String Theory I

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UNIT 3

BRST Quantization

BRS&T: Becchi-Ronet-Stora & Tyutin.

3.1 Point particle

Recall

$$S = \frac{1}{2}\int d\tau \left(\frac{1}{\eta}\dot{X}^\mu \dot{X}_\mu - \eta m^2\right) = \int d\tau \, (p_\mu \dot{x}^\mu - \eta\chi)$$

where

$$\chi = \frac{1}{2}(p^\mu p_\mu + m^2).$$

The constraint $\chi = 0$ generates the transformation

$$\delta X^\mu = \epsilon\{X^\mu, \chi\} = \epsilon p^\mu, \ \ \delta p^\mu = 0.$$

Quantization: $|\vec{k}\rangle, \ \ H = p_0 = \sqrt{\vec{p}^2 + m^2}, \ \ H|\vec{k}\rangle = \sqrt{k^2 + m^2}|\vec{k}\rangle$. where

$$\omega = \sqrt{k^2 + m^2}$$

is the dispersion relation.
Sexier approach: Let ϵ be anticommuting, say $\epsilon \to \epsilon c$, where ϵc are commuting and anticommuting respectively. Promote c to coordinate status. Let b be its conjugate momentum, so

$$S_{bc} = \int d\tau b\dot{c}.$$

The action $S' = S + S_{bc} = \int d\tau (p_\mu \dot{x}^\mu + b\dot{c})$ is invariant under

$$\begin{aligned} \delta_B X^\mu &= \epsilon c p^m u, \ \ \delta_B p^\mu = 0 \\ \delta_B b &= -\epsilon(\chi - m^2) \end{aligned}$$

Check:

$$\delta S' = \int d\tau \left[\epsilon(c\dot{p}^{\mu})p_{\mu} - \epsilon(\chi - m^2)\dot{c}\right] = \int d\tau \epsilon \frac{d}{d\tau}(\frac{1}{2}c(p^{\mu}p_{\mu} + m^2)) = 0.$$

Generated by $Q_B = c\chi$ Nilpotent:

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = 0$$

Quantization: $\{b, c\} = 1, \ \ [P_{\mu}, X^{\nu}] = -i\eta^{\mu\nu}$. The b, c theory is much like b_0, c_0 in strings. States $|\psi\rangle, \ |\chi\rangle$

$$b_0|\psi\rangle = 0, \ \ c_0|\chi\rangle, \ \ c_0|\psi\rangle = |\chi\rangle, \ \ b_0|\chi\rangle = |\psi\rangle.$$

Include momentum, $|k\rangle \otimes |\psi\rangle = |k, \psi\rangle \ \ (p^{\mu}|k\rangle = k^{\mu}|k\rangle)$

$$Q_B|k,\psi\rangle = \frac{1}{2}(k^2 + m^2)|k,\chi\rangle, \quad Q_B|k,\chi\rangle = 0,$$

where $|k,\psi\rangle$ is closed for $k^2 + m^2 = 0$ and $|k,\chi\rangle$ is closed. Set of closed states $|k,\psi\rangle, \ \ k^2 + m^2 = 0$ is a set of physical states, in agreement with the analysis above (gauge fixed).

3.2 Strings

Recall $S = \frac{1}{2\pi\alpha'}\int d^2z \partial X^{\mu}\bar{\partial}X_{\mu}$. Constraint $T = \frac{1}{\alpha'}\partial X^{\mu}\partial X_{\mu} = 0$ (and similarly for $\bar{T}$) generates conformal transformations

$$\delta X^{\mu} = \epsilon v \partial X^{\mu}$$

Now make v anticommuting, $v \to c$, then b conjugate momentum bc system we already studied.

$$S_{bc} = \frac{1}{2\pi}\int d^2z b\bar{\partial}c.$$

Let us guess that $S' = S + S_{bc}$ is invariant under the transformations

$$\delta_B X^{\mu} = i\sigma c\partial X^{\mu}, \quad \delta_B b = i\epsilon T...$$

This does not quite work. We need $T \to T + T_{bc}$ and then $\delta_B c \neq 0$. The correct transformations are

$$\delta_B X^{\mu} = i\sigma c\partial X^{\mu}, \quad \delta_B b = i\epsilon(T + T_{bc}), \quad \delta_B c = i\epsilon c\partial c.$$

Then $\delta S' = 0$. The corresponding Noether current is

$$j_B = cT + \frac{1}{2} : cT_{bc} : + \frac{3}{2}\partial^2 c.$$

Require: j_B have weight $h = 1$, so the charge $Q_B = \oint \frac{dz}{2\pi i} j_B$ has $h = 0$ (conformally invariant scalar operator). If we look at the cT part in j_B we see $h_T = 2$ therefore $h_c = -1$. Therefore the bc system must have $\lambda = 2$ and $h_b = 2$.

3.3 Mode Expansion

$$Q_B = \sum_n c_n L_{-n} + \frac{1}{2}\sum_{m,n}(m-n) : c_m c_n b_{-m-n} : -c_0 = \sum_n : c_n \left(L_{-n} + \frac{1}{2} L^{bc}_{-n} - \delta_{n0} \right) :$$

where the minus sign in front of c_0 comes from $\frac{l(1-\lambda)}{2} = -1$ and is in disagreement between mode and conformal normal ordering.

3.4 Nilpotency

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = \frac{1}{2}\sum \left([L_m^{TOT}, L_n^{TOT}] - (m-n)L_{m+n}^{TOT} \right) c_{-m} c_{-n}$$

where $L_m^{TOT} = L_m + L_m^{bc} - \delta_{m,0}$. The right hand side is $\frac{1}{12}(D + c_{bc})(m^3 - m)$ where $c_{bc} = 1 - 3(2\lambda - 1)^2 = -26$. Our BRST charge is then

$$Q_B^2 \sim \frac{1}{12}(D - 26) = 0$$

if and only if $D = 26$. Conversely, suppose $Q_B^2 = 0$. Define $L_m^{TOT} = \{Q_B, b_m\}$. Then

$$\begin{aligned} [L_m^{TOT}, L_n^{TOT}] &= [L_m^{TOT}, \{Q_B, b_m\}] = \{Q_b, [L_m, b_n] \\ &= \{Q_B, (m-n)b_{m+n}\} = (m-n)L_{m+n}. \end{aligned}$$

Physical states are annihilated by Q_B, $(Q_B|phys\rangle = 0)$. Note that $|phys\rangle + Q_B|\psi\rangle$ is also physical. they represent the same system (like A_μ and $A_\mu + \partial_\mu\psi$ in QED). Therefore, $|phys\rangle$=equivalence class $|A\rangle + Q_B|\psi\rangle$. Cohomology of the conformal group
Note:

$$(\langle A| + \langle\psi|Q_B)(|B\rangle + Q_B|\psi\rangle) = \langle A|B\rangle$$

for physical $|A\rangle$, $|B\rangle$ $(Q_B|A\rangle = Q_B|B\rangle = 0)$ where we assume $Q_B^\dagger = Q_B$. In particular, $\langle\psi|Q_B|B\rangle = 0$ therefore $\langle\psi|Q_B = 0$.

3.5 A note on BRST cohomology

Given a group with symmetry group G generated by the algebra

$$[L_i, L_j] = i f_{ij}^k L_k.$$

Introduce ghosts b_i, c^i such that

$$\{c^i, b_j\} = \delta^i_j, \ \ \{c^i, c^j\} = \{b_i, b_j\} = 0.$$

Define the BRST charge

$$\begin{aligned} Q_B &= c^i L_i - \frac{i}{2} f^k_{ij} c^i c^j b_k \\ &= c^i \left(L_i + \frac{1}{2} L^{bc}_i \right), \quad L^{bc}_i = -i f^k_{ij} c^j b_k \end{aligned}$$

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = i c^i c^j f^k_{ij} L_k - i f^k_{lm} c^l c^m \{c^i, b_k\} L_i - \frac{1}{2} f^k_{ij} f^m_{kl} c^i c^j c^l b_m = 0.$$

due to the Jacobi identity

$$\begin{aligned} [[L_i, L_j], L_k] + [[l_j, L_k], L_i] + [[L_k, L_i], L_j] &= 0 \\ i f^m_{ij}[L_m, L_k] + i f^m_{jk}[L_m, L_i] + i k^m_{ki}[L_m, L_j] &= 0 \\ -f^m_{ij} f^l_{mk} L_l - f^m_{jk} f^l_{mi} L_l - f^m_{ki} f^l_{mj} L_l &= 0 \end{aligned}$$

For strings, $[L_m, L_n] = (m-n)L_{m+n}$ and $f^k_{mn} = (m-n)\delta_{k,m+n} \quad c^m = c_{-m}$, so

$$Q_B = c_{-m} L_m - \frac{1}{2}(m-n) c_{-m} c_{-n} b_{m+n}.$$

3.6 BRST Cohomology for open strings

Open strings are easier than closed strings, but they are entirely similar. We introduce a vacuum $|\psi\rangle$ such that $b_0|\psi\rangle = 0$ and $|\chi\rangle = c_0|\psi\rangle$. Then $\langle\psi|\psi\rangle = 0$, but $\langle\chi|\psi\rangle \neq 0$ so, we define the inner product by $\langle\psi|c_0|\psi\rangle$.
Let $|\psi\rangle \otimes |k\rangle = |\psi; k\rangle, \quad \langle k|k'\rangle = (2\pi)^D \delta^D(k - k')$. Physical states will be constructed from $|\psi\rangle$, so $b_0|phys\rangle = 0$. Then $L_0|phys\rangle = \{Q_B, b_0\}|phys\rangle = 0$.

$$L_0 = \alpha' p^2 + \sum_n n b_{-n} c_n + \sum_n \alpha^\mu_{-n} \alpha_{n\mu} - 1$$

$$\hat{H} = \{|\psi\rangle, \ b_0|\psi\rangle = 0, \ L_0|\psi\rangle = 0\}$$

$$Q_B|\psi\rangle = |Z\rangle, \ b_0|Z\rangle = L_0|\psi\rangle = 0, \ L_0|Z\rangle = [L_0, Q_B]|\psi\rangle = 0.$$

Therefore $Q_B : \hat{H} \to \hat{H}$. In $\hat{H}$, $|k\rangle$ is specified by $|\vec{k}\rangle$, because k^0 is given in terms of $\vec{k}$ through $L_0 = 0$. Therefore we can define the inner product

$$\langle\vec{k}|\vec{k}'\rangle = 2k^0 (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}').$$

which is a Lorentz invariant defintion.
Example: $|\psi; \vec{k}\rangle$

$$L_0|\psi; \vec{k}\rangle = (\alpha' p^2 - 1)|\psi; \vec{k}\rangle = 0 \Rightarrow k^2 = \frac{1}{\alpha'}.$$

$$Q_B|\psi; \vec{k}\rangle = 0, \quad |\psi; \vec{k}\rangle \neq Q_B|Z\rangle$$

Therefore $|\psi;\vec{k}\rangle$ are all the cohomology classes. Same as in the light-cone quantization.
Example: $|\psi\rangle = \left(A_\mu(\vec{k})\alpha_{-1}^\mu + \beta(\vec{k})b_{-1} + \gamma(\vec{k})c_{-1}\right)|\psi;\vec{k}\rangle$

$$\langle\psi|\psi\rangle = (A_\mu^* A^{\mu *} + \beta^*\gamma + \gamma^*\beta)\langle\psi;\vec{k}|\psi;\vec{k}\rangle$$

There are 26 positive-norm states: $A_i, \beta = \gamma$ $(\alpha_{-1}^i|\psi;\vec{k}\rangle,\ (b_{-1}+c_{-1})|\psi;\vec{k}\rangle)$, 2 negative-norm states: $A_0,\ \beta = -\gamma,\ (\alpha_{-1}^0|\psi;\vec{k}\rangle,\ (b_{-1}-c_{-1})|\psi;\vec{k}\rangle)$.

$$Q_B|\psi\rangle = 0 \Rightarrow (c_{-1}k\cdot\alpha_1 + c_1 k\cdot\alpha_{-1})|\psi\rangle = 0 \Rightarrow (k_\mu A^\mu c_{-1} + \beta k_\mu \alpha_{-1}^\mu)|\psi;\vec{k}\rangle = 0$$

Therefore $k\cdot A = 0$ and $\beta = 0$. This gets rid of negative-norm states $k^0 A_0 \neq 0$ for all $k^0 \neq 0$ and $\beta = \gamma = 0$ is the other negative-norm state. 26 states remain: 2 have zero-norm:

$$k_\mu \alpha_{-1}^\mu|\psi;\vec{k}\rangle, \quad c_{-1}|\psi;\vec{k}\rangle.$$

They are orthogonal to all physical states $\langle ...|\psi\rangle = 0$.
$c_{-1}|\psi;\vec{k}\rangle$ is **exact**.
Proof: Let $|Z\rangle = \tilde{A}_\mu \alpha_{-1}^\mu|\psi;\vec{k}\rangle,\ \ k\cdot\tilde{A} \neq 0$. Then $Q_B|Z\rangle = k\cdot\tilde{A}c_{-1}|\psi;\vec{k}\rangle$. Therefore $c_{-1}|\psi;\vec{k}\rangle = \frac{1}{k\cdot\tilde{A}}Q_B|Z\rangle$.
$k\cdot\alpha_{-1}|\psi;\vec{k}\rangle$ is **exact**.
Proof: Let $|Z\rangle = b_{-1}|\psi;\vec{k}\rangle$, then $Q_B|Z\rangle = k\cdot\alpha_{-1}|\psi;\vec{k}\rangle$. In each BRST cohomolgy class there is a gauege equivalence $A^\mu = A^\mu + \alpha k^\mu$.

No-Ghost Theorem

In the light-cone gauge, we considered the space $\mathcal{H}^\perp$ is a Hilbert space. **No-Ghost Theorem:** BRST cohomology is isomorphic to $\mathcal{H}^\perp$.
Definition: $\alpha_m^\pm = \frac{1}{\sqrt{2}}(\alpha_m^0 \pm \alpha_m^1)$.
Therefore

$$[\alpha_m^+, \alpha_n^-] = -m\delta_{m+n,0}, \quad [\alpha_m^+, \alpha_n^+] = [\alpha_m^-, \alpha_n^-] = 0.$$

Q_1 is the part of Q proportional to the α_{-m}^- oscillators.

$$Q_1 = -\sqrt{2\alpha'}k^+ \sum_m a_{-m}^+ c_m.$$

$$L_m = \alpha_0^+\alpha_m^- + \sum_n \alpha_n^+\alpha_{m-n}^- + \frac{1}{2}\sum_n \alpha_n^i \alpha_{m-n}^i, \ \ Q = \sum_m L_{-m}c_m + ...$$

$$Q_1^2 = 0.$$

Definition: $R = \frac{1}{\sqrt{2\alpha'}k^+}\sum_m a_{-m}^+ b_m$

$$S = \{Q_1, R\} = \sum_m (mb_{-m}c_m + mc_{-m}b_m - \alpha_{-m}^+\alpha_m^- - \alpha_{-m}^-\alpha_m^+)$$

N.B.:

$$[Q_1, S] = [Q, \{Q, R\}] = Q_1 R Q_1 - Q_1 R Q_1 = 0$$

Theorem: $|\psi\rangle \in Large(S) \cup Kernel(Q_1) \Rightarrow |\psi\rangle$ is **exact**.
Proof: $|\psi\rangle = S|Z\rangle, \quad Q_1|\psi\rangle = Q_1|Z\rangle = 0$. Therefore, $|\psi\rangle = \{Q, R\}|Z\rangle = Q_1 R|Z\rangle$.
Corollary: Cohomology of $|\psi\rangle$ non-trivial only if $|\psi\rangle \in ker(S)$, i.e., $S|\psi\rangle = 0$.
Now $|\psi\rangle \in \mathcal{H}^\perp \Rightarrow Q_1|\psi\rangle = 0$ (trivial from the definition of Q_1). From the definition of S in terms of the oscillators, $S|\psi\rangle = 0$ only if $|\psi\rangle$ has no $\alpha^\pm_{-m},\ b_{-m},\ c_{-m}$ excitations, therefore $|\psi\rangle \in \mathcal{H}^\perp$. Therefore the Q_1 cohomology is $\mathcal{H}^\perp$.
The no-ghost theorem for Q_1: Let us go back to Q_B.
Define $U = \{Q_B - Q_1, R\} = \{Q_B, R\} - S$, therefore $\{Q_B, R\} = S + U$. Map $|\psi\rangle \in ker(S) \mapsto |Z\rangle = (1 - S^{-1}U + S^{-1}US^{-1}U + ...)|\psi\rangle$. S^{-1} makes sense for all $U|\psi\rangle$ contains $\alpha^\pm_{-m}$ excitations therefore, $U|\psi\rangle \notin ker(S)$ Clearly, $(S+U)|\psi\rangle = S|\psi\rangle = 0$. This establishes an isomorphism

$$ker(S) \cong ker(S+U).$$

We can show the cohomology of $Q_B \cong ker(S+U)$ just like we showed cohomology of $Q_1 \cong ker(S)$. Therefore, $coh(Q_B) \cong ker(S+U) \cong ker(S) \cong coh(Q_1) \cong \mathcal{H}^\perp$. Q.E.D.
Inner products: $\langle Z_1|Z_2\rangle = \langle\psi_1|\psi_2\psi\rangle$ (positive definite).