau U_L = iz\partial U_L, \\ \partial_\sigma U_R = \partial_\sigma \bar{z} \bar{\partial} U_R = -i\bar{z}\bar{\partial} U_R, \quad & \partial_\tau U_R = i\bar{z}\bar{\partial} U_R, \end{aligned}$$

so

$$\partial_\sigma Z = iz\partial U_+ i\bar{z}\bar{\partial}U, \quad \partial_\tau U = iz\partial U + i\bar{z}\bar{\partial}U = \partial_\sigma Z$$

so

$$\begin{aligned} Z(\sigma=\pi) - Z(\sigma=0) &= \int_0^\pi d\sigma\partial_\sigma Z = \int_0^\pi d\sigma\partial_\tau U = \int_0^\pi d\sigma\partial_\tau(2\alpha' p\tau) \\ &= 2\alpha' p\pi = 2\alpha'\pi\frac{n}{R} = 2\pi n R' \end{aligned}$$

In other words, the ends of the string lie at the same point in the compact dimension (in terms of the dual coordinate). Including the noncompact dimensions, this implies that end-points lie on a hyperplane (D-brane).
Notice that translation invariance is broken, or equivalently, the momentum in the compact dimension is not conserved. Since $p = \frac{n}{R}$ and in the dula theory $n \leftrightarrow w$, this is equivalent to the non-conservation of winding number in the small R theory. That is obvious. A wound closed string can break into two open strings.
Consider massless modes. These are as in the uncompactified case, i.e., the photon: $\alpha_{-1}^\mu|0;k\rangle$. We need to split it into uncompactified $\alpha_{-1}^\mu|0;k\rangle$ and compactified, $\alpha_{-1}|0;k\rangle$ components. Corresponding vertices

$$:\partial_\tau X^\mu e^{ik\cdot X} e^{inU/R} : (\sigma=0) \qquad :\partial_\tau U e^{ik\cdot X} e^{inU/R}:$$

but $n=0$ for the massless and $k^2=0$. The former is a photon *tangent* to the D-brane. The letter can be written as

$$:\partial_\sigma Z e^{ik\cdot X}: .$$

Just like the gravition $g_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ik\cdot X}$: contributes to the background and curves it, the vertex $A\partial_\sigma Z e^{ik\cdot X}$ shifts the position of the D-brane $z \to z + A\partial_\sigma Z$ ($\partial_\sigma z$ is perpendicular to the D-brane). Therefore, the D-brane is a dynamical object. Its fluctuations are described by open strings attached to it.
The D-brane is *our* Universe! Notice that the photon (and other particles) are confined to the D-brane. No wonder we never wander off into the extra dimension(s). On the other hand, gravity has to be present in the extra dimension, because gravity *creates* space.
This D-brane fills space. We can imagine more compact dimensions and have p non-compact dimensions. Then we have a Dp-brane.

Scattering

Let us compare open and closed strings. We will need to mix them in order to describe scattering by D-branes. Recall the operator product expansions for closed strings

$$X(z,\bar{z})X(0,0) \sim -\frac{\alpha'}{2}\ln|z|^2 + \dots$$

We also have

$$\partial\bar{\partial}(X(z,\bar{z})X(0,0)) \sim -\pi\alpha'\delta^2(z,\bar{z})$$

so $G(z,\bar{z}) = X(z,\bar{z})X(0,0)$ is a Green function. $\ln|z|^2$ satisfies the boundary condition of periodicity in $\sigma : z = e^{i(\sigma+\tau)}$, so $\sigma \to \sigma + 2\pi \Rightarrow z \to z$.
It is the electrostatic potential of the uniformly charged straight line. Open strings, $z = e^{i(\sigma+\tau)}$, but not $0 \leq \sigma \leq \pi$, so z is on the upper-half plane. The boundary is the real axis and that is where the vertex operators are. The Green function (and the OPE) is found in two steps. First we need to satisfy Neumann boundary conditions, $\partial_\sigma X = 0$.
This translates into $\partial_n X = 0$ (normal to boundary vanishes), which can be satisfied by adding an image charge at $\bar{z}$. This is different from electrostatics, where the tanget needs to vanish, requiring an *opposite* charge for the image. Thus,

$$G(z,\bar{z};z',\bar{z}') = -\frac{\alpha'}{2}\ln|z-z'|^2 - \frac{\alpha'}{2}\ln|\bar{z}-\bar{z}'|^2 \tag{6.9.2}$$

When both z and z' approach the boundary (i.e., they become real), we obtain

$$G(z,\bar{z};z'\bar{z}') = -\alpha'\ln|z-z'|^2 = -2\alpha'\ln|z-z'|.$$

This shows that the OPE for open strings ought to be

$$X(z)X(0) \sim -2\alpha'\ln|z|.$$

Recall the amplitudes:
Closed strings:

$$A_n = \langle : e^{ik_1\cdot X}(z_1,\bar{z}_1)e^{ik_2\cdot X}(z_2,\bar{z}_2)...e^{ik_n\cdot X}(z_n,\bar{z}_n) :\rangle.$$

View this as a function of z_1 and differentiate. We obtain

$$\partial_{z_1}A_n = \langle : ik_1\partial X e^{ik_1\cdot X}(z_1,\bar{z}_1)e^{ik_2\cdot X}(z_2,\bar{z}_2)...e^{ik_n\cdot X}(z_n,\bar{z}_n) :\rangle.$$

By making use of the OPE $\partial X(z) : e^{ik\cdot X}(0) :\sim -ik\frac{\alpha'}{2}\frac{1}{z}e^{ik\cdot X}(0)$ we obtain

$$\partial_{z_1}A_n = \frac{\alpha'}{2}A_n\sum_i\frac{k_1k_i}{z_1-z_i}.$$

Integrating ...

$$A_n \propto \prod_{i<j}|z_i-z_j|^{\alpha' k_i k_j}$$

which is the holomorphic and antiholomorphic pieces multiplied together.
For open strings, a similar argument yields $A_n \propto \prod_{i<j}|z_i-z_j|^{2\alpha' k_i k_j}$. For closed string emission from a D-brane, we need to use (6.9.2). (See (6.2.33) Polchinski)