String Theory I

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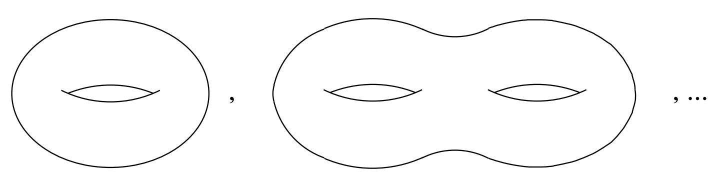
UNIT 5

Loop Amplitudes

5.1 One-loop Amplitudes

For closed string amplitudes, we consider a sphere which has six isometries (parametrized by three complex numbers $a, b, c \sim SL(2, \mathbb{C})$), so we had to put four punctures to get a modulus (conformally inequivalent surfaces).

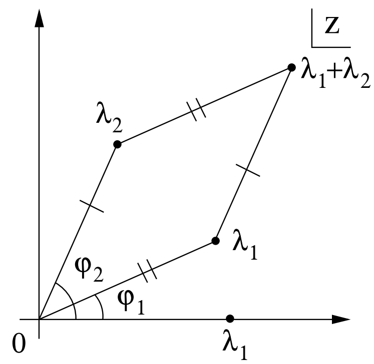
For loop amplitudes (containing *virtual* strings - quantum mechanical corrections, only present due to Heisenberg's uncertainty principle) we need to consider higher-genus Riemann surfaces (fortunately they have all been classified).



We will start with the torus. Unlike the sphere, there exist tori that are conformally inequivalent, without any punctures. So we need to study the torus by itself first. What does it represent? A virtual string that lives in the *vacuum*. Or nothing creating a pair of strings which then annihilate to produce nothing again.

Are we about to study nothing? You bet! There is energy associated with *nothing* and this energy is observable if gravity is present! It is the **Cosmological Constant**.

First let us study the geometry



$$z \approx z + \lambda_1 \approx z + \lambda_2 \therefore z \approx z + n\lambda_1 + m\lambda_2.$$

Different choices of λ_1, λ_2 lead to conformally inequivalent surfaces.
Example: $z \to \zeta z$ (rescaling) $\Rightarrow\ \lambda_1 \to \frac{1}{\zeta}\lambda_1,\ \lambda_2 \to \frac{1}{\zeta}\lambda_2$ However $\tau = \frac{\lambda_1}{\lambda_2}$ =invariant!
Now change λ_1, λ_2 thusly:

$$\begin{pmatrix} \lambda_2' \\ \lambda_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}, \quad a, b, c, d, \in \mathbb{Z}, \quad ad - bc = 1$$

Then

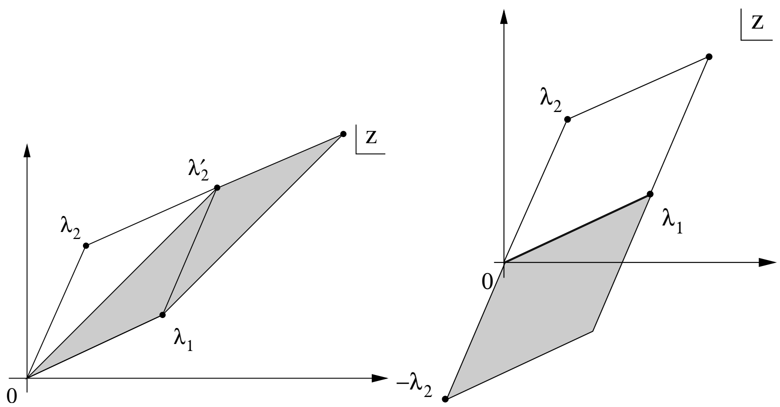
$$\begin{aligned} z &\approx z + n'\lambda_1' + m'\lambda_2' \\ &\approx z + (n'\ m') \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \\ &\approx z + n\lambda_2 + m\lambda_1 \end{aligned}$$

where

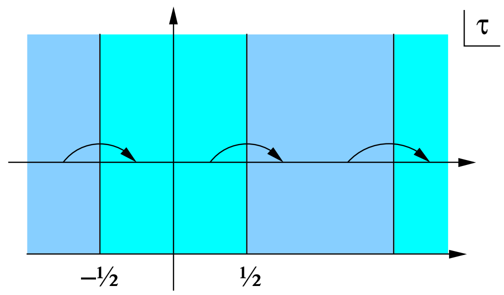
$$\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n' \\ m' \end{pmatrix}$$

Therefore it is the same torus. But $\lambda_1', \lambda_2' \to \tau' = \frac{\lambda_2'}{\lambda_1'} = \frac{a\tau+b}{c\tau+d}$.
Conclusion: $\tau,\ tau'$ define the **same** torus. τ up to SL(2,$\mathbb{Z}$) transformations uniquely labels conformally inequivalent tori. Therefore τ is a *modulus.*
We need to integrate over τ. Find the integration region. SL(2,$\mathbb{Z}$) is generated by two transformations ($S,\ T$).

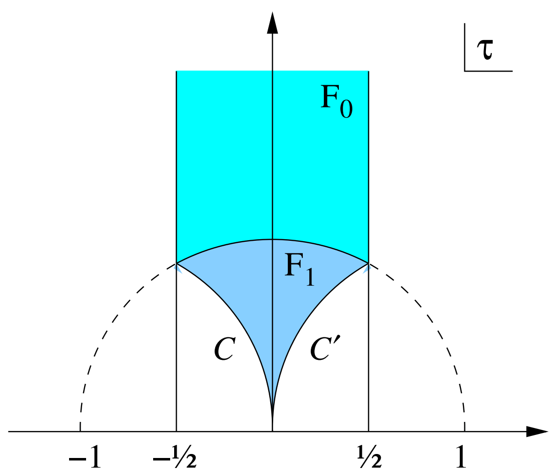
$$T:\ \tau' = \tau + 1, \quad S:\ \tau' = \frac{1}{\tau}$$



In the τ-plane, T divides it into inequivalent regions (mapping one region into the adjacent region). So concentrate on $-\frac{1}{2} \leq Re(\tau) \leq \frac{1}{2}$.

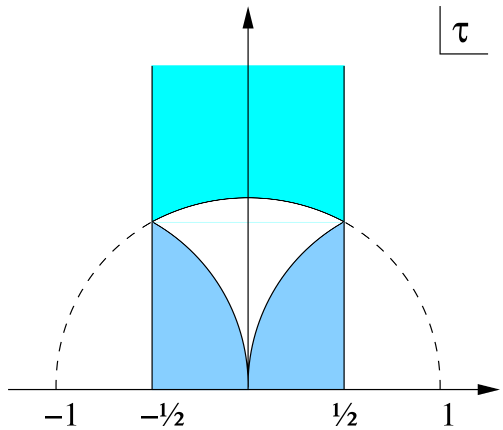


The $Im\tau$ axis, $\tau = it$, acting with S: $it' = -\frac{1}{it}$, so $t' = \frac{1}{t}$ mapping $(1, \infty) \leftrightarrow (0, 1)$. the point i is fixed and so the entire arc $\tau = e^{i\theta}$.



The region above the unit circle and in between $|Re(\tau)| = \frac{1}{2}$ is "irreducible", called the fundamental region,F_0.

S maps $\tau = \frac{1}{2} + it$ onto $\tau' = -\frac{1}{\frac{1}{2}+it} = \frac{-\frac{1}{2}+it}{\frac{1}{4}+t^2}$, $\quad \infty \to 0, \quad t = \frac{\sqrt{3}}{2} \to -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
So $F_0 \to F_1$.



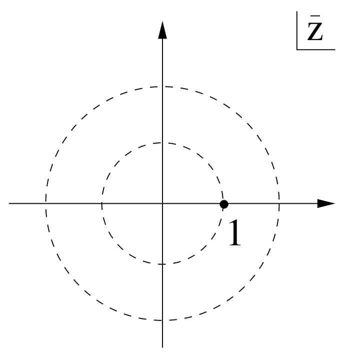
All Fundamental regions are equivalent. Integration should be over **one** fundamental region (does not matter which one).

5.2 String on a Torus

Set $\lambda_1 = 2\pi$ (without l.o.g.). If $\tau = it$, then

INSERT FIGURE HERE

This is a cylinder (closed string propogating for time t and then coming back to where it started from). In general, $\tau = \tau_1 + i\tau_2$, to $t = \tau_2$. τ_1 represents an angle $(0 \leq \tau_1 \leq 2\pi)$. The string is twisted by τ_1 before it gets identified with the initial string. This is the "cylinder" picture. We can map it onto the "sphere" picture as before, $\tilde{z} = e^{-iz}$.



$$z \approx z + 2\pi, \quad trivial$$

$$z \approx z + 2\pi\tau$$

$\tilde{z} \approx \tilde{z}e^{-2\pi i\tau}$, $|\tilde{z}| \approx |\tilde{z}|e^{2\pi\tau_2}$: unit circle $\approx$ circle of radius $e^{2\pi\tau_2}$ with twist τ_1.

Quantum Mechanics

Start with a state $|n\rangle$, evolve for "time" $2\pi\tau$, $|n\rangle \to |n(\tau)\rangle = e^{iH(2\pi\tau)}|n\rangle$. For the left-movers, $H = L_0 - \frac{D}{24}$.
Transition amplitude: $A_n = \langle n|n(\tau)\rangle = \langle n|e^{i(L_0-D/24)(2\pi\tau)}|n\rangle$ Define $Z(\tau) = \sum_n A_n(\tau) = \sum_n \langle n|e^{i(L_0-D/24)(2\pi\tau)}|n\rangle$. If n is an eigenfunction of the "Hamiltonian", $(L_0 - D/24)|n\rangle = E_n|n\rangle$ then $Z(\tau) = \sum e^{2\pi\tau i E_n} = \mathrm{Tr}\,(e^{2\pi\tau i(L_0-D/24)})$.
We need to include the right-movers. Define

$$Z(\tau) = \mathrm{Tr}\,(e^{2\pi i\tau(L_0-D/24)}e^{-2\pi i\bar{\tau}(\tilde{L}_0-D/24)}).$$

If $q = e^{2\pi i\tau}$, then

$$Z(\tau) = \mathrm{Tr}\,(q^{(L_0-D/24)}\bar{q}^{(\tilde{L}_0-D/24)})$$

Let us calculate it. Recall ...

$$L_0 + \tilde{L}_0 = \frac{\alpha'}{2}p^2 + N + \tilde{N}, \quad L_0 - \tilde{L}_0 = N - \tilde{N}$$

Therefore

$$Z(\tau) = (q\bar{q})^{-D/24}\mathrm{Tr}\,(e^{-2\pi\tau_2(\alpha'/2p^2)}q^N\bar{q}^{\tilde{N}})$$

where $N = \sum_n \alpha^\mu_{-n}\alpha_{\mu n}$, $\tilde{N} = \sum_n \tilde{\alpha}^\mu_{-n}\tilde{a}_{\mu n}$. For each n and μ, $\alpha^\mu_{-n}\alpha_{\mu n}$ has eigenvalues $nN_{\mu n} \in \mathbb{N}_0$, where $N_{\mu n}$ is the occupation number.
Therefore

$$\mathrm{Tr}\,q^N\bar{q}^{\tilde{N}} = \prod_{\mu n}\sum_{N,\tilde{N}}(q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}}) = \prod_{\mu,n}\left(\sum_{N=0}^{\infty} q^{nN_{\mu n}}\right)\left(\sum_{\tilde{N}=0}^{\infty}\bar{q}^{n\tilde{N}_{\mu n}}\right)$$

Each sum is a geometric series, therefore

$$\mathrm{Tr}\,q^N\bar{q}^{\tilde{N}} = \prod_{n=1}^{\infty}(1-q^n)^{-2D}.$$

To calculate $\mathrm{Tr}\,e^{-2\pi\tau_2(\alpha'/2p^2)}$, make the space finite and Euclidean. Then p^μ has discrete eigenvalues (n/L), where L is the size of the box.

$$\sum_n f(n/L) = L\sum_{\frac{1}{L}} f(2\pi n/L) = L\int \frac{dp_x}{2\pi}f(p_x).$$

Repeat for other dimensions and we get

$$\mathrm{Tr}\ \to L^D\int\frac{d^Dp}{(2\pi)^D}f(p)$$

The t-component $E = p_0\tilde{\psi}ip_0$ to make the integral finite, so

$$\mathrm{Tr}\,e^{-2\pi\tau_2(\alpha'p^2/2)} = iL^D\int\frac{d^Dp}{(2\pi)^D}e^{-2\pi\tau_2(\alpha'p^2/2)}$$

$$
\begin{aligned}
&= iL^D \left(\int \frac{dp}{2\pi} e^{-2\pi\tau_2(\alpha' p^2/2)} \right)^2 \\
&= iL^D \left(2\pi\sqrt{\alpha'\tau_2} \right)^{-D}
\end{aligned}
$$

Putting everything together,

$$
Z(\tau) = iL^D \left(2\pi\sqrt{\alpha'\tau_2} |q^{1/24} \prod_{n=1}^{\infty} (1-q^n)|^2 \right)^{-D}.
$$

We now need to check the modular invariance. $\tau \to \tau + 1 \Rightarrow q \to q$, obvious invariance! $\tau \to -1/\tau$ is not at all obvious! In order to show the invariance we may use the powerful machinery developed by Jacobi centuries ago, known as the Jacobi-Theta functions, Θ.

Θ-functions

These are functions with nice modular properties.
Definition:

$$
\vartheta(\nu,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n\nu} = \sum_{n=-\infty}^{\infty} q^{n^2/2} z^n, \;\; q = e^{2\pi i\tau}, \; z^{2\pi i\nu}
$$

The Jacobi-Theta functions satisfy the heat equation

$$
\frac{i}{\pi}\frac{\partial^2\vartheta}{\partial\nu^2} + 4\frac{\partial\vartheta}{\partial\tau} = 0.
$$

Periodicity properties:

$$
\begin{aligned}
\vartheta(\nu+1,\tau) &= \vartheta(\nu,\tau) \\
\vartheta(\nu+\tau,\tau) &= e^{-\pi i\tau - 2\pi i\nu}\vartheta(\nu,\tau)
\end{aligned} \tag{5.2.1}
$$

These **uniquely** define ϑ up to a multiplicative constant.
Check: $\nu \to \nu+1 \Rightarrow z \to z \therefore \vartheta(\nu+1,\tau) = \vartheta(\nu,\tau)$. $\nu \to \nu+\tau \Rightarrow z \to qz$

$$
\begin{aligned}
\vartheta(\nu+\tau,\tau) &= \sum q^{n^2/2} z^n q^n = q^{-1/4} \sum q^{(n+1)^2/2} z^n \\
&= q^{-1/4} z^{-1} \sum q^{(n+1)^2/2} z^{n+1} \\
&= q^{-1/4} z^{-1} \vartheta(\nu,\tau) \\
&= e^{-\pi i\tau - 2\pi i\nu}\vartheta(\nu,\tau)
\end{aligned}
$$

Equations (5.2.1) admit a different form of solution (which must be the same by uniqueness).

$$
\vartheta(\nu,\tau) = \prod_{m=1}^{\infty} (1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2})
$$

Check: $\nu \to \nu + 1$ is trivial to check $\nu \to \nu + \tau$: $z \to qz$ therefore

$$\prod \to \prod \times \frac{1 + z^{-1}q^{-1/2}}{1 + zq^{1/2}} = (\prod) \times z^{-1}q^{-1/2}$$

as before. The normalization constants also match. Check the limit $q \to 0$. ϑ is usually called ϑ_3. A general ϑ function is given by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i(n+a)^2\tau + 2\pi i(n+a)(\nu+b)} = e^{\pi i a^2\tau + 2\pi i a(\nu+b)}\vartheta_3(\nu + a\tau + b, \tau)$$

where

$$\vartheta_3(\nu, \tau) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\nu, \tau).$$

Interesting ϑ:

$$\begin{aligned} \vartheta_1 &= -\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\nu, \tau) = i \sum_{n=-\infty}^{\infty} (-)^n q^{(n-1/2)^2/2} z^{n-1/2} \\ &= 2e^{i\pi\tau/4} \sin \pi\nu \prod_{m=1}^{\infty} (1 - q^m)(1 - zq^m)(1 - z^{-1}q^m) \end{aligned}$$

The product forms are useful to find zeros of ϑ_3.

$$z = -q^{\pm(m-1/2)} \Rightarrow e^{2\pi i\nu} = e^{\pi i\tau(2m-1)+\pi i}$$

Therefore

$$\begin{aligned} \nu &= \frac{1}{2}(\tau + 1),\ \ \nu + 1,\ \nu + 2, ... \\ &= n_1 + n_2\tau,\ \ \nu + \tau,\ \nu + 2\tau, ... \end{aligned}$$

Modular Transformations

$T:\ \tau \to \tau + 1$:

$$\begin{aligned} \vartheta_3(\nu, \tau) &= \sum e^{\pi i n^2\tau + \pi i n^2 + 2\pi i n\nu} \\ &= \sum e^{\pi i n^2\tau + 2\pi i n(\nu+1/2) + \pi i n^2 - \pi i n} \\ &= e^{\pi i n^2\tau + 2\pi i n(\nu+1/2)}, \quad e^{\pi i n(n-1)} = e^{2\pi i k} = 1 \\ &= \vartheta_3(\nu + 1/2, \tau) \end{aligned}$$

$S:\ \tau \to -\frac{1}{\tau}$: This transformation is more difficult. First, convert the sum to an integral:

$$\vartheta_3(\nu, \tau) = \sum q^{n^2/2} z^n = \int_{-\infty}^{\infty} dx q^{x^2/2} z^x \sum_n \delta(x - n)$$

and

$$\sum_n \delta(x-n) = \sum_{m=-\infty}^{\infty} e^{2\pi i x m}.$$

Proof: If $x \in \mathbb{Z}$, then clearly $\sum \to \infty$. If $x \notin \mathbb{Z}$, then

$$\begin{aligned}\sum_{m=-\infty}^{\infty} e^{2\pi i x m} &= \frac{1}{1-e^{2\pi i x}} + \frac{1}{1-e^{-2\pi i x}} - 1 \\ &= 2Re(1-e^{2\pi i x})^{-1} - 1 = 2Re\frac{e^{-\pi i x}}{-2i\sin(\pi x)} - 1 = 0\end{aligned}$$

Also,

$$\int_{-1/2}^{1/2} \sum_{m=-\infty}^{\infty} e^{2\pi i x m} = \sum_{m\neq 0} \frac{1}{2\pi i m} e^{2\pi i x}\Bigg|_{-1/2}^{1/2} + 1$$

If $m \neq 0$, then $e^{2\pi i m x}\big|_{-1/2}^{1/2} \sim \sin \pi m = 0$, therefore $\int_{-1/2}^{1/2} = 1$. By periodicity, $\sum e^{2\pi i x m} = \sum_n \delta(x-n)$. Therefore

$$\begin{aligned}\vartheta_3(\nu,\tau) &= \sum_m \int_{-\infty}^{\infty} dx q^{x^2/2} z^x e^{2\pi i x m} \\ &= \sum_m \int_{-\infty}^{\infty} dx e^{\pi i \tau x^2} e^{2\pi i \nu x} e^{2\pi i m x} \\ &= \sum_m \int_{-\infty}^{\infty} dx e^{\pi i \tau (x+(\nu+m)/\tau)^2 - \pi i (\nu+m)^2/\tau} \\ &= \frac{1}{\sqrt{-i\tau}} e^{-i\pi\nu^2/\tau} \sum_m e^{-\pi i m^2/\tau - 2\pi i \nu m/\tau} \\ &= \frac{1}{\sqrt{-i\tau}} e^{-\pi i \nu^2/\tau} \vartheta_3(\nu/\tau| - 1/\tau)\end{aligned}$$

Therefore

$$\vartheta_3\left(\frac{\nu}{\tau}| - \frac{1}{\tau}\right) = \sqrt{-i\tau} e^{\pi i \nu^2/\tau} \vartheta_3(\nu|\tau).$$

Similarly for ϑ_1, we obtain

$$\begin{aligned}\vartheta_1(\nu|\tau+1) &= e^{\pi i/4}\vartheta_1(\nu|\tau) \\ \vartheta_1\left(\frac{\nu}{\tau}| - \frac{1}{\tau}\right) &= -\sqrt{-i\tau} e^{\pi i \nu^2/\tau}\vartheta_1(\nu|\tau)\end{aligned}$$

ϑ_1 can be related to the partition function $Z(\tau)$ as follows. Recall

$$\vartheta_1(\nu|\tau) = 2e^{\pi i \tau/4} \sin \pi\nu \prod_{m=1}^{\infty} (1-q^m)(1-zq^m)(1-z^{-1}q^m).$$

To get $Z(\tau)$, we can not just set $z=1$, because then $\sin\pi\nu=0$, so $\vartheta_1=0$. We need to differentiate with respect to ν first.

$$\partial_\nu\vartheta_1(\nu|\tau) = 2\pi e^{\pi i\tau/4}\left(\cos\pi\nu\prod(...)+\sin\pi\nu\partial_\nu\prod(...)\right)$$

Now set $\nu=0$ and the second term vanishes. Therefore

$$\partial_\nu\vartheta_1(\nu|\tau) = 2\pi e^{\pi i\tau/4}\left[\prod_{m=1}^{\infty}(1-q^m)\right]^3 = 2\pi\left[q^{1/24}\prod_{m=1}^{\infty}(1-q^m)\right]^3.$$

Notice the appearance of $1/24$ in the exponent! It is needed for the modular invariance! Therefore

$$\eta(\tau) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m) = \left(\frac{\partial_\nu\vartheta_1(\nu|\tau)}{2\pi}\right)^3$$

where $\eta(\tau)$ is the Dedekind η-function.

Modular Properties of the Dedekind function

$\tau\to\tau+1$

$$\eta(\tau+1) = e^{\pi 1/12}\eta(\tau)$$

$$\begin{aligned}\frac{1}{\tau}\partial_\nu\vartheta_1\left(0|-\frac{1}{\tau}\right) &= -\sqrt{-i\tau}\partial_\nu\vartheta_1(0|\tau)\\ \partial_\nu\vartheta_1\left(0|-\frac{1}{\tau}\right) &= (-i\tau)^{3/2}\partial_\nu\vartheta_1(0|\tau)\end{aligned}$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)$$

So we see that ϑ and η are not invariant under modular transformations. We really only care about the partition function, which depends on η. Let us check how the partition function transforms under the modular transformations from knowing how η transforms. The partition function is

$$Z(\tau) = iL^D\left(\frac{1}{2\pi\sqrt{\alpha'\tau_2}}|\eta(\tau)|^{-2}\right)^D$$

Under $\tau\to\tau+1$, τ_2 does not change, nor does $|\eta(\tau)|$. Under $\tau\to-\frac{1}{\tau}$, $\tau_2\to\frac{\tau_2}{|\tau|^2}$, so $\frac{1}{\sqrt{\tau_2}}\to\frac{|\tau|}{\sqrt{\tau_2}}$ and $|\eta(\tau)|^{-2}\to\frac{1}{|\tau|}|\eta(\tau)|^{-2}$
Therefore $\frac{1}{\sqrt{\tau_2}}|\eta(\tau)|^{-2}$ is modular invariant and so is $Z(\tau)$.

5.3 The bc system

Recall $L_0 = \sum n : b_{-n} c_n : -1$. This is in the "sphere" picture. To go back to the cylinder picture, use

$$L_0 \to L_0 + \frac{c}{12}$$

where $c = 13$. Therefore

$$L_0 = \sum n : b_{-n} c_n : + \frac{1}{12}$$

which is the generator of the t-translations for left-movers.

$$Z_{bc} = q^{1/2} \bar{q}^{1/2} \mathrm{Tr}\ q^{L_0} \bar{q}^{\tilde{L}_0}$$

To calculate $\mathrm{Tr}\ q^{L_0}$, start with $b_{-1}c_1 + c_{-1}b_1$. $b_{-1}c_1$ which has eigenvectors

$$|0\rangle = |\psi\rangle, \quad |1\rangle = b_{-1}|\psi\rangle$$

with respective eigenvalues $0, 1$.
We can show $(b_{-1}c_1)^2$ is a projection operator by using the bc algebra,

$$\{b_{-m}, c_n\} = \delta_{0,m+n}, \ \ \{b_m, b_n\} = \{c_m, c_n\} = 0.$$

$$(b_{-1}c_1)^2 = b_{-1}c_1 b_{-1}c_1 = b_{-1}(1 - b_{-1}c_1)c_1 = b_{-1}c_1.$$

Therefore, the eigenvalues $0, 1$ are the only eigenvalues. Now

$$\mathrm{Tr}\ |\psi\rangle\langle\psi| = \langle\chi|\psi\rangle = \langle\psi|c_0|\psi\rangle$$

is the only sensible definition. Define $\langle 0| = \langle\psi|$, $\langle 1| = \langle\psi|c_1$.

$$\mathrm{Tr}\ |1\rangle\langle 1| = \langle 1|1\rangle = \langle\psi|c_1 c_1 b_{-1}|\psi\rangle = -1.$$

The eigenvectors of $c_{-1}b_1$ are $|\psi\rangle$ and $c_{-1}|\psi\rangle$ so overall, the $b_{-1}c_1 + c_{-1}b_1$ eigenvalues of $|\psi\rangle, \ b_{-1}|\psi\rangle, \ c_{-1}|\psi\rangle, \ c_{-1}b_{-1}|\psi\rangle$ are $0, 1, 1, 2$ respectively. Therefore

$$\mathrm{Tr}\ q^{b_{-1}c_1 + c_{-1}b_1} = q^0 - q^1 - q^1 + q^2 = (1-q)^2.$$

Similarly, we obtain

$$\mathrm{Tr}\ q^{n(b_{-n}c_n + c_{-n}b_n)} = (1-q^n)^2.$$

Therefore

$$\mathrm{Tr}\ q^{L_0} = \prod (1-q^n)^2,$$

and

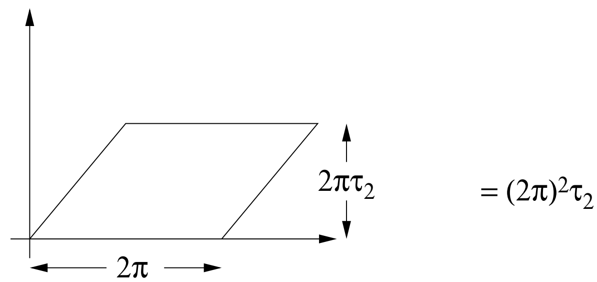
$$Z_{bc}(\tau) = (q\bar{q})^{1/2} \prod_{n=1}^{\infty} |1-q^n|^4 = |\eta(\tau)|^4.$$

5.4 Vacuum Energy

Combine $Z(\tau)$ for X^μ with $Z_{bc}(\tau)$ to get the total partition function.

$$Z_{total}(\tau) = iL^D \left(\frac{1}{2\pi\sqrt{\alpha'\tau_2}} |eta(\tau)|^{-1} \right)^D |\eta(\tau)|^4 = iL^D (2\pi\sqrt{\alpha'\tau_2})^{-D} |\eta(\tau)|^{-2(D-2)}.$$

So $D \to D-2$ which is good since only the transverse modes should contribute. This is not modular invariant. However, note that we have a conformal transformation left: rigid translations (similar to the sphere where we had $SL(2,\mathbb{C})$ invariance). There is a one to one correspondence between points on the torus and translations. Therefore the number of translations is proportional to the volume of the torus.



We need to *average* over translations, so divide by the volume of the group $((2\pi)^2\tau_2)$. We also have a reflection symmetry, so we need to multiply by $1/2$. If we have vertex operators, we need to fix *one* of the positions (c.f. a sphere where we fixed three positions). Finally the one-loop vacuum string amplitude is

$$Z_{physical} = iL^D \int_{F_0} \frac{d\tau d\bar{\tau}}{2(2\pi)^2\tau_2} (2\pi\sqrt{\alpha'\tau_2})^{-D} |\eta(\tau)|^{-2(D-2)}.$$

This is the cosmological constant. Why? C.f. with the point particle.

$$E = \sqrt{\vec{p}^2 + m^2}.$$

On a circle of length ℓ the temperature is $T \sim 1/\ell$. The partition function is given by

$$\begin{aligned} Z(\ell) &= L^D \int \frac{d^D p}{(2\pi)^D} e^{-\chi^\ell}, \quad \chi = \frac{1}{2}(-E^2 + \vec{p}^2 + m^2) = \frac{1}{2}(p^2 + m^2) \\ &= iL^D \, (2\pi\ell)^{-D/2} \, e^{-m^2\ell/2}. \end{aligned} \tag{5.4.1}$$

What information can we gain by evaluating the vacuum amplitude for the string? The vacuum amplitude defined earlier is given by,

$$Z = \int_0^\infty \frac{dl}{2\pi} Z(l).$$

where

$$Z(l) = L^D \int \frac{d^D p}{(2\pi)^D} e^{-\chi l} = iL^D \left(\frac{1}{\sqrt{2\pi l}} \right)^D e^{-m^2 l/2}$$

There is an ultraviolet divergence when $l = 0$. Cut off the integration region at ϵ and later let $\epsilon \to 0$. Apply a Wick rotation ($p_0 \to ip_0$) and integrate over p_0.

$$\begin{aligned} Z(l) &= i \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \exp[-\frac{l}{2}(p_0^2 + \vec{p}^2 + m^2)] \\ &\sim \frac{1}{\sqrt{l}} \exp[-\frac{l}{2}(\vec{p}^2 + m^2)] \end{aligned}$$

Now do the integral over l:

$$\begin{aligned} Z &= \int_{\epsilon}^{\infty} \frac{dl}{l} \frac{1}{\sqrt{l}} \exp[-\frac{l}{2}(\vec{p}^2 + m^2)] \\ &\sim \sqrt{\vec{p}^2 + m^2} \int_{\epsilon}^{\infty} \frac{dx}{x^{3/2}} \, \mathrm{e}^{-\mathrm{x}/2} \\ &\sim \sqrt{\vec{p}^2 + m^2} * \text{constant} \end{aligned}$$

$$\begin{aligned} Z(l) &\sim L^D \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} \sqrt{\vec{p}^2 + m^2} \\ &\sim L^D \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} E(\vec{p}) \end{aligned}$$

This is equivalent to integrating over all possible modes of the string. We can define the energy density as Λ.

$$\Lambda = \frac{Z}{L^D} \sim \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} E(\vec{p}) \qquad \text{“Cosmological Constant”}$$

This implies the cosmological constant is the sum of the vacuum energies from each of the states of the string.
How does the Cosmological constant play a role in one loop correction of the vacuum?

$$\Lambda \sim \int_{F_0} \frac{d\tau \, d\bar{\tau}}{\tau_2} (2\pi \sqrt{\alpha' \tau_2})^{-D} |\eta(\tau)|^{-2(D-2)}$$

Notice that there is no ultraviolet divergence, because τ is not allowed to approach zero ($|\tau| \geq 1$ in F_0). One the other hand, if $\tau_2 \to \infty$ (very long torus)

$$|q| = |e^{2\pi i \tau}| = e^{-2\pi \tau_2} \to 0$$

so we may expand

$$|\eta(\tau)|^{-2} = |q|^{-1/12} |1 - q + ...|^{-2} = e^{\pi \tau_2/6}(1 + 4Req + ...).$$

The dominant contribution

$$\int^{\infty} \frac{d\tau_2}{\tau_2}(\sqrt{4\pi^2\alpha'\tau_2})^{-26}e^{4\pi\tau_2}$$

c.f. with $(2\pi\tau_2)^{-D/2}\exp(-m^2\ell/2)$. Since $\ell = 2\pi\alpha'\tau_2$ the square of the mass becomes $m^2 = -4/\alpha'$ and we have a tachyon! This diverges due to $m^2 < 0$. Other terms in the sum are due to other modes with $m^2 \geq 0$, so they all converge.

5.5 Thermodynamics

In this section we look at various thermodynamic properties of the partition function.

$$\tau_1 = 0 \qquad \tau = i\tau_2 \qquad q = e^{-2\pi i\tau_2}$$

In the limits above what can we find out about the partition function?

$$\begin{aligned} Z(\tau_2) &= |q|^{-D/12}\mathrm{Tr}\left(q^{L_0}\bar{q}^{\tilde{L}_0}\right) \\ &= \mathrm{e}^{-\pi \mathrm{D}\tau_2/6}\mathrm{Tr}\left(\mathrm{e}^{-2\pi\tau_2 \mathrm{H}}\right) : \qquad \mathrm{H} = \mathrm{L}_0 + \tilde{\mathrm{L}}_0 \\ &= \mathrm{e}^{-\pi \mathrm{D}\tau_2/6}\sum_{\mathrm{n}} \mathrm{e}^{-2\pi\tau_2 \mathrm{E_n}} : \qquad \frac{1}{\mathrm{T}} = 2\pi\tau_2 \end{aligned}$$

High-temperature limit: $\tau_2 \to 0$. Then $q \to 1$ and $Z(\tau_2)$ is dominated by high-weight states. By modular invariance, $\tau_2 \to -1/\tau_2$, it is related to $Z(1/\tau_2)$.
Small-temperature limit: $\tau_2 \to \infty \Rightarrow q \to q' = \mathrm{e}^{-2\pi/\tau_2} \to 0$

$$Z(\tau_2) = \sum_n \mathrm{e}^{-\mathrm{E_n}/\mathrm{T}} \simeq \mathrm{e}^{-\mathrm{E_0}/\mathrm{T}} + \sum_{\mathrm{n}\neq 0} \mathrm{e}^{-\mathrm{E_n}/\mathrm{T}} + \ldots$$

Only the $E_0 = 0$ will survive in the sum.

$$Z\left(\frac{1}{\tau_2}\right) = \mathrm{e}^{-\frac{\pi \mathrm{D}}{6\tau_2}}$$

$$\tau_2 \to 0 \ : \qquad Z(\tau_2) = Z\left(\frac{1}{\tau_2}\right) = \mathrm{e}^{-\frac{\pi \mathrm{D}}{6\tau_2}}$$

Entropy

$$Z = \sum_E \rho(E)\mathrm{e}^{-\mathrm{E}/\mathrm{T}} \qquad \text{saddle point approximation}$$

There exists a stationary exponent when

$$d(\ln\rho - E/T) = 0.$$

Define the entropy as $S = \ln\rho$. Therefore $dE = TdS$.

$$Z = \sum \mathrm{e}^{\mathrm{S}-\mathrm{E}/\mathrm{T}} \qquad \mathrm{d}(\mathrm{S} - \mathrm{E}/\mathrm{T}) = 0 \Rightarrow \mathrm{dS} = \frac{\mathrm{dE}}{\mathrm{T}}$$

Free energy

$$F = -T \ln Z = -T\left(S - \frac{E}{T}\right) = E - TS$$

The entropy can be found from the Free energy, $S = -\frac{\partial F}{\partial T}$. Comparing with $Z(\tau_2)$ we see there is extra factor of $e^{-D/12T}$. Modular invariance implies $Ze^{-D/12T}$ is invariant. Therefore

$$\begin{aligned} Z(\tau)\mathrm{e}^{-\mathrm{D}/12\mathrm{T}} &= Z\left(\frac{(2\pi)^2}{T}\right)\mathrm{e}^{-\pi^2\mathrm{DT}/12} \\ Z(\tau) &= \mathrm{e}^{\mathrm{D}/12\mathrm{T}}\mathrm{e}^{-\pi^2\mathrm{DT}/12}\mathrm{Z}\left(\frac{(2\pi)^2}{\mathrm{T}}\right) \end{aligned}$$

$Z(1/T)$ is a slowly varying function in the saddle point approximation. When we go from statistical mechanics to thermodynamics $Z\left(\frac{1}{T}\right)$ is ignored.

$$\ln Z(T) = \frac{D}{12T} - \frac{\pi^2 DT}{12}$$

$$F = -\frac{D}{12} + \frac{\pi^2 DT^2}{12} = \frac{D}{12}(\pi^2 T^2 - 1), \quad S = -\frac{\partial F}{\partial T} = \frac{D\pi^2 T}{6},$$

$$E = F + TS = \frac{D}{12}(\pi^2 T^2 + 1) \qquad S \leq \pi E.$$

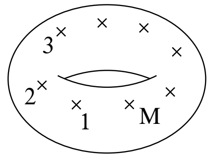
At high temperature the energy is proportional to the square of the temperature ($E \propto T^2$). This represents the "Casimir energy in 2-D", which in four-dimensions is given by $E \propto T^4$. We may express S in terms of E

$$\begin{aligned} \frac{D}{12}\pi^2 T^2 &= E - \frac{D}{12}, \\ S^2 &= \left(\frac{D\pi^2}{6}\right)^2 T^2 = \frac{D^2}{26}\pi^4 \frac{12}{\pi^2 D}\left(E - \frac{D}{12}\right) = 4\frac{D\pi^2}{12}\left(E - \frac{D}{12}\right), \\ S &= 2\pi\sqrt{\frac{D}{12}\left(E - \frac{D}{12}\right)} \qquad \text{“Cardy”} \end{aligned}$$

or $S \leq \pi E$ which is the same as the black hole entropy, where Bekenstein's equation is $S \leq S_B \sim E$. If the equation is generalized to 4-D, it gives the Friedman-Robertson-Walker universe equation for the Hubble constant.

5.6 Amplitudes on a torus

For M tachyons, $V_i =: e^{ik_i \cdot X}(z_i)$ together with the right movers, the amplitude can be writen as



$$A(z_1, z_2, ..., z_M) = \text{Tr } V_1 \ .... \ V_M q^{(L_0 - D/24)} \times (c.c.)$$

where

$$X_L = x + \frac{\alpha'}{2} pz + i\sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{n} \alpha_n e^{-inz}, \quad [x, p] = i$$

At the moment, let us omit the complex conjugate. We can insert it when needed. There are two parts to the amplitude: the p-integral and the N-modes.

$$\int \frac{d^D p}{(2\pi)^D} \langle p | e^{-\pi\tau_2 \alpha' p^2} e^{ik_1 \cdot x/2} e^{i\alpha' k_1 \cdot pz/2} \ \cdots \ e^{ik_m \cdot x/2} e^{i\alpha' k_M \cdot pz_M/2} |p\rangle.$$

using $X_L = x + \frac{\alpha'}{2} pz + i\sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{n} \alpha_n e^{-inz}$. To evaluate the integral, commute all $e^{ik_1 \cdot x/2}$ through. When they hit $|p\rangle$, they change $|p\rangle \to |p + k_1 + \ldots + k_M\rangle$. Then $\langle p | p + k_1 + \ldots + k_M \rangle = \delta^D(k_1 + \ldots + k_M)$ (conservation of momentum).
As we push the factors through, the amplitude becomes

$$\int \frac{d^D p}{(2\pi)^D} e^{i\pi\tau p^2} e^{i(k_1 z_1 + \ldots + k_M z_M)\cdot p} \delta^D(k_1 + \ldots + k_M)$$

This is a Gaussian integral. We need to shift the momentum $p \to p - Im(k_1 z_1 + \ldots + k_M z_M)/\pi\tau_2$. The amplitude becomes

$$A \sim (2\pi\sqrt{\alpha' \tau_2})^{-D/2} \delta^D(k_1 + \ldots + k_M) e^{-\alpha'/\pi\tau_2 [\text{Im}(k_1 z_1 + \ldots + k_M z_M)]^2} \prod_{i<j} e^{-\alpha'/2 k_i \cdot k_j (z_i - z_j)} \times c.c.$$

Using conservation of momentum, we may also write

$$(k_1 z_1 + \ldots + k_M z_M)^2 = -\sum_{i<j} k_i \cdot k_j (z_1 - z_j)^2.$$

Next, do the oscillators. We can do each oscillator separately, so fix μ, n and do $\alpha_{-n^\mu}, \ \alpha_n^\mu$. The trace is

$$T_n^\mu = \text{Tr} \prod_{i=1}^{M} \exp\left(\frac{1}{n} k_{i\mu} \alpha_{-n}^\mu\right) \exp\left(-\frac{1}{n} k_{i\mu} \alpha_n^\mu e^{-niz}\right) q^{\alpha_{-n}^\mu \alpha_{n\mu}}$$

where there is no sum on μ!

To evaluate the trace we need to insert the identity $I = \sum_E |E\rangle\langle E|$ and then use $\mathrm{Tr}\ A = \sum_E \langle E|A|E\rangle$. For a complete set of states, choose

$$|\kappa\rangle = e^{\frac{1}{n}\kappa\cdot\alpha_{-n}}|0\rangle,$$

where $\kappa \in \mathbb{C}$. The identity operator is

$$I = \frac{1}{n\pi}\int d^2\kappa e^{-|\kappa|^2/n}|\kappa\rangle\langle\kappa|.$$

To show this, consider an eigenstate of the number operator $N = \sum \alpha_{-n}\alpha_n$ (dropped the μ to avoid confusion).

$$|\ell\rangle = \frac{1}{\sqrt{\ell!}}\frac{(\alpha_{-n})^\ell}{\sqrt{n}}|0\rangle, \ \ \langle\ell|\ell\rangle = 1.$$

Then

$$\begin{aligned}\langle\ell|\kappa\rangle &= \frac{1}{\ell!}\langle\ell|(\frac{\kappa}{n})^\ell\alpha_{-n}^\ell|0\rangle, \\ &= \left(\frac{\kappa}{n}\right)^\ell\frac{1}{\sqrt{\ell!}},\end{aligned}$$

$$\langle\ell|I|\ell'\rangle = \frac{1}{n\pi}\frac{1}{\sqrt{(\ell!)^2}}\int d^2\kappa\, e^{-|\kappa|^2/n}\left(\frac{\bar{\kappa}}{n}\right)^\ell\left(\frac{\kappa}{n}\right)^{\ell'}$$

For $\ell \neq \ell'$, $\langle\ell|I|\ell'\rangle = 0$ which can be shown by using polar coordinates. For $\ell = \ell'$

$$\langle\ell|I|\ell\rangle = \frac{1}{n\pi\alpha'}\int d^2ke^{-|\kappa|^2/n}\left(\frac{|\kappa|}{n}\right)^{2\ell} = 1,$$

which implies

$$< \ell|I|\ell' >= \delta_{\ell\ell'}.$$

Also $q^N|\kappa\rangle = |\kappa q^n\rangle$.
PROOF:

$$\begin{aligned}q^N\alpha_{-n}q^{-N} &= \alpha_{-n} + \ln q[N,\alpha_{-n}] + \frac{1}{2}(\ln q)^2[N,[N,[\alpha_{-n}]] \\ &= \alpha_{-n} + n\ln q\alpha_{-n} + \frac{n^2}{2}(\ln q)^2\alpha_{-n} + ... \\ &= e^{n\ln q}\alpha_{-n} = q^n\alpha_{-n} \\ q^N(\alpha_{-n})^\ell q^{-N}) &= q^{n\ell}(\alpha_{-n})^\ell \\ q^N e^{\frac{1}{n}\kappa\alpha_{-n}}q^{-N} &= e^{\frac{1}{n}q^n\kappa\alpha_{-n}},\end{aligned}$$

where we used the Glauber identity ($e^Ae^B = e^Be^Ae^{[A,B]}$). Therefore

$$q^N|\kappa\rangle = |\kappa q^N\rangle.$$

The trace becomes

$$T_n^\mu = \frac{1}{n\pi}\int d^2\kappa e^{-|\kappa|^2/n}\langle\kappa|\prod_{i=1}^{m} e^{\frac{1}{n}k_i\cdot\alpha_{-n}e^{niz_i}}e^{-\frac{1}{n}k_i\cdot\alpha_n e^{-niz}}e^{\frac{1}{n}q^n\kappa\alpha_{-n}}\,|0\rangle.$$

As we push $\exp\left(-\frac{1}{n}k_1\alpha_n e^{-niz}\right)$ through, the commutators contribute to a new factor given by

$$\exp\left(-\frac{1}{n}k_1 q^n\kappa e^{-niz_1}\right).$$

After moving the exponentials over to act on the states, the commutators contribute an overall factor of

$$\prod_{i<j}\exp\left(-\frac{1}{n}k_i\cdot k_j e^{ni(z_i-z_j)}\right),\quad \prod_i \exp\left(-\frac{1}{n}k_i q^n\kappa e^{-niz_i}\right)$$

Finally push $\exp\left(\frac{1}{n}\bar{x}\alpha_n\right)$ through. We get a factor $\exp\left(\frac{1}{n}k_i\bar{\kappa}e^{niz_i}\right)$ from each vertex and $\exp\left(\frac{1}{n}q^n|\kappa|^2\right)$ from $|q^n\kappa\rangle$. Therefore

$$T_n^\mu = \frac{1}{n\pi}\int d^2\kappa e^{-(1-q^n)|\kappa|^2/n}\prod_{i<j}\exp\left(-\frac{1}{n}k_i\cdot k_j e^{ni(z_i-z_j)}\right)\prod_i\exp\left(-\frac{1}{n}k_i q^n\kappa e^{-niz_i}\right)$$

This is a Gaussian integral: $\frac{1}{\pi}\int d^2\kappa e^{-c|\kappa|^2}\,e^{a\kappa+b\bar{\kappa}} = \frac{1}{c}e^{-ab/c}$.
Therefore

$$T_n^\mu \sim \frac{1}{1-q^n}\prod_{i<j}e^{-\frac{1}{n}k_i\cdot k_j e^{-ni(z_i-z_j)}}\exp\left\{-\frac{(\sum k_i e^{niz_i})(\sum k_i e^{-niz_i})q^n}{n(1-q^n)}\right\}.$$

Use

$$\sum_{i<j}k_i\cdot k_j = -\frac{1}{2}\sum k_i 2 = -M.$$

$$\prod_{n,\mu}T_n^\mu = \prod_{m=1}^{\infty}(1-q^m)^{-D}\prod_{i<j}e^{-k_i\cdot k_j}\frac{e^{mi(z_i-z_j}+q^m e^{-mi(z_i-z_j)}-2q^m}{m(1-q^m)}$$

where we see the whole second product is a new contribution. If we use the identity

$$\sum_{m=1}^{\infty}\frac{1}{m}\frac{x^m}{1-y^m} = -\sum_{n=0}^{\infty}\ln(1-xy^n)$$

we find

$$\begin{aligned}\prod_{n,\mu}T_n^\mu &= \prod_{m=1}^{\infty}(1-q^m)^{-D}\prod_{i<j}e^{-k_i\cdot k_j\sum_{n=0}^{\infty}\ln(1-q^n e^{i(z_i-z_j)})+\ln(1-q^{n+1}e^{-i(z_i-z_j)})-2\ln(1-q)}\\ &= \prod_{m=1}^{\infty}(1-q^m)^{-D}\prod_{i<j}(1-qe^{i(z_i-z_j)})^{\alpha' k_i\cdot k_j}\prod_{n=1}^{\infty}\left[\frac{(1-q^n e^{i(z_i-z_j)})(1-q^n e^{-i(z_i-z_j)})}{(1-q^n)^2}\right]^{\alpha' k_i\cdot k_j}.\end{aligned}$$

The first factor $\prod_{i<j}(1-qe^{i(z_i-z_j)})^{\alpha' k_i\cdot k_j}$ is combines with $\prod_{i<j} e^{-\alpha/2(z_i-z_j)k_i\cdot k_j}$ to give

$$\prod_{i<j}\sin(z_i-z_j)^{\alpha' k_i\cdot k_j/2}.$$

Then

$$\sin(z_i-z_j)\prod_{n=1}^{\infty}\left[\frac{(1-q^n e^{i(z_i-z_j)})(1-q^n e^{-i(z_i-z_j)})}{(1-q^n)^2}\right]^{\alpha' k_i\cdot k_j} \sim \frac{\vartheta_1\left(\frac{z_i-z_j}{2\pi}|\tau\right)}{\partial_\nu\vartheta_1(0|\tau)},$$

where

$$\vartheta_1(\nu|\tau) = 2e^{\pi i\tau/4}\sin\pi\nu\prod(1-q^m)(1-zq^m)(1-z^{-1}q^m).$$

As we recall, ϑ_1 has nice modular properties, given by

$$\begin{aligned}\vartheta_1(\nu|\tau+1) &= e^{i\pi/4}\vartheta_1(\nu|\tau),\\ \vartheta_1\left(\frac{\nu}{\tau}|-\frac{1}{\tau}\right) &= -i\sqrt{-i\tau}e^{\pi i\nu^2/\tau}\vartheta_1(\nu|\tau).\end{aligned}$$

Let us compare the amplitude to the sphere amplitude. The M-point amplitude for the sphere was

$$A = \langle V_1 ... V_M\rangle \sim \prod_{i<j}|z_i-z_j|^{\alpha' k_k\cdot k_j}.$$

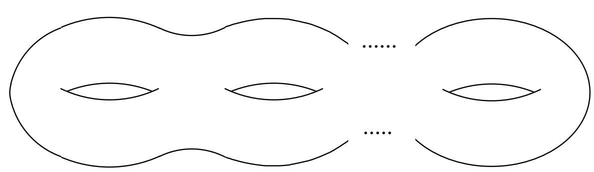
For small $|z_i-z_j|$, we may make the approximation

$$\vartheta_1\left(\frac{z_i-z_j}{2\pi}|\tau\right)\sim\sin\pi\nu\sim z_i-z_j.$$

So, the torus is similar to the sphere at short distances.

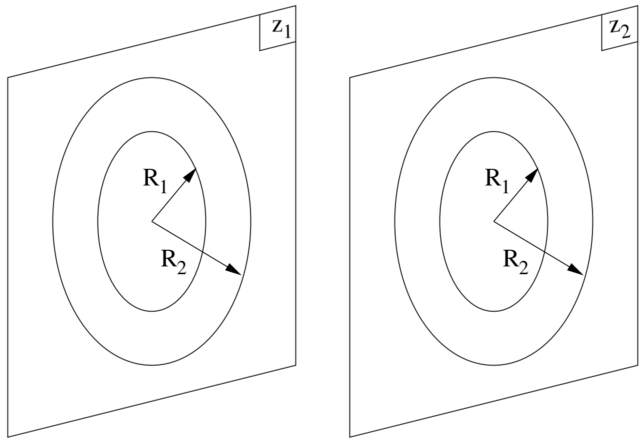
5.7 Higher Genus Surfaces

All two-dimensional surfaces may be classified by their genus. These surfaces are classified the number of handles they posess. There is no classification for surfaces of dimension greater than two. The genus of a sphere and torus are zero and one respectively.

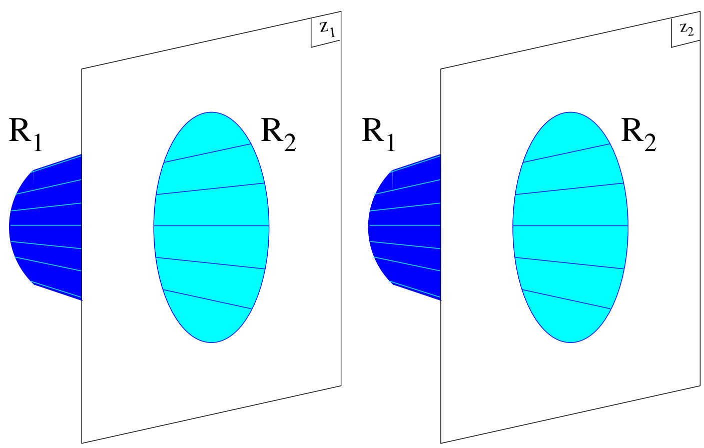


Plumbing Fixture

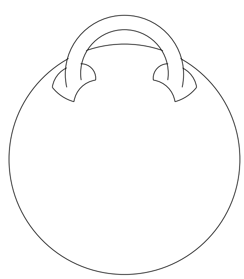
This is a method of creating higher genus surfaces from lower genus surfaces.



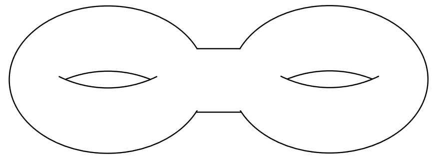
Coordinate patches z_1 and z_2 may belong to different or same surface. Cut a disk of radius R_1 around the origin of each. Then glue annular regions $R_1 < |z_i| < R_2$. $(i = 1, 2)$ by identifying



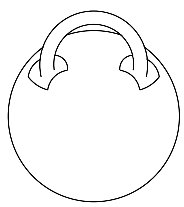
Example 1: z_1 and z_2 on the same plane. Add a handle to create a torus.



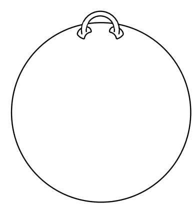
Example 2: z_1 and z_2 on two tori.

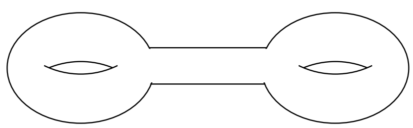


Plumbing is a reversible process. One can cut handles and then patch disks to create a lower genus surface. Let us look at an interesting case, $q \to 0$. Then for fixed R_1, $R_2 \to 0$, so an annulus maps to a semi-infinite cylinder, an object with **physical** meaning!
Examples:

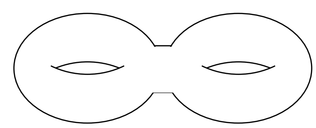


equivalent to





equivalent to



This is the boundary of the moduli space. Recall that violations to the BRST symmetry may also come from the boundary of the moduli space, so pinched handles are important.

Amplitudes

We will start with a trivial example: two spheres connected by a cylinder.

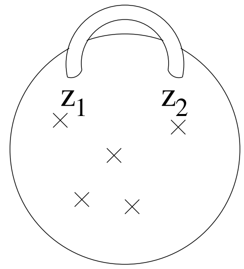
$$\begin{aligned} A_1 &= \langle 0|V_1...V_M|E\rangle = \langle 0|V_1...V_M V_E|0\rangle \\ A_2 &= \langle E|V_{M+1}...V_N|0\rangle = \langle 0|V_E^* V_{M_1}...V_N|0\rangle \end{aligned}$$

where

$$|E\rangle = V_E|0\rangle \quad \text{and} \quad \mathrm{I} = \sum_{\mathrm{E}} |\mathrm{E}\rangle\langle \mathrm{E}|$$

$$A = \sum_E A_1 A_2 = \langle 0|V_1...V_M V_{M+1} V_N|0\rangle = \langle 0|V_1...V_N|0\rangle$$

This is just a N-point amplitude on a sphere, obviously since $S^2 \cup S^2 = S^2$
Next, let us move to an example which is nontrivial, but known.



If $q = 1$, then the identification: $z_1 z_2 = 1$ which is the two patches on the sphere around zero and infinity respectively. We might as well choose $z_1 = 0,\ z_2 = \infty$. Then the amplitude,A, becomes

$$A = \langle E|V_1...V_M|E\rangle = \langle 0|V_E^* V_1...V_M V_E|0\rangle$$

and the trace of the amplitude becomes

$$\text{Tr } A = \sum_E A = \text{Tr } V_1...V_M.$$

Now let us have $z_1 z_2 = q$, need $z_2 \to z_2 q$, a conformal transformation under which

$$V_E \to \left(\frac{\partial z_2'}{\partial z_2}\right)^h \left(\frac{\partial \bar{z}_2'}{\partial \bar{z}_2}\right)^{\tilde{h}} V_E = q^h \bar{q}^{\tilde{h}} V_E.$$

$\{E\}$ is the set of h's (weights), where

$$L_0|E\rangle = L|E\rangle, \quad \tilde{L}_0|E\rangle = \tilde{L}|E\rangle.$$

Therefore

$$\sum_E A = \text{Tr } \left(V_1...V_M q^{L_0} \bar{q}^{\tilde{L}_0}\right), \text{ for } q \neq 1.$$

Generalize ...

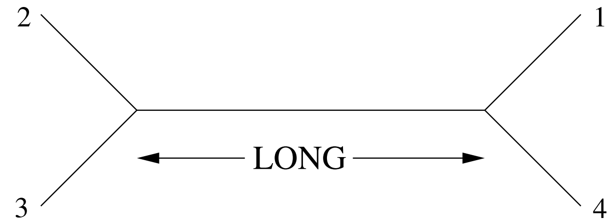
$$A = \sum_E \langle V_1...V_M|E\rangle\langle E|V_{M+1}...V_N|0\rangle q^h \bar{q}^{\tilde{h}}$$

to obtain an amplitude of a higher genus surface.

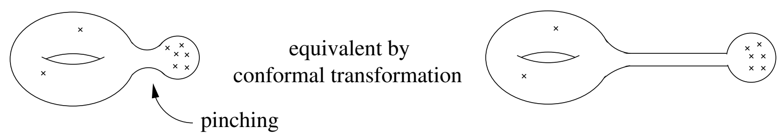
Divergences

We have already seen that there are no ultra-violet divergences that plague quantum field theories (short distance effects- breakdown of theory, i.e., new physics at short distances). String theory is the ultimate theory, no matter how short the distance.
Example: The torus
Puzzle: as vertices come together on a torus, we get a singularity. Is this a short distance effect? No! It is like the case of a sphere (topology plays no

role). Recall the singularity in $\langle V_1(\infty)V_2(1)V_3(z)V_4(0)\rangle$ as $z \to 1$. We obtain poles, which after a conformal transformation seem to come from a particle propagating for a long time.

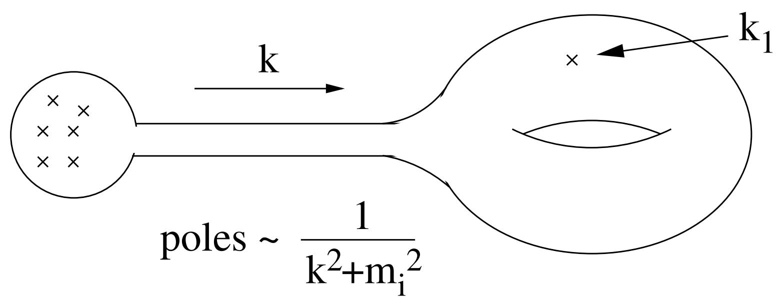


That is a long distance effect, i.e., an "infrared divergence" which is expected. $z \to 1$ means the distance becomes small on the worldsheet (bogus?). A long intermediate state is a **spacetime** concept (REAL!). Similarly, on a torus



Special Case I

All vertices but one come together.



Conservation of momentum implies $k = k_1$, but $k_1^2 = -m^2$, so $1/(k^2 + m^2) = 1/0$, a singularity (infrared).
We get rid of it in quantum field theory as follows. Let $k^2 \neq -m^2$. Then

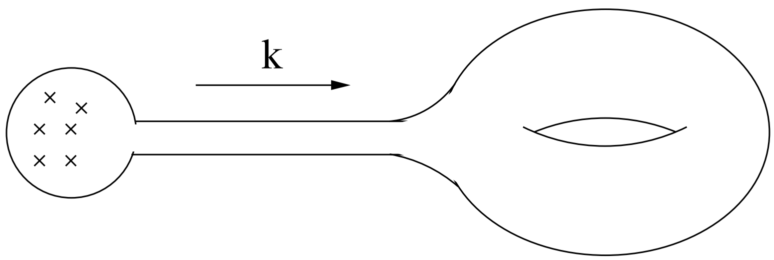
$$\begin{aligned} &= \delta + \frac{\delta^2}{k^2+m^2} + \frac{\delta^3}{(k^2+m^2)^2} + \dots \\ &= \frac{\delta}{1-\frac{\delta}{k^2+m^2}} = \frac{\delta(k^2+m^2)}{k^2+m^2-\delta} \end{aligned}$$

Notice that the pole has been shifted by δ, so the mass is corrected by quantum effects: $\delta m^2 = -\delta$. This means that we should have started with a particle of mass $m^2 - \delta$ and **not** m^2. It was a poor perturbative expansion (singular

perturbation theory). It is the **same** in string theory. We got $1/0$ because we asked the **wrong** question.
Note: Massless particles receive no quantum corrections as required by gauge invariance.

Special Case II

All vertices come together



So the pole from the massless modes goes as $1/k^2 = 1/0$! (Bad)
Again, we are asking the wrong question. To calculate the contribution of the massless modes, after we insert $\sum_E |E\rangle\langle E|$, we pick $E = 0$. That is an insertion of $\partial X^\mu \bar{\partial} X_\mu$. If we regulate the integral, e.g., by cutting the length of the connecting cylinder by L_{max}, we obtain

$$Amp \sim C \int d^2 z \partial X^\mu \bar{\partial} X_\mu$$

where C is an infinite constant.
Now this can also come from a perturbation in the action $\int d^2 z \partial X^\mu \bar{\partial} X_\mu$ which tells us that the metric in **spacetime**, instead of being flat, $G_{\mu\nu} = \eta_{\mu\nu}$ (Lorentz) should be $G_{\mu\nu} = (1 - C)\eta_{\mu\nu}$. The flat background is **not** a good zeroth order approximation because of the gravitational effects of the string.