String Theory I

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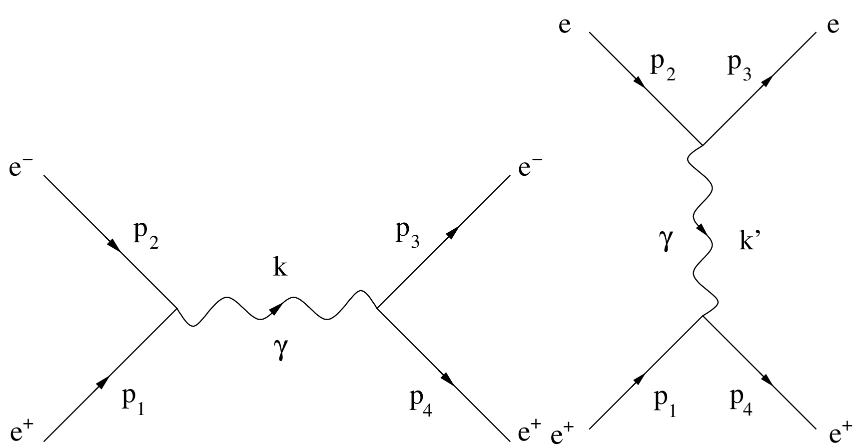
UNIT 4

Tree-level Amplitudes

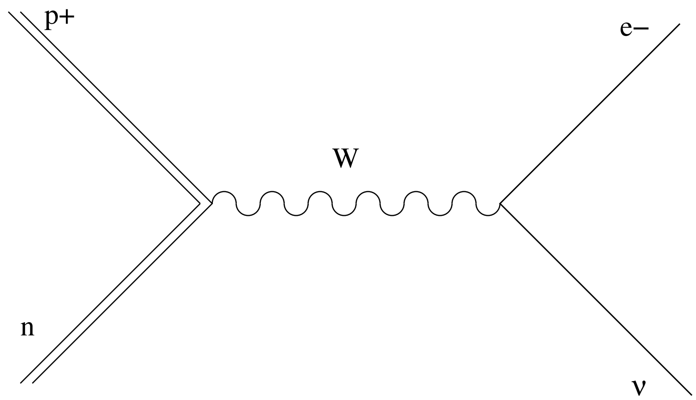
4.1 String Interactions

In particle theory, we need to introduce a multi-particle space (Fock space) where creation and annhilation are possible. In string theory, the tools we have developed for one string are sufficient for the description of multi-string states and interactions! The entire quantum theory of strings is based on these tools!

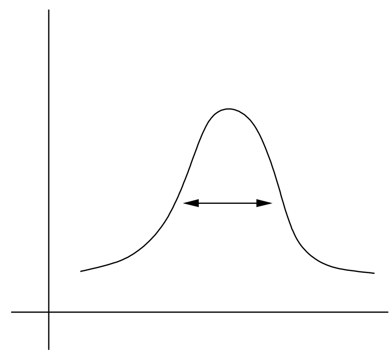
Example of particle interactions



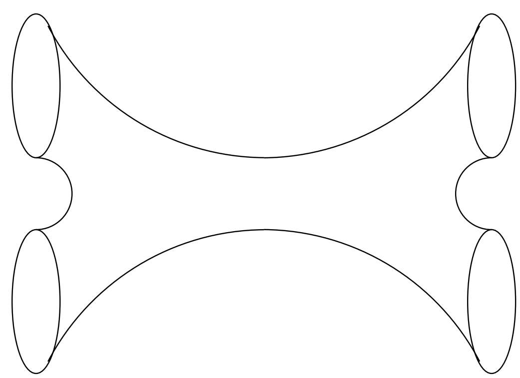
$1/k^2$: inverse of the Klein-Gordon operator $\Box\phi = 0$, $\Box^{-1} \sim 1/k^2$ There is a pole at $k^2 \sim 0$, e.g., β-decay



Amplitude$\sim \frac{1}{k^2 - m_W^2}$, pole at $k^2 = m_W^2$, resonance.

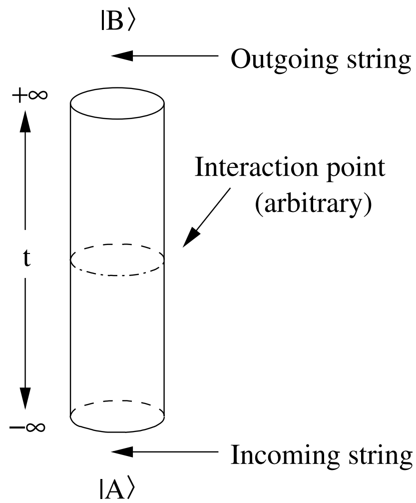


Strings

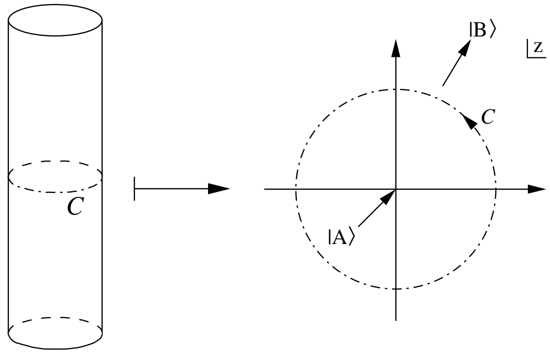


The interaction consists of strings joining and splitting. Where do they join? This is a stupid question. It depends on the time slicing. Therefore this is a *fuzzy* interaction. Moreover, the shape (geometry) of the surface is not important, only the topology is important. There is one diagram for *all* tree diagrams.

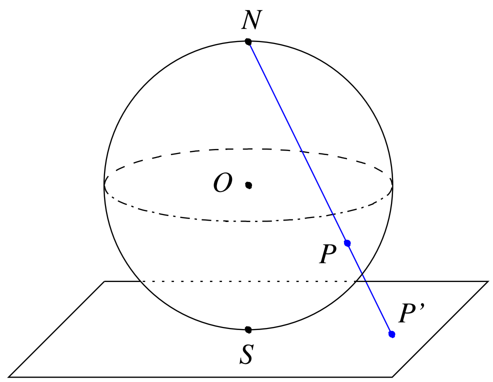
Example



There is an arbitrary interaction point. The amplitude is constructed by joining two semi-infinite cylinders. Map the cylinders to a plane:



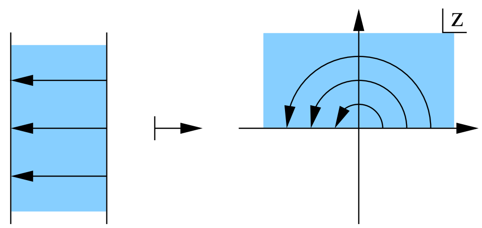
cylinder $\to \mathbb{C}\bigcup\{\infty\} = S^2$ (sphere). This is done through *stereographic* projection (sphere=fat cylinder).



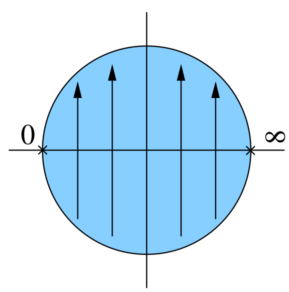
Amplitude: sphere with states (operator insertions) at North and South poles. Notice the equivalence of the two poles (clinder $z \to \frac{1}{z}$).

Open Strings

Make a strip by cutting the cylinder in half along the axis.



We then map the strip to the upper-half plane which can then be mapped to the unit circle via the mapping $z \to \frac{z-i}{z+i}$.



Each string is a semi-infinite cylinder (or strip), which is mapped to a disk. When we put two on a sphere, they were simply represented by insertion of $A(z)$ at $z = 0,\ z = \infty$.
Guess: For scattering of N strings we can do the same, i.e., on a sphere select points $z_1, z_2, ..., z_N$ and insert operators $A_i(z_i)$. Then the amplitude is

$$A \sim \langle 0|A_1(z_1)A_2(z_2)...A_N(z_N)|0\rangle.$$

Now $A(0)$ is equivalent to $\oint_{\mathcal{C}} \frac{dz}{2\pi i} A(z)$.
For conformal invariance, we require that all A_i have dimension $h_i = 1$ so that $\int dz A_i(z)$ have zero dimension (conformally). Then we should define

$$Amp \sim \int dz_1 ... dz_N \langle 0 | A_1(z_1) ... | 0 \rangle .$$

In fact, the measure should read $\int d^2 z_1 ... d^2 z_N$, but we will not be writing the $\bar{z}$ piece explicitly. The proper dimension of $A_i(z, \bar{z})$ should be $h_i = 1,\ \bar{h}_i = 1$. In general,

$$A(z) \sim : \partial^{m_1} X \partial^{m_2} X ... e^{ik \cdot X} :,$$

where $h = m_1 + m_2 + ... \alpha' k^2 = 1$. We shall work with the simplest case $A(z) = e^{ik \cdot X},\ k^2 = \frac{1}{\alpha'}$. The rest is similar.
Complication: The amplitude is conformally invariant: $z \to z + \epsilon v(z)$ where $v(z)$ is analytic. $v(z)$ should be analytic everywhere in $\mathbb{C} \cup \{\infty\}$. We need to check that the transformation is analytic at infinity. So let $z \mapsto \frac{1}{z} = z'$.

$$\delta z' = -\frac{1}{z^2} \delta z = -\epsilon \frac{1}{z^2} v(z) = -\epsilon z'^2 v\left(\frac{1}{z'}\right).$$

therefore $v(z) = a + bz + cz^2$ so that $z'^2 v\left(\frac{1}{z'}\right)$ is analytic. This is a six-parameter family of transformations. It includes SO(3) (rotation group). Special Cases:
• $z \mapsto z + \epsilon a$ generated by L_{-1}. Recall $[L_m, A] = z^{m+1} \partial A + h(m+1) z^m A$ where $h = 1$ for BRST invariance. So $[L_{-1}, A] = \partial A - \frac{1}{z} A$ i.e., L_{-1} generates translations in z.
Finite transformation: $z \mapsto z + a$,

$$A(z) \to e^{aL_{-1}} A(z) e^{-aL_{-1}} = A(z + a).$$

•$z \mapsto z + \epsilon b z = (1 + \epsilon b) z$ generated by L_0.

$$[L_0, A] = z \partial A + A.$$

$$A(z) \to e^{bL_0} A(z) e^{-bL_0} = A(e^b z).$$

Finite transformation: $z \to e^b z$.
•$z \mapsto z + \epsilon c z^2$ generated by L_1.

$$[L_1, A] = z^2 \partial A + zA.$$

Finite transformations: $z \to \frac{z}{1 - cz} = z'$

$$A(z) \to e^{cL_1} A(z) e^{-cL_1} = A\left(\frac{z}{1 - cz}\right).$$

Combination of all three: $z \mapsto \frac{az + b}{cz + d},\ \ ad - bc = 1$ defines the group SL$(2, \mathbb{C})$ whose algebra is

$$[L_1, L_{-1}] = 2L_0, \quad [L_1, L_0] = L_1, \quad [L_{-1}, L_0] = -L_{-1}.$$

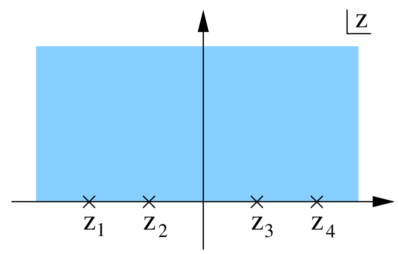
This is a closed algebra (no constant term) and is common in *all* conformal field theories.
How come a matrix entered acting on a number z? **Answer**: Consider the vector

$$(z_1,\ z_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + dz_2 \\ cz_1 + dz_2 \end{pmatrix}.$$

Let $z = z_1/z_2$. Then

$$z \mapsto \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{az + b}{cz + d}.$$

For **open strings**: the Real axis is a boundary, and the group of symmetries becomes SL$(2,\mathbb{R})$, $a, b, c, d \in \mathbb{R}$. Then under $z \to \frac{az+b}{cz+d}$, ∂ is invariant. The upper-half plane maps to itself.
Amplitude for open strings:



$$Amp \sim \langle V(z_1)V(z_2)...V(z_N)\rangle, \quad z_i \in \mathbb{R},$$

where the product is time ordered and thus the z_is are ordered. How do we integrate over z_i? Due to SL$(2,\mathbb{R})$ symmetry, we have redundency, so naive integral would be proportional to the volume of SL$(2,\mathbb{R})$ which is infinite! We need to fix the gauge by choosing three points. Easiest to fix them to $(0, 1, \infty)$. This is an arbitrary choice, but all choices are equivalent by the SL$(2,\mathbb{R})$ symmetry. We will integrate over the rest of the parameters.

Example 1: Three tachyons

Consider three tachyons, $V_i(z) =: e^{ik_i \cdot X(z)}$: The amplitude is given by

$$A \sim \langle 0|V_1(z_1)V_2(z_2)V_3(z_3)|0\rangle,$$

where

$$X^\mu(z) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}a_m^\mu\left(z^{-m} + \bar{z}^{-m}\right).$$

Since $z \in \mathbb{R}$, X^μ reduces to

$$X^\mu(z) = x^\mu - i\alpha' p^\mu \ln|z| + i\sqrt{2\alpha'}\sum_{m\neq 0}\frac{1}{m}a_m^\mu z^{-m}.$$

Since we can fix three points, let us choose $(z_1 = \infty,\ z_2 = 1,\ z_3 = 0)$, then

$$V_3(z_3 = 0)|0\rangle = |0; k_3\rangle, \quad \langle 0|V_1(z_1 = \infty) = \langle 0; -k_1|.$$

The amplitude becomes

$$A \sim \langle 0; -k_1|V_2(z_2 = 1)|0; k_3\rangle = \langle 0; -k_1|0; k_2 + k_3\rangle = \delta^D(k_1 + k_2 + k_3).$$

One can derive this for arbitrary z_1, z_2, z_3, due to the SL(2,$\mathbb{R}$) symmetry.

Example 2: Two tachyons and one vector

Consider two tachyons, $V_i(z) =: e^{ik_i\cdot X(z)}:$ and a vector, $V_j(z) =: A^\mu \partial X_\mu e^{ik_j\cdot X(z)}:$, where $k_j^2 = 0$. We may act the vertex operators on the vacuum states

$$\begin{aligned} A &\sim \langle 0; -k_1|V_2(1)A^\mu\partial X_\mu|0; k_3\rangle \sim \langle 0; -k_1|e^{ik_2\cdot x}e^{\sqrt{\frac{\alpha'}{2}}a_1\cdot k_2}A_\mu\alpha^\mu_{-1}|0; k_3\rangle \\ &\sim \sqrt{2\alpha'}A\cdot k_2\delta^D(k_1 + k_2 + k_3). \end{aligned} \tag{4.1.1}$$

A is transverse to it's momentum $(A \cdot k_3 = 0)$ therefore, the amplitude is

$$A \sim \sqrt{\frac{\alpha'}{2}}A\cdot(k_2 - k_1)\delta^D(k_1 + k_2 + k_3),$$

where the dot product represents the coupling of the electromagnetic potential to the charged scalar. We may check the gauge invariance of the amplitude. Using the gauge transformation $A^\mu \to A^\mu + \omega k_3^\mu$, the amplitude becomes

$$\delta(A) \sim k_3\cdot(k_2 - k_1) = k_2^2 - k_1^2 = 0.$$

Example 3: Four tachyons

This is the first nontrivial amplitude. Due to the SL$(2,\mathbb{R})$ symmetry, we may fix three operators. Now we have an extra operator we can not fix. We must integrate over its parameter. After we operate vertex operators on the vacuum states, the amplitude is given by

$$A \sim \langle 0; -k_1| :e^{ik_2\cdot X}(1)::e^{ik_3\cdot X}(z): |0; k_3\rangle.$$

This is a time-ordered product and we must integrate over z from $[0, 1]$. The amplitude becomes

$$A \sim \int_0^1 dz\langle 0; -k_1| :e^{ik_2\cdot X}(1)::e^{ik_3\cdot X}(z): |0; k_4\rangle.$$

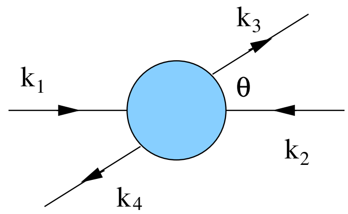
Using the mode expansion of X^μ,

$$A \sim \int_0^1 dz\langle 0; -k_1|e^{ik_2\cdot x}e^{\sqrt{2\alpha'}\sum_{m>0}k_2\cdot\alpha_m/m}e^{ik_3\cdot x}e^{i\frac{\alpha'}{2}k_3\cdot p\ln|z|}e^{\sqrt{2\alpha'}\sum_{n>0}k_3\cdot\alpha_{-n}/n}|0; k_4\rangle.$$

Using the Hausdorff formula, $e^A e^B = e^{[A,B]} e^B e^A$,

$$\begin{aligned} A &\sim \int_0^1 dz \langle 0; -k_1 | z^{2\alpha' k_3 \cdot k_4} e^{-2\alpha' k_2 \cdot k_3 \sum z^m/m} | 0; k_3 + k_4 \rangle \\ &\sim \int_0^1 dz z^{2\alpha' k_3 \cdot k_4} (1-z)^{-2\alpha' k_2 \cdot k_3} \delta^D (k_1 + k_2 + k_3 + k_4) \end{aligned}$$

Define the Mandelstam variables



$$s = (k_1 + k_2)^2 = (k_3 + k_4)^2 = -2k_3 \cdot k_4 - \frac{2}{\alpha'}, \quad t = -(k_2 + k_3)^2, \quad u = -(k_2 + k_4)^2,$$

and

$$s + t + u = -\frac{4}{\alpha'}.$$

The amplitude expressed in terms of Mandelstam variables becomes

$$A \sim \int_0^\infty dz z^{-\alpha' s - 2} (1-z)^{-\alpha' t - 2} \delta^D (k_1 + k_2 + k_3 + k_4) \sim B(-\alpha' s - 1, \alpha' t - 1),$$

where B is the Euler-beta function with the property

$$B(x, y) = \frac{\Gamma(y)\Gamma(y)}{\Gamma(x+y)}.$$

This is known as the *Veneziano* amplitude. Note, there are poles at $-\alpha' s - 1 = 0$ and $\alpha' t - 1 = 0$. Let us focus on the first pole ($-\alpha' s - 1 = 0$).

$$\begin{aligned} A &\sim \Gamma(-\alpha' s - 1) = \frac{\Gamma(-\alpha' s)}{\alpha' s + 1} + \ldots \quad (\Gamma(x+1) = x\Gamma(x)) \\ &\sim -\frac{1}{-\alpha' s + 1} + \ldots \end{aligned}$$

The pole is due to an intermediate tachyon ($s = -1/\alpha'$). Unitarity requires This checks, since The next pole is at $\alpha' s = 0$.

$$\Gamma(-\alpha' s - 1) = -\frac{\Gamma(-\alpha' s)}{\alpha' s + 1} = \frac{\Gamma(-\alpha' s + 1)}{(\alpha' s + 1)(\alpha' s)} = \frac{1}{\alpha' s} + \ldots$$

The amplitude becomes

$$A \sim \frac{1}{\alpha' s} \frac{\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha' t - 2)} = \frac{\Gamma(-\alpha' t - 2)}{\alpha' s} + \ldots = \frac{u - t}{2s} + \ldots$$

where we used the condition $s+t+u=-4/\alpha'$.
Check unitarity: The amplitude is gauge invariant. Summing over the polarizations $\sum \epsilon^{\mu}\epsilon^{\nu}=\eta^{\mu\nu}$ gives the amplitude

$$A \sim \alpha' \frac{(k_1-k_2)(k_3-k_4)}{2k^2} = \frac{u-t}{2s}.$$

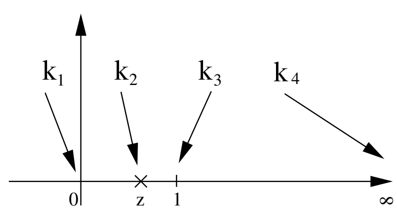
All the poles in $\alpha' s$: $\alpha' s = -1, 0, 1, 2, ...$ which are the masses of the open string states. ($\alpha' s^2 = N-1$ from $L_0 - 1 = 0$). Curious Result: same structure of poles we obtain for $\alpha' t$, since the amplitude is symmetric in s and t. This would also be true of a field theory amplitude.
Alternate derivation of the poles: It is instructive to find the poles without performing the integral for two reasons. (a) We can not always do the integral. (b) We can see what type of world-sheet contributes to the pole (physical picture for an effective field theory).
Let $z \to 0$

$$A \sim \int_0 dz z^{-\alpha' s-2} + ... = \left.\frac{z^{-\alpha' s=1}}{-\alpha' s-1}\right|_0 + ... = -\frac{1}{\alpha' s+1} + analytic.$$

Taylor expansion:

$$A \sim \int_0 z^{-\alpha' s-2}(1-z)^{-\alpha' t-2} = \int_0 dz z^{-\alpha' s-2}(1+(\alpha' t+2)z+...),$$

where the first and second terms in the expansion represent the $\alpha' s = -1$ and $\alpha' s = 0$ poles respectively. The other poles are acquired through higher order terms in the expansion. Poles in $\alpha' t$ are obtained from $z \to 1$.



There is no reason to restrict $\int dz$ to $\int_0^1 dz$. We would like to extend the integral to $\int_{\infty}^{\infty} dz$. The integral becomes $\int_{-\infty}^{0} + \int_0^1 + \int_1^{\infty}$.
$\int_{-\infty}^{0}$: ordering $(k_1+k_2+k_3+k_4)$ which is $\int_0^1$ with $k_2 \leftrightarrow k_1$. The effect is switching t and u. This can be seen through the transformation $z \mapsto 1-\frac{1}{z}$ which maps $(0,1) \mapsto (-\infty, 0)$. Therefore, if $\int_0^1 = I(s,t)$ then $\int_{-\infty}^{0} = I(t,u)$. Similarly, $\int_1^{\infty} = I(s,u)$. Therefore, the integral becomes

$$\int_{-\infty}^{\infty} = I(s,t) + I(s,u) + I(t,u).$$

Now the amplitude is completely symmetric in s, t, u.

BRST invariance

If $V(z)$ has weight $h=1$, then $\int dzV(z)$ has weight $h=0$. It is BRST invariant. Let us check this.

$$\begin{aligned}[Q,V(z)] &= \sum c_{-n}[L_n,V(z)] = \sum c_{-n}(z^{n+1}\partial V(z)+(n+1)z^nV(z)) \\ &= c(z)\partial V(z)+\partial c(z)V(z) = \partial(c(z)V(z))\end{aligned}$$

Therefore

$$[Q,\int V] = \int \partial(c(z)V(z)) = 0.$$

What happens with the three Vs that we fixed? To turn them into $h=0$ operators, we multiply them by $c(z)$. Then $c(z)V(z)$ has the weight $h=0$.

$$\{Q,cV\} = \{Q,c\}V - c[Q,V] = c\partial cV - cc\partial V - c\partial cV = 0.$$

Now in the amplitude, we have three $c(z)$s, $z_i = 0,1,\infty$. The amplitude must be defined with respect to the SL(2, $\mathbb{R}$) invariant vacuum. Recall:

$$b_0|\psi\rangle = 0, \quad |\chi\rangle = c_0|\psi\rangle.$$

$$L_m^{bc} = \sum_n (2m-n) : b_n c_{m-n} : -\delta_{m,0}$$

So,

$$L_0^{bc} = \sum_n n : b_{-n}c_n : -1,\ L_1^{bc} = \sum_n (2-n) : b_n c_{-n} :,\ L_{-1}^{bc} = \sum_n (-2-n) : b_n c_{-n-1} : .$$

The operators act on the states

$$L_0^{bc}|\psi\rangle = -|\psi\rangle, \quad L_1^{bc}|\psi\rangle = 0, \quad L_{-1}^{bc}|\psi\rangle = b_{-1}|\chi\rangle,$$

So $|psi\rangle$ is *not* invariant. Let $|0\rangle = b_{-1}|\psi\rangle$.

$$[L_0^{bc},b_{-1}] = b_{-1},\ [L_1^{bc},b_{-1}] = 2b_0,\ [L_{-1}^{bc},b_{-1}] = 0.$$

Therefore,

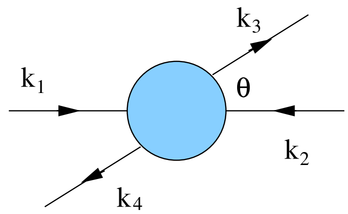
$$L_0^{bc}|0\rangle = b_{-1}|\psi\rangle - b_{-1}|\psi\rangle = 0,\ L_{-1}^{bc}|0\rangle = b_{-1}b_{-1}|\psi\rangle = 0,\ L_{-1}^{bc}|0\rangle = b_{-1}b_{-1}|\chi\rangle = 0.$$

So, $|0\rangle$ *is* SL(2, $\mathbb{R}$) invariant.
The ghost contribution is

$$\langle 0|c(\infty)c(1)c(0)|0\rangle, \quad c(z) = \sum_n c_n z^{-n+1}.$$

$$c(0)|0\rangle = c_1|0\rangle = |\psi\rangle, \quad \langle 0|c(\infty) = \langle\psi|, \quad \psi|c(1)|\psi\rangle = \langle\psi|c_0|\psi\rangle = 1.$$

High Energy



$$k_1^\mu = (E/2, \vec{p}), \;\; k_2^\mu = (E/2, -\vec{p}), \;\; k_3^\mu = (-E/2, -\vec{p'}), \;\; k_4^\mu = (-E/2, \vec{p'}).$$

where $\left(\frac{E}{2}\right)^2 - \vec{p}^2 = m^2, \;\; |\vec{p'}| = p$. The Mandelstam variables become

$$s = -(k_1+k_2)^2 = E^2, \;\; t = -(k_1+k_3)^2 = (4m^2-E^2)\sin^2\frac{\theta}{2}, \;\; u = -(k_1+k_4)^2 = (4m^2-E^2)\cos^2\frac{\theta}{2}.$$

The high energy limit is equivalent to the small angle limit, where $s \to 0$ and t is fixed. The gamma function is approximated by

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}.$$

The amplitude is

$$A \approx \frac{\Gamma(-\alpha' s - 1)\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha' s - \alpha' t - 2} \approx \frac{s^{-\alpha' s - 1}}{s^{-\alpha' s - \alpha' t - 2}} e^{\alpha' t + 1}\Gamma(-\alpha' t - 1) \sim s^{\alpha' t + 1}\Gamma(-\alpha' t - 1).$$

This is the Regge behavior. At the poles $\alpha' t - 1 \sim -n$, the amplitude goes as $A \sim s^n$ which is the exchange of a particle of spin n.
For a fixed angle,θ=fixed: $s, t \to \infty, \;\; s/t =$ fixed. The amplitude becomes

$$\begin{aligned}
A &\sim \frac{s^{-\alpha' s - 1} t^{-\alpha' t - 1}}{(s+t)^{-\alpha' s - \alpha' t}} \sim \frac{s^{-\alpha' s} t^{-\alpha' t}}{u^{\alpha' u}} \sim e^{-\alpha'(s\ln s + t\ln t + u\ln u)} \\
&\approx e^{-\alpha'(s\ln(s/s) + t\ln(t/s) + u\ln(u/s))} \\
&\approx e^{-\alpha' s(\frac{t}{s}\ln\frac{t}{s} + \frac{u}{s}\ln\frac{u}{s}} \\
&\approx e^{-\alpha' s\left(-\sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2}\right)} \\
&\approx e^{-Cs}, \quad C > 0.
\end{aligned}$$

unlike in field theory, where the amplitude goes as $A \sim s^{-n}$. Therefore the underlying smooth extended object of size $\sqrt{\alpha'}$.

4.2 A Short Course in Scattering Theory

We define the $\langle in|$ state in the *real* infinite past ($t \to -\infty$), and the $|out\rangle$ state in the infinite future ($t \to \infty$). These states are both described by free particles. There is an isomorphism

$$|in\rangle = S|out\rangle, \quad S = \lim_{t\to\infty} e^{iHt/\hbar}..$$

To conserve probabilities, S must be unitary, $S^\dagger S = 1$ (c.f. unitarity of evolution operator, $U = e^{iHt/\hbar}$). The transition probability ($S = I + iT$) is

$$|\langle i_{-\infty}|f_\infty\rangle|^2 = |\langle i|T|f\rangle|^2,$$

where $|i>$ and $|f\rangle$ represent states in the same Hilbert space. We will discard the I because it represents $|i\rangle \to |i\rangle$ (forward scattering i.e., along the beam: undetectable).
Unitarity

$$S^\dagger S = I = I + i(T - T^\dagger) + T^\dagger T.$$

Therefore

$$\langle i|T|f\rangle - \langle i|T^\dagger|f\rangle^* = i\langle i|T^\dagger T|f\rangle.$$

Insert complete sets of physical states

$$\begin{aligned} \langle i|T|f\rangle - \langle i|T^\dagger|f\rangle^* &= i\sum_n \langle i|T^\dagger|n\rangle\langle n|T|f\rangle, \\ 2Im < i|T|f\rangle &= \sum_n < i|T|n\rangle\langle f|T|n\rangle^*. \end{aligned} \tag{4.2.1}$$

Viewed as a function of s, $\langle i|T|f\rangle$ has poles in s. Away from the pole, $\langle i|T|f\rangle$ is real, so the left hand side vanishes.
Near the pole, we obtain a behavior $\sim \frac{1}{s+m^2}$ (pole at $s = -m^2$). to find the imaginary part, first *regulate* the amplitude

$$\frac{1}{s+m^2} \to \frac{1}{s+m^2+i\epsilon}$$

for small ϵ. Then

$$Im\frac{1}{s+m^2} \to Im\frac{1}{s+m^2+i\epsilon} = \frac{-\epsilon}{(s+m^2)^2+\epsilon^2} = -\pi\delta(s+m^2).$$

Therefore, for

INSERT FIGURE HERE

the imaginary part is

INSERT FIGURE HERE

This is in agreement with unitarity.

4.3 N-point open-string tree amplitudes

$$Amp \sim \langle : e^{ik_1\cdot X}(z_1) : ... : e^{k_2\cdot X}(z_n) :\rangle = A(z_1, ..., z_n)$$

Consider

$$\partial_1 A(z_1, ..., z_n) \sim \langle \partial_{z_1} : e^{ik_1\cdot X}(z_1) : ... : e^{k_2\cdot X}(z_n) :\rangle$$

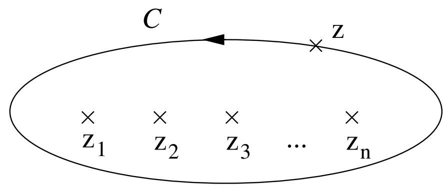
To evaluate this, consider the OPE

$$ik\cdot\partial X(z) : e^{ik_1\cdot X}(z_1) := \frac{\alpha' k_1^2}{2(z-z_1)} e^{ik_1\cdot X(z_1)} : +\partial_1 : e^{ik_1\cdot X}(z_1) : +...$$

So, first replace $\partial_1 : e^{ik_1\cdot X}(z_1) :$ by $\partial X(z) : e^{ik_1\cdot X}(z_1) :$ in Amp and define

$$f^\mu(z) = \langle \partial X^\mu(z) : e^{ik_1\cdot X}(z_1) : ... : e^{ik_n\cdot X}(z_n) :\rangle$$

The singularity structure of $f^\mu(z)$ can be deduced from OPEs



$$\partial X(z) : e^{ik_1\cdot X}(z_1) := -\frac{i\alpha' k_1^\mu}{2(z-z_1)} e^{ik_1\cdot X(z_1)} : +...$$

Therefore,

$$f^\mu(z) = -\frac{i\alpha'}{2} A(z) \sum_{i=1}^{n} \frac{k_1^\mu}{z-z_i} + ...$$

Behavior at $z\to\infty\ : z' = \frac{1}{z}$

$$\partial X^\mu = \frac{\partial z'}{\partial z}\partial' X^\mu = -\frac{1}{z^2}\partial' X^\mu$$

which implies

$$f^\mu(z) = -\frac{1}{z^2}\langle \partial' X^\mu + ...\rangle.$$

Therefore, as $z\to\infty$, $f^\mu(z) \sim \frac{1}{z^2} \to 0$ $(\langle \partial' X^\mu : ... :\rangle$ analytic at $\infty)$ Therefore the holomorphic piece vanishes and

$$f^\mu(z) = -\frac{i\alpha'}{2} A \sum_{i=1}^{n} \frac{k_i^\mu}{z-z_i}.$$

Now define a contour $\mathcal{C}$ surrounding all z_i's. There are two ways to evaluate the contour integral, $\oint \frac{dz}{2\pi i} f(z)$. Cauchy $\Rightarrow \oint \frac{dz}{2\pi i} f^\mu(z) = -\frac{i\alpha'}{2} A \sum_{i=1}^{n} \frac{k_i^\mu}{z-z_i}$, or in the transformed coordinate $\frac{z'=1}{z}$, $\mathcal{C}$ encircles $z'=0$ where $f^\mu(z)$ is analytic. Therefore

$$\oint \frac{dz}{2\pi i} f^\mu(z) = 0 \Rightarrow \sum_{i=1}^{n} k_i^\mu = 0$$

The momentum is conserved.
Now consider $ik \cdot f$ and compare with the OPE

$$ik \cdot \partial x(z) : e^{ik \cdot X}(z_1) := \frac{\alpha' k_1^2}{2(z - z_1)} : e^{ik_1 \cdot X}(z_1) : + \partial_1 : e^{ik_1 \cdot X}(z_1) : + \ldots$$

which implies

$$ik \cdot f = \frac{\alpha'}{2} \frac{k_1^2}{z - z_1} A + \frac{\alpha'}{2} \sum_{i \neq 1} \frac{k_1 \cdot k_i}{z - z_i} A$$

Therefore

$$\partial_1 A = \frac{\alpha'}{2} \frac{k_1^2}{z - z_1} A + \frac{\alpha'}{2} \sum_{i \neq 1} \frac{k_1 \cdot k_i}{z - z_i} A.$$

Therefore

$$\partial_1 \ln A = \frac{\alpha'}{2} \sum_{i \neq 1} \frac{k_1 \cdot k_i}{z_1 - z_i} A.$$

Repeating for other points,

$$\partial_j \ln A = \frac{\alpha'}{2} \sum_{i \neq j} \frac{k_j \cdot k_i}{zi - z_j} A.$$

By integrating we obtain $\ln A = \sum_{i<j} \ln |z_i - z_j|^{k_i \cdot k_j} + const$ where we added the $\bar{z}$ piece. Therefore,

$$A \propto \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j}.$$

For open strings, $\alpha' \to 2\alpha'$, so

$$A \propto \prod_{i<j} |z_i - z_j|^{2\alpha' k_i \cdot k_j}.$$

SL$(2,\ \mathbb{R})$ Invariance

$$z \to z' = \frac{az + b}{cz + d}, \qquad czz' + dz' = az + b \to z = \frac{dz' - b}{a - cz'}, \quad ad - bc = 1.$$

Therefore,

$$z_i - z_j = \frac{dz_i' - b}{a - cz_i'} - \frac{dz_j - b}{a - cz_j'} = \frac{z_i' - z_j'}{(a - cz_i')(a - cz_j')}. \tag{4.3.1}$$

Therefore,

$$A \propto \prod_{i<j} |z_i - z_j|^{2\alpha' k_i \cdot k_j} = \prod_{i<j} |z_i' - z_j'|^{2\alpha' k_i \cdot k_j} \prod (a - cz_i')^{2\alpha' k_i^2}, \quad k_i^2 = \frac{1}{\alpha'}. \tag{4.3.2}$$

$$dz_i = \frac{dz_i}{(a - cz_i')^2}.$$

If we let $z_j \rightarrow z_i$ in (4.3.1), we find that the amplitude is invariant under SL(2, $\mathbb{R}$) transformations. The measure is given by

$$\prod dz_i = \prod dz_i' \prod (a - cz_i')^{-2},$$

however, the last factor cancels with the overall factor in (4.3.2).

4.4 Closed Strings

For open strings we found four tachyons,

$$\begin{aligned} A_{open} &\sim \int_{-\infty}^{\infty} dz\, z^{2\alpha' k_3 \cdot k_4} (1-z)^{2\alpha' k_2 \cdot k_3} \delta^D(k_1 + k_2 + k_3 + k_4) \\ &= \int_{-\infty}^{0} + \int_0^1 + \int_1^{\infty} \end{aligned}$$

where

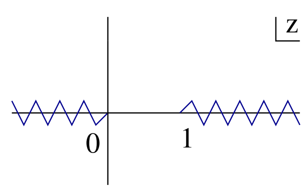
$$\int_0^1 = I(s,t) = \int_0^1 dz\, z^{-\alpha' s - 2} (1-z)^{-a' t - 2} \delta^D(k_1 + k_2 + k_3 + k_4)$$

$$\int_{-\infty}^{0} = I(t,u), \quad \int_1^{\infty} = I(s,u), \;\; z \in \mathbb{R}.$$

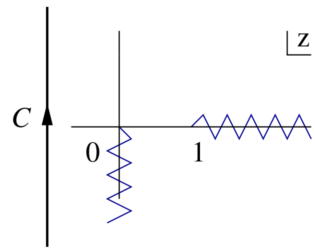
For closed strings, z is the entire $\mathbb{C}$ and we need to multiply the holomorphic and anti-holomorphic pieces, so

$$A_{closed} \sim \int d^2 z\, |z|^{-\alpha' s/2 - 4} |1 - z|^{-\alpha' t/2 - 4}.$$

Note: $s \rightarrow s/4$ is due to the different expansion of the $X^{\mu}s$. The tachyon mass is $m^2 = -\frac{4}{\alpha'}$, whereas for the open string it is, $m^2 = -\frac{1}{\alpha'}$.
To calculate the amplitude for the closed string, treat z and $\bar{z}$ as independent variables and deform the contour of integration until it coincides with the real axis. Then $z, \bar{z} \in \mathbb{R}$. We must take care with the branch cuts.
There are three cases.
(i) $\bar{z} < 0$: Contour for z:



$\mathcal{C}$ has branch cuts on the same side and therefore contributes nothing.
(ii) $\bar{z} > 1$:
There is no contribution for the same reason as in (i).
(iii) $0 < \bar{z} < 1$:



$$A_{closed} \sim \oint dz\ z^{-\alpha' s/4-2}(1-z)^{-\alpha' t/4-2} \times \int_0^1 d\bar{z}\ \bar{z}^{-\alpha' s/4-2}(1-\bar{z})^{-\alpha' t/4-2}$$

Contribution from the upper side of $\mathcal{C}$ is

$$\int_1^\infty d\eta |\eta|^{-\alpha' s/4-2} e^{-i\pi(\alpha' t/4+2)}|1-\eta|^{-\alpha' t/4-2} \times I(s/4, t/4).$$

The lower side gives

$$\int_1^\infty d\eta\ |\eta|^{-\alpha' s/4-2} e^{+i\pi(\alpha' t/4+2)}|1-\eta|^{-\alpha' t/4-2} \times I(s/4, t/4).$$

Therefore the amplitude for the closed string is

$$A_{closed} \sim \sin\frac{\pi\alpha' t}{4} I(t/4, u/4) I(s/4, t/4).$$

This can be cast in a symmmetric form by using the transformation properties of the Gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad for \frac{-\alpha' t}{4} - 1$$

So, since $s + t + u = 4m^2 = -16/\alpha'$

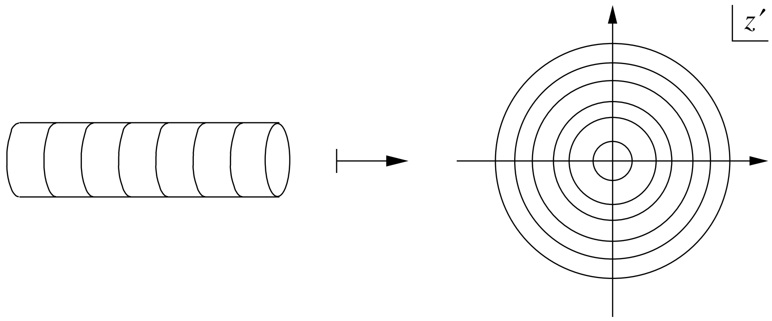
$$\Gamma(-\alpha' t/4 - 1)\Gamma(2 + \alpha' t/4) = \frac{\pi}{\sin(\alpha' t\pi/4)}.$$

Therefore the amplitude is given by

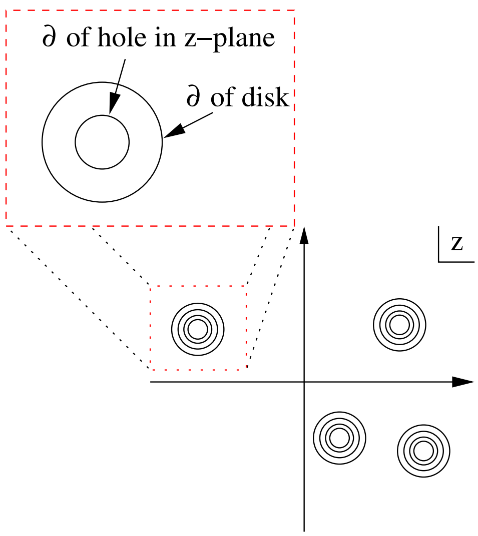
$$A_{closed} \sim \pi \frac{\Gamma(-\alpha' s/4 - 1)\Gamma(-\alpha' t/4 - 1)\Gamma(-\alpha' u/4 - 1)}{\Gamma(-\alpha' s/4 - \alpha' t/4 - 1)\Gamma(-\alpha' t/4 - \alpha' u/4 - 1)\Gamma(-\alpha' u/4 - \alpha' s/4 - 1)}.$$

4.5 Moduli

Build closed-string four-point amplitude as follows. In the z-plane, drill holes. This will represent the diagram on the left with amputated legs. Now attach the legs by telescopically collapsing each semi-infinite tube to a disc:



Next, patch the discs on the z-plane. This produces a sphere with four punctures. There will be regions of overlap where $z' = f(z)$.



By conformal transformations, I can fix three points (due to SL$(2,\mathbb{C})$ symmetry). The fourth point cannot be fixed. Call it z. Punctured spheres with two different z's, are **not** related by a conformal transformation. There are inequivalent surfaces.
They are parametrized by **two** parameters, z_1 and $\bar{z}_1 \in \mathbb{C}$. These parameters are called moduli and their space, moduli space (although it should be called modulus space) (c.f. vector space). They are also called Teichmuller parameters. They label conformally inequivalent surfaces. To calculate amplitudes, we need to integrate over the moduli.

E.g., the four-point amplitude, $\langle V_1(\infty)V_2(1)V_3(z_1)V_4(0)\rangle$ need to integrate over $z_1 \to \int d^2z_1$. In general, N-point amplitudes integrate $\int d^2z_1...d^2z_{N-3}$ at $z = \infty,\ 1,\ 0$ we specified $V \sim c\tilde{c} : e^{ik\cdot X}$:. We can do the same for the unfixed V's to put them all on equal footing.
Thus, let $V_i = c\tilde{c} : e^{ik_i\cdot X}$:, $\forall i$. Since we introduced an extra $c,\ \tilde{c}$, we need to compensate for it with a $b,\ \tilde{b}$ insertion.
To do this work as follows. Shift $z_1 \to z_1 + \delta z_1$. This is implemented in the z'-plane by a coordinate transformation

$$z' \to z' + \delta z_1 v^z(z', \bar{z}').$$

where v^z is of course **not** conformal (depends on z as well as $\bar{z}$). Introduce the Beltrami differential.

$$\psi = \partial_{\bar{z}} v^z$$

There is a similar differential for the complex conjugate

$$\bar{\psi} = \partial_z v^{\bar{z}}$$

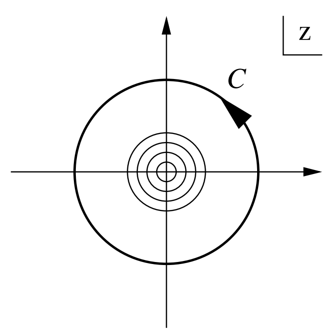
If v^z represents a conformal transformation, then $\psi,\ \bar{\psi} = 0$. Thus ψ encodes information about conformally inequivalent surfaces.
We will insert $\frac{1}{2\pi}\int d^2z'(p\psi + \tilde{b}\bar{\psi})\times$ anti-holomorphic in the amplitude. We integrate over the patch that we will use so

$$Amp \sim \int d^2z_1 \langle V_1V_2V_3V_4 \left(\frac{1}{2\pi}\int d^2z'(p\psi + \tilde{b}\bar{\psi}) \times (anti)\right)\rangle$$

Since $\partial_{\bar{z}} b = 0$, the integral is $\sim \int d^2z' \left(\partial_{\bar{z}}(bv^z) + \partial_z(\tilde{b}v^{\bar{z}})\right)$. Therefore it can be written as (divergence theorem)

$$B_1 = \frac{1}{2\pi i}\oint_{\mathcal{C}} \left(dz' b v^z - d\bar{z}'\tilde{b}v^{\bar{z}}\right)$$

where $\mathbb{C}$ is in the overlap region of z and z'.



Explicitly,

$$v^z = \frac{\partial z'}{\partial z_1}.$$

In the overlap region, $z = z' + z_1$, so $v^z + 1 = 0$, therefore $v^z = -1$. Therefore

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} = dz' b v^z = -b_{-1}, \quad b(z) = \sum b_n z^{-n-2}, c = \sum c_n z^{-n+1}.$$

$$\int dz_1 b_{-1} V_3 = \int dz_1 b_{-1} c\tilde{c} : e^{ik_3\cdot X}(z_1) := \tilde{c} : e^{ik_3\cdot X}(z_1)$$

where

$$b_{-1}c = \oint_{\mathcal{C}} dz' b(z') c(z_1) = 1.$$

$$\int dz_1 b_{-1}\tilde{c} : e^{ik_3\cdot X}(z_1) :=: e^{ik_3\cdot X}(z_1) :$$

so the b-insertions kill $c\tilde{c}$ from V_3 and the amplitude is as before.

4.6 BRST Invariance

$$\{Q_B, B_1\} = \frac{1}{2\pi i}\oint_c dz' \left(T v^z - d\bar{z}\bar{T} v^{\bar{z}}\right)$$

Recall

$$T(z')V_3(z_1) = \frac{h}{(z'-z_1)^2} + \frac{1}{z'-z_1}\partial V_3 + ..., \quad h = 0!$$

Therefore $\{Q_B, B_1\}V_3 \sim \oint dz' \partial V_3 = 0$ unless the moduli space has ∂ (not true here, but argument is general and sometimes $\partial \neq 0$).