String Theory I

GEORGE SIOPSIS AND STUDENTS

Department of Physics and Astronomy
The University of Tennessee
Knoxville, TN 37996-1200
U.S.A.
e-mail: siopsis@tennessee.edu

Last update: 2006

Contents

UNIT 2

Conformal Field Theory

2.1 Massless scalars in two dimensions

We will start by looking at the Polyakov action with one change. The metric has been replaced with a Euclidean metric $\delta_{a,b}$ with signature (+,+).

$$S = \frac{T}{2}\int d\tau d\sigma(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu)$$

where T, $\dot{X}^\mu$, X'^μ represent the string tension, $\partial_\tau X^\mu$, $\partial_\sigma X^\mu$ respectively.
We can derive the equation of motion by varying the action with respect to the coordinate X. We find:

$$\delta_X S = 0 \rightarrow X''_\mu + \ddot{X}^\mu = \nabla^2 X^\mu = 0, \qquad \text{where} \qquad \nabla^2 = \partial^2_{\sigma_1} + \partial^2_{\sigma_2}$$

We can define z and $\bar{z}$ as linear combinations of σ_1 and σ_2. These will represent the new worldsheet coordinates

$$z = \sigma_1 + i\sigma_2, \qquad \bar{z} = \sigma_1 - i\sigma_2.$$

The bar denotes complex conjugate. We can also invert the coordinate transformation

$$\sigma_1 = \frac{z+\bar{z}}{2}, \qquad \sigma_2 = \frac{z-\bar{z}}{2}.$$

Define differentiation:

$$\partial_z = \partial = \frac{1}{2}(\partial_1 + i\partial_2) \qquad \text{and} \qquad \partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_1 - i\partial_2)$$

∇^2 can be written as $4\partial\bar{\partial}$, and the volume element is given by

$$d^2z = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} d\sigma d\tau = 2d\sigma d\tau.$$

Also define

$$\int d^2z\ \delta^2(z,\bar{z}) = 1$$

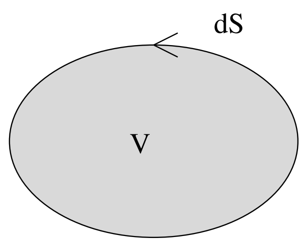
so that $\delta(\sigma^1)\delta(\sigma^2) = \delta^2(z,\bar{z})$. The action in complex coordinates is

$$S = T\int d^2z \partial X^\mu \bar{\partial} X_\mu\ .$$

After varying the action with respect to the coordinate X_μ, we get the equation of motion

$$\partial\bar{\partial}X^\mu = 0.$$

This implies $\bar{\partial}X$ is a holomorphic function, ie a function of z. Also, ∂X is an antiholomorphic function, ie a function of $\bar{z}$.
Another useful result is the divergence theorem for complex coordinates. First, let's look at the divergence theorem for three dimensions i.e., electrostatics. The divergence theorem states that for any well behaved vector field $\mathbf{E}(\mathbf{x})$ defined within a volume V surrounded by the closed surface S the relation



$$\int_V d^3x\ \boldsymbol{\nabla}\cdot\boldsymbol{E} = \oint_S \boldsymbol{E}\cdot\boldsymbol{n}\ da$$

holds between the volume integral of the divergence of $\mathbf{E}$ and the surface integral of the outwardly directly normal component of $\mathbf{E}$.
In 2D:

$$\int d^2z\ (\partial E^z + \bar{\partial}E^{\bar{z}}) = i\oint_{\partial R}(E^z dz - E^{\bar{z}}d\bar{z}) \qquad \text{where} \qquad \hat{n} = (-d\bar{z}, dz).$$

We can now write the mode expansions in terms of the complex coordinates.

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{z-\bar{z}}{2}\frac{p^\mu}{p^+} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\{\alpha_n^\mu e^{\frac{2n\pi i z}{\ell}} + \tilde{\alpha}_n^\mu e^{\frac{-2n\pi i\bar{z}}{\ell}}\}$$

As one can see, this mode expansion can be broken into two pieces (left and right handed).

$$X^\mu(z,\bar{z}) = X_L^\mu(z) + X_R^\mu(\bar{z})$$

We see the left handed piece cooresponds to a holomorphic function and the right handed piece to a antiholomorphic function. We will never look at the mode expansion itself. Instead we will always look at various derivatives of the mode expansion.

$$\partial X_L^\mu = \frac{i}{2}\frac{p^\mu}{p^+} - \frac{\pi}{\ell}\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu \exp\left[\frac{in\pi z}{\ell}\right],$$

$$\bar{\partial} X_R^\mu = \frac{i}{2}\frac{p^\mu}{p^+} - \frac{\pi}{\ell}\sqrt{2\alpha'}\sum_{n\neq 0}\tilde{\alpha}_n^\mu \exp\left[\frac{in\pi \bar{z}}{\ell}\right]$$

We can absorb the p^μ by defining α_0^μ a certain way.

$$\text{Let } \alpha_0^\mu = -i\frac{\ell}{2\pi\sqrt{2\alpha'}}\frac{p^\mu}{p^+}$$

Now the derivatives on the fields simplify.

$$\partial X_L^\mu = -\frac{\pi}{\ell}\sqrt{2\alpha'}\sum_{all\ n}\alpha_n^\mu \exp\left[\frac{in\pi z}{\ell}\right],\ \ \bar{\partial} X_R^\mu = -\frac{\pi}{\ell}\sqrt{2\alpha'}\sum_{all\ n}\tilde{\alpha}_n^\mu \exp\left[\frac{in\pi \bar{z}}{\ell}\right]$$

2.2 Solution to the Boundary-Value Problem with Green function

The solution to the Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on the bounding surface S can be obtained by means of Greens theorem. In general, we want to solve the equation,

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -\mathbf{4}\pi\delta(\mathbf{x}-\mathbf{x}')$$

where G(x,x') is the potential and the delta function is a point source. The solution for G is given as

$$G(\mathbf{x},\mathbf{x}') = \frac{\mathbf{1}}{|\mathbf{r}-\mathbf{r_0}|} + \mathbf{F}(\mathbf{x},\mathbf{x}')$$

with F satisfying the Laplace equation inside the volume V:

$$\nabla'^2 F(\mathbf{x},\mathbf{x}') = \mathbf{0}$$

In two dimensions the Poisson equation is given by

$$\begin{aligned}\nabla^2 G(z,0;\bar{z},0) &= -2\pi\delta(z)\delta(\bar{z}) \qquad \text{where} \qquad \nabla^2 = \partial\bar{\partial}\\ &= -2\pi\delta^2(z,\bar{z})\end{aligned}$$

where

$$G(z,\bar{z}) = \ln|z|^2 = \ln|z| + \ln|\bar{z}|$$

This is just the solution for the potential to a line charge in two dimensions. We can prove that G is a solution of the Poisson equation in problem 1.

$$\partial\bar{\partial}G = \partial\bar{\partial}\ln|z|^2 = \partial\frac{1}{\bar{z}} + \bar{\partial}\frac{1}{z} = 2\pi\delta^2(z,\bar{z}), \quad z = 0$$

Now that we see G is directly related to the potential, we can take the gradient to get the electric field.

$$E_z = \partial G, \quad E_{\bar{z}} = \bar{\partial}G$$

$$\int_V d^3x\ \nabla\cdot E = \oint_S E\cdot d\hat{n} = 2\pi R(\frac{1}{R}) = 2\pi$$

2.3 Amplitudes

We are interested in calculating vacuum expectation values

$$\langle 0|X^{\mu_1}(z_1)X^{\mu_2}(z_2)\cdots X^{\mu_n}(z_n)|0\rangle.$$

Let us start with the simplest nontrivial example, the two-point amplitude.

The two-point amplitude

We will focus only on the left-movers.

$$X^\mu = X^\mu_L = i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n \exp\left[\frac{-2\pi i n z}{\ell}\right].$$

$$\begin{aligned} A^{\mu\nu} &= \langle 0|X^\mu(z)X^\nu(z')|0\rangle \\ &= -\frac{\alpha'}{2}\sum_{n,m\neq 0}\frac{1}{nm}\exp\left[\frac{-2\pi i(n-m)z}{\ell}\right]\langle 0|\alpha^\mu_n\alpha^\nu_m|0\rangle \end{aligned}$$

In order evaluate the vacuum expectation value, break the sum into four pieces: $m,n>0$; $m>0,n<0$; $m<0,n>0$; $m,n<0$. Only the $m<0,n>0$ terms survive, because the others either kill the vacuum or produce an inner product of orthogonal states. To evaluate, we express the operators in terms of their commutator. This is possible because the second term in the commutator kills the vacuum.

$$[\alpha^\mu_n,\alpha^\nu_m] = n\eta^{\mu\nu}\delta_{m+m,0}.$$

After applying the Kroneker delta our sum reduces to

$$A^{\mu\nu} = \frac{\alpha'}{2}\sum_{n>0}\frac{1}{n}\exp\left[-\frac{2\pi i n(z-z')}{\ell}\right].$$

This is a sum of the form $\sum \frac{\omega^n}{n}$, but only converges for $|\omega| < 1$.

$$\left|\exp\left[-\frac{2\pi i(z-z')}{\ell}\right]\right| = \left|\exp\left[\frac{2\pi(z-z')}{\ell}\right]\right| < 1 \Rightarrow z_2 - z_2' < 0, z_2 < z_2'$$

So we see the Xs must be time ordered to give a correct result. Therefore, when we calculate any two-point function, we must use the time ordered product of the two operators

$$\langle 0|X(z)X(z')|0\rangle \Rightarrow \langle 0|T[X(z)X(z')]|0\rangle.$$

We define time ordering as

$$T[X^\mu(z)X^\nu(z')] = \begin{cases} X^\mu(z)X^\nu(z') & \text{for } z_2 < z_2' \\ X^\nu(z)X^\mu(z') & \text{for } z_2 > z_2' \end{cases},$$

or in terms of the step function the product becomes

$$T[X^\mu(z)X^\nu(z')] = \theta(z_2' - z_2)X^\mu(z)X^\nu(z') + \theta(z_2 - z_2')X^\nu(z)X^\mu(z').$$

Evaluating the sum:

$$A^{\mu\nu} = \frac{\alpha'}{2}\eta^{\mu\nu}\sum_n \frac{e^n}{n} = \frac{\alpha'}{2}\eta^{\mu\nu}\ln|1-e^{-\beta}|$$

By applying the D'Alembertian to $A^{\mu\nu}$, we show it is a two-point Green function

$$\begin{aligned}
\partial\bar{\partial}T[X^\mu(z)X^\nu(z')] &= (\partial^2 + \bar{\partial}^2)T[X^\mu(z)X^\nu(z')] \\
&= T[\partial_1^2 X^\mu X^\nu] + T[\partial_1(-\delta(z_2 - z_2')X^\mu X^\nu + \delta(z_2 - z_2')X^\mu X^\nu)] \\
&= T[\partial_1^2 X^\mu X^\nu] + T[\partial_2\delta(z_2 - z_2')[X^\mu, X^\nu]] + T[\partial_2 X^\mu X^\nu)] \\
&= T[\partial_1^2 X^\mu X^\nu] + \delta(z_2' - z_2)[\partial_2 X^\mu, X^\nu] + T[\partial_2^2 X^\mu, X^\nu] \\
&= T[(\partial_1^2 X^\mu + \partial_2^2 X^\mu)X^\nu] + \delta(z_2' - z_2)[\partial_2 X^\mu, X^\nu] \\
&= \delta(z_2' - z_2)[\partial_2 X^\mu, X^\nu] \\
&= \pi\alpha'\delta^2(z - z')\eta^{\mu\nu}
\end{aligned}$$

$A^{\mu\nu}$ must be of the form of a Green function, $A^{\mu\nu} = \eta^{\mu\nu}G(z, z')$.
For future reference X^μ will imply only the holomorphic piece, unless specified otherwise. We may express the time-ordered product in terms of the normal ordered product minus the singularity.

$$T[X^\mu X^\nu] =: X^\mu X^\nu : -\frac{\alpha'}{2}\ln|z - z'|\eta^{\mu\nu}$$

As $z \to z'$:

$$X^\mu(z)X^\nu(z') = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z - z'| + \sum_{k>0}^{\infty}\frac{(z - z')^k}{k!} : X^\nu\partial_k X^\mu(z') :$$

$$\sum_k \frac{1}{k!}\left(-\frac{1}{2}\frac{\alpha'}{2}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right)^k = \exp\left[-\frac{\alpha'}{4}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right]$$

Define an operator $\mathcal{O}$ such that $\mathcal{O} = X^\mu X^\nu$.

$$:\mathcal{O} := \exp\left[-\frac{\alpha'}{4}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right]\mathcal{O}$$

This needs to be inverted.

$$\mathcal{O} = \exp\left[\frac{\alpha'}{4}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right] : \mathcal{O} :$$

Now $\mathcal{O}$ should have no singularities. We have to define the product between two $\mathcal{O}$s. This product will have singularities, unless the product of the two is normal ordered. The product of two time ordered operators, $:\mathcal{O}_1::\mathcal{O}_2:$, has singularities, whereas the time ordered product, $:\mathcal{O}_1\mathcal{O}_2:$, contains none.

$$:\mathcal{O}_1[X]\mathcal{O}_2[Y] := \exp\left[-\frac{\alpha'}{2}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right] : \mathcal{O}_1 :: \mathcal{O}_2 :$$

invert

$$:\mathcal{O}_1[X] :: \mathcal{O}_2[Y] := \exp\left[\frac{\alpha'}{2}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta Y^\nu}\right] : \mathcal{O}_1\mathcal{O}_2 :$$

Example: Let $\mathcal{O}_1 = \mathcal{O}_2 = \partial X^\mu(z)\partial X_\mu(z) = T(z)$

$$: T(z) :: T(z') :=: \partial X^\mu(z)\partial X_\mu(z) :: \partial' X^\nu(z')\partial' X_\nu(z') :$$

There are two possible double contractions and four possible single contractions,

$$\begin{aligned} : T(z) :: T(z') : &= 2 * \frac{\alpha'}{2}^2 \eta^{\mu\nu}\eta_{\mu\nu}(\partial\partial'\ln|z-z'|)^2 - 4 * \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'| : \partial X_\mu \partial' X_\nu : + : T(z)T(z') : \\ &= \frac{\alpha'^2}{2}\frac{d}{(z-z')^4} - \frac{2\alpha'}{(z-z')^2} : T : - \frac{\alpha'}{z-z'} : \partial' T(z') : + : T(z)T(z') : \end{aligned}$$

Example: Let $\mathcal{O}_1 =: e^{ik_1 X(z)} :$, $\mathcal{O}_2 =: e^{ik_2 X(z')} :$, $\frac{\delta}{\delta X^\mu}\mathcal{O}_n = ik_{n\mu}\mathcal{O}_n$

$$\begin{aligned} : \mathcal{O}_1\mathcal{O}_2 : &= \exp\left[\frac{\alpha'}{2}\ln|z-z'|\eta^{\mu\nu}(ik_{1\mu})(ik_{2\nu})\right] : \mathcal{O}_1 :: \mathcal{O}_2 : \\ &= \exp\left[-\frac{\alpha'}{2}k_1\cdot k_2\ln|z-z'|\right] : \mathcal{O}_1 :: \mathcal{O}_2 : \\ &= (z-z')^{-\frac{\alpha'}{2}k_1\cdot k_2} : \mathcal{O}_1 :: \mathcal{O}_2 : \end{aligned}$$

$$\begin{aligned} \Rightarrow : \mathcal{O}_1 :: \mathcal{O}_2 : &= (z-z')^{\frac{\alpha'}{2}k_1\cdot k_2} : \mathcal{O}_1\mathcal{O}_2 : \\ &= (z-z')^{-\frac{\alpha'}{2}k_1\cdot k_2} : e^{i(k_1+k_2)\cdot X}(1+\mathcal{O}(z-z')) : \end{aligned}$$

In this example we see the time-ordered product for two vertex operators representing tachyons.

2.4 Noether's Theorem

For every symmetry in a theory, there must be some conserved current, ie

$$\partial_\mu j^\mu = 0 \Rightarrow \text{Symmetry(S)}.$$

We can integrate over the zeroth component of the conserved current to get the charge.

$$Q = \int d^3x \partial_0 j^0 \rightarrow \frac{dQ}{dt} = \int_R d^3x j^0 = \int d^3x \nabla \cdot \vec{j} = \int d\vec{s} \cdot \vec{j} = 0$$

Q generates transformations.

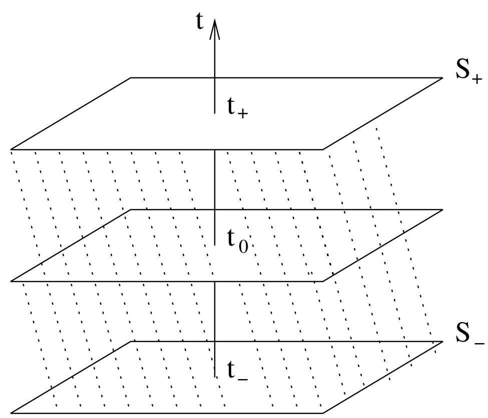
$$\delta A = i\epsilon[Q, A] = i\epsilon(QA - AQ) \qquad \epsilon << 1$$

Example: Let $Q = H$

$$\delta A = i\epsilon[H, A] = \dot{A}$$

or $Q = \vec{p}$

$$\vec{\nabla} A = i[\vec{p}, A]$$

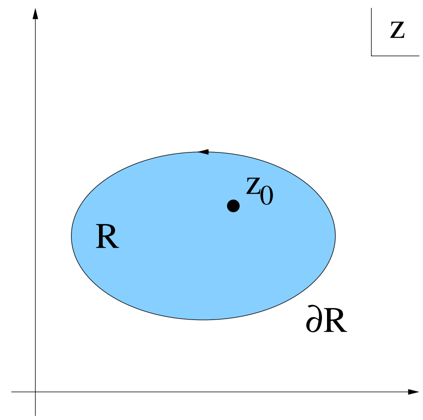


Look at $A(t_0)$:
Region R is bounded by $[S_+, S_-]$.

$$\begin{aligned}
\delta A &= i\epsilon(Q(t_+)A(t_0) - A(t_0)Q(t_-)) \\
&= i\epsilon\{\int_{S_+} d^3x\, j^0 A(t_0) - \int_{S_-} d^3x\, j^0 A(t_0)\} \\
&= i\epsilon \int_{\partial R} dS_\mu j^\mu A(t_0) \\
&= i\epsilon \int_{\partial R} dS_\mu T[j^\mu A(t_0)] \\
\delta A &= i\epsilon \int_R d^4x\, \partial_\mu T[j^\mu A(t_0)] \quad \text{"Ward Identity"}
\end{aligned}$$

Show the Ward Identity explicitly in two dimensions:

$$\begin{aligned}\delta A &= \frac{i\epsilon}{2\pi}\int_R d^2z\,\partial_a T[j^a A(z_0)], \quad j=(j_z,j_{\bar z})\\ &= \frac{i\epsilon}{2\pi}\oint_{\partial R}(dz j_z - d\bar z j_{\bar z})A(z_0)\end{aligned}$$



$\partial_a j^a = \partial_{\bar z} j_z + \partial_z j_{\bar z} = 0$ for special case j_z is holomorphic, and $j_{\bar z}$ is antiholomorphic.

$$\oint \frac{dz}{2\pi} j_z A(z_0) = \lambda(z_0): \quad j_z A(z_0) = \ldots + \frac{\lambda(z_0)}{z-z_0} + \ldots$$

$$\delta A = \epsilon\lambda + \epsilon\bar\lambda = -\epsilon(\lambda - \bar\lambda)$$

Example: Define the transformation on X and the current density.

$$X^\mu \to x^\mu + \epsilon a^\mu: \quad j_a^\mu = \frac{i}{\alpha'}\partial_a A^\mu \quad j_z^\mu = \frac{i}{\alpha'}\partial X^\mu$$

$$\bar\partial j_z^\mu = 0 \qquad \text{remember X is holomorphic}$$

Let $A(z_0) =: e^{ik\cdot X}:$

$$\begin{aligned}\delta A &= :e^{ik\cdot(X+\epsilon a)}:\\ &= (1+i\epsilon k\cdot a)A(z_0)\end{aligned}$$

$$\begin{aligned}j_z^\mu A(z_0) &= \frac{i}{\alpha'}\frac{\alpha'}{2}\eta^{\mu\nu}\partial\ln|z-z_0|(ik_\nu):A(z_0): + \text{regular terms}\\ &= -\frac{1}{2}\frac{k^\mu}{z-z_0}A(z_0)\\ &\Rightarrow \lambda = \frac{1}{2}k^\mu A(z_0) = -\bar\lambda: \quad \text{Residue}\end{aligned}$$

$$\epsilon(\lambda + \bar{\lambda}) = \epsilon(k^\mu A(z_0))$$
$$\epsilon a^\mu(\lambda - \bar{\lambda}) = \epsilon a^\mu k_\mu A(z_0)$$

Example: Let us look at translations.

$$\begin{aligned} \delta z = -\epsilon, \delta X^\mu &= X^\mu(z-\epsilon) - X^\mu(z) \\ &= -\epsilon \partial X_\mu \end{aligned}$$

Noether Current

$$S = -\frac{1}{2\pi\alpha'}\int \partial X^\mu \bar{\partial} X_\mu$$

$$\begin{aligned} \delta z = -\epsilon(z,\bar{z}) \qquad \delta S &= \frac{1}{2\pi\alpha'}\int \partial(\epsilon\partial X^\mu)\bar{\partial}X_\mu + \partial X^\mu \bar{\partial}(\epsilon \partial X_\mu) \\ &= \frac{1}{\pi\alpha'}\int \bar{\partial}\epsilon(\partial X^\mu \partial X_\mu) \end{aligned}$$

2.5 Conformal Invariance

$$\begin{aligned} T &= \frac{1}{\pi\alpha'} : \partial X^\mu \partial X_\mu : \quad \bar{\partial} T = 0 \quad \text{conserved, T is holomorphic} \\ \bar{T} &= \frac{1}{\pi\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu : \quad \partial \bar{T} = 0 \;\; \bar{\text{T}} \text{ is antiholomorphic} \end{aligned}$$

$$T_{\tau\tau} = -\frac{1}{2\alpha'} : \dot{X}^2 + x'^2 := T_{\sigma\sigma}$$
$$T_{\tau\sigma} = T_{\sigma\tau} = -\frac{1}{\alpha'}\dot{X}\cdot X' : \quad \text{Traceless} \quad T^a_a = 0.$$

For an arbitrary function $v(z)$:

$$j(z) = iv(z)T(z), \quad \bar{\partial} j = 0.$$

There are an infinite number of conservation laws. We may calculate the operator product expansion for the stress tensor with the field X^μ

$$\begin{aligned} T(z)X^\mu(z') &= \; : \partial X^\nu \partial X_\nu : X^\mu(z') = \frac{1}{\alpha'}\eta^{\mu\nu}\partial \ln|z-z'|\alpha'\partial X_\nu(z') + ... \\ &= \frac{1}{z-z'}\partial X^\mu(z') + ... \\ \bar{T}(\bar{z})X^\mu(\bar{z}') &= \frac{1}{\bar{z}-\bar{z}'}\bar{\partial} X^\mu(\bar{z}') + ... \end{aligned}$$

From $j = iv(z)T(z), \partial^a j_a = 0$

$$\bar{\partial} j = 0 \text{ or } \partial \bar{\text{j}} = 0$$

$$\bar{\partial}(v(z)T(z)) = 0$$

$v(z)T(z)$ is a conserved current, where

$$vTx^\mu \sim \frac{v\partial x^\mu}{z - z'}.$$

Let z transform as $\delta z = z + \epsilon V$, then

$$x^\mu \longrightarrow x^\mu - \epsilon v \partial x^\mu$$

and $T(z)A(z')$ can be expanded as Laurent series

$$T(z)A(z') = \frac{a_{-1}}{z - z'} + \frac{a_{-2}}{(z - z')^2} + ... + \text{regular terms},$$

then λ is

$$\begin{aligned}
\lambda &= \oint \frac{dz}{2\pi} iv(z)T(z)A(z') \\
&= \oint \frac{dz}{2\pi} i\left[\frac{a_{-1}v(z)}{z - z'} + \frac{a_{-2}v(z)}{(z - z')^2} + ..\right] \\
&= ia_{-1}v(z') + ia_{-2}\partial v(z') + \frac{i}{2!}a_{-3}\partial^2 v(z') + \cdots. \text{ ie.} \\
\delta A &= -\epsilon a_{-1}v - \epsilon a_{-2}\partial v - \frac{\epsilon}{2!}a_{-3}\partial^2 v - \cdots, \\
&= -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} a_{-n-1}\partial^n v.
\end{aligned}$$

Therefore, to find a_{-n}, for all n, we need to find how $A(z)$ transforms under the conformal transformation $z \to z + \epsilon v(z)$. The simplest case is a scaling: $v(z) = z,\ z \to z + \delta z = (1+\epsilon)z$. Find $A(z)$ that has a simple scaling property (eigenfunction), $\delta A = -h\epsilon A$, for finite ζ.

$$\begin{aligned}
A' &= (1+\epsilon)^{-h} A \\
&= \zeta^{-h} A.
\end{aligned}$$

or $A(z, \tilde{z})$ transforms as

$$A(z, \bar{z}) \to \zeta^{-h}\bar{\zeta}^{-\tilde{h}} A(z, \bar{z}').$$

For $\zeta = re^{i\theta}$,

$$A'(z', \bar{z}') \to r^{-(h+\tilde{h})} e^{-(h-\tilde{h})\theta} A(z, z')$$

$h + \tilde{h}$ is the dimension of A, and determines the transformation under scaling. $h - \tilde{h}$ determines the transformation under spin. If A is order h, then ∂A is order $h + 1$, i.e.

$$\begin{aligned}
\frac{\partial A}{\partial z} &= \frac{\partial z'}{\partial z}\frac{\partial A}{\partial z'} \\
&\to (1 - \epsilon z)(1 - h\epsilon z)\partial A \\
&\to (1 - (h+1)\epsilon z)\partial A.
\end{aligned}$$

or

$$\bar{\partial} A, \quad \tilde{h} \rightarrow \tilde{h}+1$$

Compare the coefficient of ∂v with the equations for δA. This implies $a_{-2} = hA$. For simplicity, let $v(z) = 1$.

Special Case: Do a translation transformation on $z,\ z \rightarrow z+\epsilon$.Then $\delta A = A(z-\epsilon) - A(z) = \epsilon \partial A \rightarrow a_{-1} = \partial A$. For an arbitary $v(z),\ z \rightarrow z + \epsilon v(z),\ \delta A$ is given by

$$\begin{aligned} \delta A &= -h\epsilon \partial v A \\ A' &= (\tfrac{\partial \zeta}{\partial z})^{-h} (\tfrac{\partial \bar{\zeta}}{\partial \bar{z}})^{-\tilde{h}} A \\ A' &= (1+\epsilon \partial v)^{-h} A. \end{aligned}$$

If δA has only two singularity terms in the form below, call A a primary field, ie.

$$T(z)A(z') = \frac{\partial A}{z-z'} + \frac{h \partial A}{(z-z')^2}.$$

Some examples for $T(z)A(z')$

$$\begin{aligned} T(z)X^\mu(z') &= -\frac{1}{\alpha'} : \partial X^\nu \partial X_\nu : X^\mu(z') = \frac{1}{2}\eta^{\mu\nu} \partial \ln(z-z') \partial X_\nu \\ &= \frac{1}{z-z'} \partial X^\mu, \text{ ie. } h = \tilde{h} = 0 \\ T(z)\partial^2 X^\mu &= \frac{1}{2}\eta^{\mu\nu} \partial \partial^2 \ln(z-z') \partial X_\nu \\ &= \frac{2}{(z-z')^3} \partial X^\mu(z) \\ &= \frac{2}{(z-z')^3} [\partial X^\mu(z') + (z-z')\partial^2 X(z') + \frac{1}{2!}(z-z')^2 \partial^3 X^\mu(z') + ... \\ &= \frac{2}{(z-z')^3} \partial X^\mu + \frac{2}{(z-z')^2} \partial^2 X^\mu(z') + \frac{1}{z-z'} \partial \partial^2 X^\mu + ... \\ T(z) : e^{ikX} : &= \frac{1}{\alpha'} \left(\frac{\alpha'}{2} \eta^{\mu\nu} k_\nu \partial \ln(z-z') \right)^2 : e^{ik\cdot X} : + \eta^{\mu\nu} k_\nu \partial \ln(z-z') : \partial X_\mu e^{ik\cdot X} : \\ &= \frac{\alpha'}{4} \frac{k^2}{(z-z')^2} : e^{ik\cdot X} : + \frac{1}{z-z'} : \partial e^{ik\cdot X} \end{aligned}$$

For $A = e^{ik\cdot X},\ T(z)A(z')$ implies $h = \frac{\alpha' k^2}{4}$ and $\bar{T}(\bar{z})A(\bar{z}')$ implies $\tilde{h} = \frac{\alpha' k^2}{4}$.Therefore A has weight $(h, \tilde{h}) = (\frac{\alpha' k^2}{4},\ \frac{\alpha' k^2}{4})$. If a translation is applied to z $(z \rightarrow z + \epsilon v)$, then $e^{ik\cdot X(z)} \rightarrow e^{ik\cdot X(z-\epsilon v)}$ and $\delta(e^{ik\cdot X(z)}) = -\epsilon v \partial e^{ik\cdot X(z)}$, which means that $h = 0$. We have just shown that h is nonzero, and arrives from normal ordering (quantum effect).

$$\begin{aligned} \partial X e^{ik\cdot X} &\rightarrow h = 1 + \tfrac{\alpha' k^2}{4} \\ \partial^2 X e^{ik\cdot X} &\rightarrow h = 2 + \tfrac{\alpha' k^2}{4} \\ \partial^m X e^{ik\cdot X} &\rightarrow h = m + \tfrac{\alpha' k^2}{4} \\ \partial^{m_n} X^{\mu_n} ... \partial^{m_2} X^{\mu_2} \partial^{m_1} X^{\mu_1} e^{ik\cdot X} &\rightarrow h = m_n + ... + m_2 + m_1 + \tfrac{\alpha' k^2}{4} \end{aligned}$$

It looks like these make a good *basis* for operators.

X^μ	$h=0$	$\tilde{h}=0$	$(0,0)$
∂X^μ	$h=1$	$\tilde{h}=0$	$(1,0)$
$\bar{\partial} X^\mu$	$h=0$	$\tilde{h}=1$	$(0,1)$
$\partial^2 X^\mu$	$h=2$	$\tilde{h}=0$	$(2,0)$
$e^{ik\cdot X}$	$h=\frac{\alpha' k^2}{4}$	$\tilde{h}=\frac{\alpha' k^2}{4}$	$(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4})$
$\partial X e^{ik\cdot X}$	$h=1+\frac{\alpha' k^2}{4}$	$\tilde{h}=1+\frac{\alpha' k^2}{4}$	$(1+\frac{\alpha' k^2}{4}, 1+\frac{\alpha' k^2}{4})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$T(z)$ has weight $h=2$, but is not a primary field. The TT OPE is given by:

$$T(z)T(z') = \frac{D}{2(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{2}{\alpha'(z-z')} : \partial^2 X_\mu \partial X^\mu :$$

where $\partial^2 X_\mu \partial X^\mu$ can be written as $\frac{1}{2}(\partial(\partial X_\mu \partial X^\mu))$ if $T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : +V_\mu \partial^2 X^\mu$

2.6 Free CFTs

In this section we will explore various different conformal field theories. We may classify all CFTs by knowing their central charges and three-point functions. We will find the central charges for the following free conformal field theories. We will leave the three-point functions for the reader.

Linear dilaton

The TT OPE is given by

$$\begin{aligned} T(z)T(z') &\sim \frac{D}{2(z-z')^4} + V_\mu V^\mu \frac{\alpha'}{2} \partial^2 \partial'^2 \ln(z-z') + ... \\ &= \frac{D}{2(z-z')^4} + \frac{6 V_\mu V^\mu \alpha'}{2(z-z')^4} + ... \\ &= \frac{c}{2(z-z')^4} + ... \, , \end{aligned}$$

where $c = D + 6\alpha' V_\mu V^\mu$. Therefore the central charge for the Linear Dilaton theory can be any number. When the number of dimensions is one or two the theory is *exactly* solvable. Also, if we compactify some dimensions V^μ can live in the compact subspace, because there is no need for Lorentz invariance there.

$$\begin{aligned} T(z)X^\mu(z') &\sim \partial \ln(z-z') \partial X^\mu + V^\mu \frac{\alpha'}{2} \partial^2 \ln(z-z') + ... \\ &= \frac{1}{z-z'} \partial X^\mu + \frac{V^\mu \alpha'}{2(z-z')^2}, \quad h=0 \text{ and } X^\mu \text{ is not a primary field,} \end{aligned}$$

$$v(z)T(z)X^\mu(z') \sim \frac{V^\mu\alpha'}{2(z-z')^2}[v(z')+(z-z')\partial v+...]+\frac{v}{z-z'}\partial X^\mu+....$$

For $\delta X^\mu=-\epsilon,\ \lambda=\frac{1}{2}V^\mu\alpha'\partial v+v\partial X^\mu$.

bc theory

Let b and c be anticommuting fields, i.e. spinors.The action can be writen as

$$S=\frac{1}{2\pi}\int d^2zb\bar\partial c$$

The equations of motion are given by: $\bar\partial c=0$, $\bar\partial b=0$, where b and c are holomorphic. If we let $\ell=2\pi$, then we can write b and c as

$$b(z)=i\sum b_ne^{inz},\ c(z)=i\sum c_ne^{inz},$$

where

$$\begin{aligned}
\{b(z),c(z')\} &= \delta(\sigma-\sigma')_{\text{equaltime}},\ z_2=z_2'\text{ and }0<\sigma<2\pi\text{ or}\\
\{b_m,c_n\} &= \delta_{m+n,0}\\
\langle 0|b(z)c(z')|0\rangle &= \frac{1}{1-e^{i(z-z')}}\\
&\sim \frac{1}{z-z"}+\text{regular terms},\ \text{then}\\
:b(z)c(z'): &= b(z)c(z')-\frac{1}{z-z'}
\end{aligned}$$

If b has weight $h_b=\lambda$, then $h_c=1-\lambda$.This is known since the action has weight 0 and the volume element has weight $(-1,-1)$. From the transformation $\delta z=\epsilon(z)$, b will change to

$$\begin{aligned}
b' &= \left(\frac{\partial z'}{\partial z}\right)^\lambda b(z-\epsilon)\\
&= (1-\lambda\partial\epsilon)(b-\epsilon\partial b),\ \text{then}\\
\delta b &= -\lambda\partial b-\epsilon\partial b,\ \text{and}\\
\delta c &= -(1-\lambda)\partial c-\epsilon\partial c\\
\delta S &= \int\bar\partial\epsilon((\partial b)c-\lambda\partial(bc)),
\end{aligned}$$

where $(\partial b)c-\lambda\partial(bc)$ is the Noether current in this case, then T is

$$\begin{aligned}
T &= :(\partial b)c:-\lambda\partial(:bc:),\ \text{and}\\
c(z)b(z') &= \frac{1}{z-z'}+...\\
b(z)c(z') &= \frac{1}{z-z'}+...
\end{aligned}$$

$$
\begin{aligned}
T(z)T(z') &= -\left(\frac{1}{z-z'}\right)^2 - 2\lambda\partial\left(\frac{1}{z-z'}\partial\frac{1}{z-z'}\right) + \lambda^2\partial\partial'\frac{1}{(z-z')^2} \\
&= \frac{-1+6\lambda-6\lambda^2}{2(z-z')^4} + \ldots \\
&= \frac{c}{2(z-z')^4} \\
c &= -2+12\lambda-12\lambda^2 = 1-3(2\lambda-1)^2.
\end{aligned}
$$

From the Linear Dilaton theory $c = D + 6\alpha V^2$. Let the charges from the two theories be equal, ie. $D + 6\alpha V^2 = 1 - 3(2\lambda - 1)^2$ and solve for V. For the case where $D = 1$, one will obtain

$$V = \frac{1}{\sqrt{2\pi}}(2\lambda - 1).$$

Can the bosons be equivalent to the fermions? We will see later. Let us explore a special case, $\lambda = \frac{1}{2}$. We find $V = 0,\ c = 1$ and, $b,\ c$ can be writen as a linear combination of scalar fields Ψ_1 and Ψ_2

$$b(z) = \frac{1}{\sqrt{2}}(\Psi_1 + i\Psi_2), \quad c(z) = \frac{1}{\sqrt{2}}(\Psi_1 - i\Psi_2).$$

The action may be expressed in terms of the Ψs

$$S = \frac{1}{4\pi}\int d^2z(\Psi_1\bar{\partial}\Psi_1 + \Psi_2\bar{\partial}\Psi_2),$$

with a stress tensor

$$T = -\frac{1}{2}\Psi_i\partial\Psi_i, \quad i = 1, 2.$$

Another interesting case is for $\lambda = 2$ and V=0. Then the central charge, c becomes $c = -26$ from $c = 1 - 3(2\lambda - 1)^2$. This is the result obtained in chapter 1.

$\beta\gamma$ theory

The next example, the bosonic case, let β and γ be commuting scalar fields. The $\beta\gamma$ action is given by:

$$S = \frac{1}{\sqrt{2\pi}}\int d^2z\beta\bar{\partial}\gamma,$$

where

$$\bar{\partial}\beta = \bar{\partial}\gamma = 0$$

The procedure is the same as for the spinor case.

$$
\begin{aligned}
\beta(z)\gamma(z') &= \frac{1}{z-z'} + \ldots \\
\gamma(z)\beta(z') &= -\frac{1}{z-z'} + \ldots \\
c &= -1 + 3(2\lambda - 1)^2
\end{aligned}
$$

Note that if $\lambda = \frac{3}{2}$, then $c = 11$. If we combine the central charges for the four theories, the number of physical dimensions reduces from twenty-six to ten.

$$\left.\begin{array}{cccc} X^\mu & D & (b,c) & -26 \\ \Psi^\mu & \frac{d}{2} & (\beta,\gamma) & 11 \end{array}\right\} D + \frac{d}{2} - 26 + 11 = 0 \Longrightarrow D = 10.$$

From

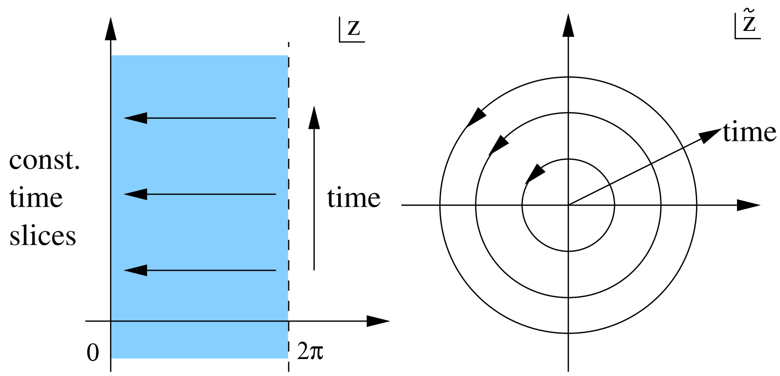
$$X_L^\mu(z) = x^\mu - \frac{\alpha'}{2}p^\mu z + i\sqrt{\frac{\alpha'}{2}}\sum \frac{1}{n}\alpha_n^\mu e^{inz}$$

$$\begin{aligned} \langle X_L(z), X_L(z')\rangle &= \ln(1 - e^{i(z-z')}) \\ &\sim \ln(z - z') + ... \end{aligned}$$

2.7 Virasoro Algebra

The worldsheet of a free closed string moving through space-time looks like a cylinder whose radius may fluctuate depending on the excitation mode. We can map this cylinder to the complex plane. This map would be equivalent to sqeezing one end of the string so it looks like a cone.Then smash the cone into a plane. Now the worldsheet coordinates can be expressed as complex coordinates, where r represents τ and the phase, θ, represents the position on the string, σ ($z \mapsto \tilde{z}$, cylinder $\mapsto$ plane). For closed strings $\lambda = 2\pi, 0 < \sigma < 2\pi$

$$\tilde{z} = e^{-iz}$$



Instead of expanding in Fourier modes, do a Laurent Expansion.

$$X^\mu \sim \sum \frac{1}{n}\alpha_n^\mu \tilde{z}^{-n}$$

What happens to $T(z)$:

$$\begin{aligned} T(z) &= \frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \\ &= \sum T_m e^{imz} \\ &\stackrel{?}{=} \sum T_m \tilde{z}^{-m} \end{aligned}$$

$T(z)$ doesn't transform as easily as guessed. Subtract the term with central charge to make it a primary field (tensor). Then transform each coordinate.

$$T(z)T(z') = \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{1}{z-z'}\partial T(z') + \dots$$

multiply both sides by $v(z)$

$$v(z)T(z)T(z') = v(z)[\dots]$$

$$\delta T = -\epsilon\lambda = \epsilon\frac{c}{2}\frac{1}{3!}\partial^3 v - 2\epsilon\partial v T - \epsilon v\partial T$$

$$z \to z + \epsilon v(z) \;\; z \to \tilde{z} = e^{-iz}$$

$$T(z) = (\frac{\partial\tilde{z}}{\partial z})^2 T(\tilde{z}) + \frac{c}{12}\{\tilde{z}, z\},$$

where $\{\tilde{z}, z\}$ is known as a Schwarzian derivative, defined as

$$\{\tilde{z}, z\} = \frac{2\tilde{z}'''\tilde{z}' - 3(\tilde{z}'')^2}{2(\tilde{z}')^2},$$

which equals $1/2$ for our example.

$$T = -\tilde{z}^2\tilde{T} + \frac{c}{24}$$

We can invert this and solve for $\tilde{z}$,

$$\begin{aligned}\tilde{T}(\tilde{z}) &= -\tilde{z}^{-2}\tilde{T} + \frac{c}{24}\tilde{z}^{-2} \\ &= \sum L_m z^{-m-2}, \qquad\qquad L_m = -T_m + \frac{c}{24}\delta_{m,0}\end{aligned}$$

Invert the equation and solve for L.

$$L_m = \oint \frac{d\tilde{z}}{2\pi i}\tilde{z}^{m+1}\tilde{T}$$

The Hamiltonian $\int_0^{2\pi}\frac{d\sigma}{2\pi}T$ can now be written in terms of L. Including left and right movers, the Hamiltonian is

$$H = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24},$$

and the number operator is $N = L_0 - \bar{L}_0$. We can see that $\bar{\partial}L_m = 0$ since $\bar{\partial}[\tilde{z}^{m+1}\tilde{T}] = 0$. This implies all L_ms generate symmetries.

Commutators

Recall the Ward identity for any operator A : $\delta A = i\epsilon[Q, A]$ implies Q can be written as an integral of the current, j. $Q = \oint \frac{dz}{2\pi i} j(z)$ The OPE of T with some primary field of weight h is given by:

$$T(z)A(z') = \frac{h}{(z-z')^2}A(z') + \frac{\partial A(z')}{z-z'} + \dots$$

Look at the variation of A:

$$\delta A = -\epsilon h \partial v A - \epsilon v \partial A$$

choose $v = \tilde{z}^{m+1} \quad j = \tilde{z}^{m+1}\tilde{T} \quad Q = \oint j \sim L_m$

$$\Rightarrow \delta A = i\epsilon[L_m, A]$$

$$[Q, A] = h(m+1)\tilde{z}^m + \tilde{Z}^{m+1}\partial A = [L_m, A]$$

expand A in a Laurent expansion:

$$\begin{aligned} A &= (\frac{\partial \tilde{z}}{\partial z})^{-h} \sum A_m \tilde{z}^{-m} \\ &= \sum A_m \tilde{z}^{-m-h} \end{aligned}$$

This is the expansion for a primary field in the $\tilde{z}$ coordinates. Look at the commutator of L with A in these cordinates.

$$\begin{aligned} [L_m, A_n] &= h(m+1)A_{n+m} - (n+m+h)A_{n+m} \\ &= [(h-1)m - n]A_{m+n} \end{aligned}$$

We got an algebra from an OPE. We know the algebra, but what are the representations of the algebra.

$$\delta T = \text{expected} + \frac{c}{12}\partial^3 v$$

$$[L_m, T] = \text{expected} + \frac{c}{12}(m+1)m(m-1)\tilde{z}^{m-2}$$

$$\begin{aligned} [L_m, L_n] &= \text{expected} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\ &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \end{aligned}$$

This is the **Virasoro algebra**. Let us focus on the special case $m = 0$. The action of the Hamiltonian L_0:

$$[L_0, A_n] = -nA_n$$

even if $A_n = L_n$. If

$$L_0|\psi\rangle = E|\psi\rangle, \qquad |\psi'\rangle = L_n|\psi\rangle$$

$$\begin{aligned} L_0|\psi'\rangle &= [L_0, L_n]|\psi\rangle + L_n L_0|\psi\rangle \\ &= -nL_n|\psi\rangle + EL_n|\psi\rangle \\ &= (E-n)L_n|\psi\rangle \\ &= (E-n)|\psi'\rangle \end{aligned}$$

For $n > 0 \Rightarrow L_n|\psi\rangle$ has lower *energy* then $|\psi\rangle$. Is the spectrum of L_0 unbounded? This needs to be fixed.
Let us look at the $n = -1, 0, 1$ Virasoro subalgebra. Is this analogous to the raising and lowering operators for angular momentum.

$$[L_0, L_1] = -L_1, \;\; [L_0, L_{-1}] = L_{-1}, \;\; [L_1, L_{-1}] = 2L_0$$

We notice that this closed algebra is independent of c, therefore for every CFT we should get this subalgebra.This is a Lie algebra SL(2,$\mathbb{R}$) and is not compact. Look at Quantum Mechanics:

$$[L_0, L_+] = L_+, \;\; [L_0, L_-] = -L_-, \;\; [L_+, L_-] = 2L_0$$

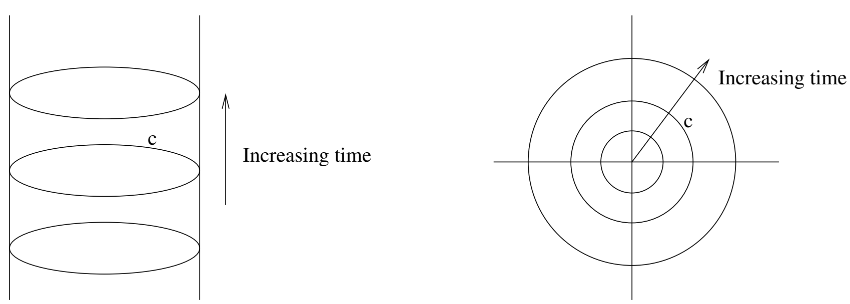
This is close to the algebra above, but not the same.The difference is actually very important!

2.8 Mode Expansions

Free Scalars

Now that we know how to transform to these $\tilde{z}$ coordinates, we can look at the same calculations and look for similarities.The mode expansion is given by,

$$X_L^\mu = x^\mu - i\frac{\alpha'}{2}p^\mu \ln\tilde{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\frac{1}{m}\alpha_m^\mu \tilde{z}^{-m}.$$



The two-point function which has to be radially ordered is given by,

$$\begin{aligned}\langle X_L^\mu(\tilde{z})X_L^\nu(\tilde{z}')\rangle &= \frac{\alpha'}{2}\sum\frac{1}{m}(\frac{\tilde{z}'}{\tilde{z}})\eta^{\mu\nu}|\tilde{z}'|<|\tilde{z}| \quad \rightarrow \text{ time ordering...i.e. radial ordering}\\ &= \frac{\alpha'}{2}\ln|1-\frac{\tilde{z}'}{\tilde{z}}|\eta^{\mu\nu}.\end{aligned}$$

The normal ordered product is given by,

$$:XX:=XX=\frac{\alpha'}{2}\ln|z-z'| \text{ in z picture.}$$

Now we can compare our definition for :: to switching a and $a^\dagger$ around.

$$\begin{aligned}:X_L^\mu(\tilde{z})X_L^\nu(\tilde{z}'): &= :(X_L^+ + X_L^-)^\mu(X_L^+ + X_L^-)^\nu:\\ &= X_L^\mu(\tilde{z})X_L^\nu(\tilde{z}') + [X^{(-)}(z'),X^{(+)}(z)]\\ &= X_L^\mu(\tilde{z})X_L^\nu(\tilde{z}') - \frac{\alpha'}{2}\sum_n\frac{1}{n}\eta^{\mu\nu}\frac{\tilde{z}'}{\tilde{z}}^n\end{aligned}$$

$$[x^\mu, -i\frac{\alpha'}{2}p^\mu\ln\tilde{z}] = XX + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|\tilde{z}-\tilde{z}'|.$$

From normal ordering

$$X^\mu(z)X^\nu(z') = \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-z') + :X^\mu(z)X^\nu(z'):,$$

the operator product can be writen as product = singularity + normal ordering product From

$$L^m = \oint\frac{dz}{2\pi i}z^{m+1}T(z), \qquad \text{charge}$$

where $z^{m+1}T(z)$ is a conserved current and

$$\begin{aligned}\partial X^\mu &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\alpha_m^\mu z^{-m-1}\\ P^\mu &= \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu\\ X^\mu &= x^\mu + p^\mu\ln z + i\sum_{m\neq 0}\frac{1}{m}\alpha_m^\mu z^{-m}\\ T(z) &= \frac{1}{\alpha'}:\partial X^\mu\partial X_\mu:\\ &= \frac{1}{2}\sum_{n_1,n_2}z^{-n_1-1}z^{-n_2-1}:\alpha_{n_1}^\mu\alpha_{n_2\mu}:\\ L_m &= \frac{1}{2}\sum_n:\alpha_{m-n}^\mu\alpha_{n\mu}:\end{aligned}$$

The Hamiltonian (L_0) is given by:

$$L_0 = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{n\mu}$$

What about the equal-time commutator? That means $|z| = |z'|$. To calculate $X^\mu(z)X^\nu(z')$, we need to approach $|z| \to |z'|$ from $|z| > |z'|$. To calculate $X^\nu(z)X^\mu(z')$, we need to approach $|z| \to |z'|$ from $|z| < |z'|$. Thus, the commutation relations are

$$\begin{aligned}
[X^\mu(\sigma), \Pi^\nu(\sigma')] &= 2\pi i \delta(\sigma - \sigma') \\
[X^\mu(\sigma), X^\nu(\sigma')] &= 0, \text{ where } \mathrm{X} = \mathrm{X_L(z)} + \mathrm{X_R(\bar{z})} \\
[X_L^\mu(z), X_L^\nu(z')] &= X_L^\mu(z)X_L^\nu(z') - X_L^\nu(z')X_L^\mu(z) \\
&= \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-z') - \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-z') \\
&= \frac{\pi i}{2}\alpha'\eta^{\mu\nu}\frac{d}{dz}(\text{step function}) \\
&= \frac{\pi i}{2}\alpha'\eta^{\mu\nu}\delta(z-z'),
\end{aligned}$$

where differentiating a step function, a delta function is obtained.
N.B. The commutator $[X_L, X_L] \neq 0$ means X_L is **not** a coordinate. It is a combination of a coordinate and momentum.
Another interesting example

$$\begin{aligned}
e^A e^B &= e^{[A,B]}e^B e^A, \text{ then} \\
: e^{ik_1X_1} :: e^{ik_2X_2} : &= e^{\pm i\pi\frac{\alpha'}{2}k_1\cdot k_2} : e^{ik_2X_2} :: e^{ik_1X_1} :,
\end{aligned}$$

For the special case $D = 1$, $X^\mu = \frac{\alpha'}{2}\Psi$, $k_1^\mu = \pm\sqrt{\frac{2}{\alpha'}} = k_2^\mu$ and let $\mathcal{O}_1 = e^{\pm i\psi}$ and $\mathcal{O}_2 = e^{\pm i\psi}$, then

$$\mathcal{O}_1\mathcal{O}_2 = e^{\pm i\pi}\mathcal{O}_2\mathcal{O}_1 = -\mathcal{O}_2\mathcal{O}_1, \quad \text{(Quantum group!)}$$

or

$$\begin{aligned}
\{\mathcal{O}_1\mathcal{O}_2, \mathcal{O}_2\mathcal{O}_1\} &= 0 \\
\psi(z)\psi(z') &\sim \ln(z-z'), \text{ then} \\
: e^{i\psi}(z) :: e^{-i\psi}(z') : &\sim e^{-\ln(z-z')} : e^{i\psi}(z)e^{-i\psi}(z') : \\
&\sim \frac{1}{z-z'} : e^{i\psi}(z)e^{-i\psi}(z') : \\
: e^{i\psi}(z) :: e^{i\psi}(z') : &\sim e^{\ln(z-z')} : e^{i\psi}(z)e^{i\psi}(z') : \\
&\sim (z-z') : e^{i\psi}(z)e^{i\psi}(z') : .
\end{aligned}$$

Example: bc **CFT**

We can write b and c in terms of ψ: $b =: e^{i\psi}$: and $c =: e^{i\psi}$:. The stress tensor is given as

$$T_\psi =: \partial\psi\partial\psi : +V\partial^2\psi.$$

From $b(z) = \sum b_m z^{-m-\lambda}$ and $c(z) = \sum c_m z^{-m-1+\lambda}$, we may calculate the OPE, $b(z)c(z') \sim \frac{1}{z-z'}$. The anticommutator between b and c is

$$\{b_m, c_n\} = \delta_{m+n,0}.$$

b_n and c_n are annihilation operators for $n > 0$. For the zero modes, $m, n = 0$ the anticommutator is

$$\{b_0, c_0\} = 1$$

If we let $|\Psi\rangle$ be a null state of b, ie. $b_0|\psi\rangle = 0$ and $c_0|\psi\rangle = |\chi\rangle$, then

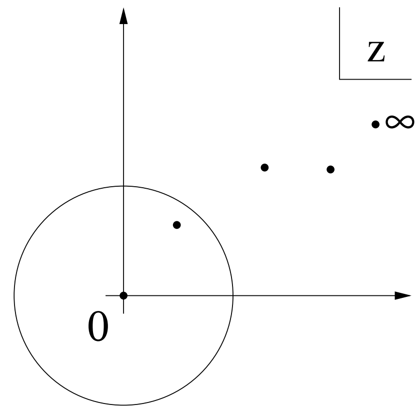
$$\begin{aligned}
c_0|\chi\rangle &= c_0c_0|\psi\rangle = 0, \qquad \{b_m, b_n\} = \{c_m, c_n\} = 0 \\
b_0|\chi\rangle &= b_0c_0|\psi\rangle = \{b_0, c_0\}|\psi\rangle = |\psi\rangle, \text{ then} \\
\langle\psi|\psi\rangle &= \langle\chi|b_0b_0|\chi\rangle = 0 \\
\langle\chi|\chi\rangle &= \langle\psi|b_0b_0|\psi\rangle = 0, \text{ and} \\
\langle\psi|\chi\rangle &\neq 0.
\end{aligned}$$

So $|\psi\rangle$ and $|\chi\rangle$ are independent *vacua*. Hilbert space: act on $|\psi\rangle,\ |\chi\rangle$ with the creation operators $b_{-n},\ c_{-n},\ n > 0$. By convention we will group b_0 and c_0 with creation and annihilation operators respectively. Then $|\psi\rangle$ is the vacuum, $|\psi\rangle = |0\rangle$.
N.B. Define $\langle 0| = \langle\chi|$ and $\langle\psi|\chi\rangle = \langle\chi|\psi\rangle = 1$.

2.9 Vertex operators

Vertex operators are one-to-one corresponding to their states, $I \simeq |0>$ for an operator ∂X^μ and momentum operator p^μ

$$\begin{aligned}
\partial X^\mu(z) &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu z^{-m-1} \\
p^\mu|0\rangle &= 0, \\
p^\mu|0;k\rangle &= k^\mu|0;k\rangle, \text{ then} \\
\partial X^\mu(z)|0\rangle &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=1}^{\infty} \alpha_{-m}^\mu z^{-m-1}|0\rangle \\
&= -i\sqrt{\frac{\alpha'}{2}}(\alpha_{-1}^\mu + \alpha_{-2}^\mu z + \alpha_{-3}^\mu z^2 + ...)|0\rangle \text{ letting } z \to 0 \\
\partial X^\mu(0) &= -i\sqrt{\frac{\alpha'}{2}}\alpha_{-1}^\mu|0\rangle
\end{aligned}$$



From $|A\rangle \simeq A(z)$ or $|A\rangle = A(0)|0\rangle,\ \partial X^\mu$ operating at a bra is

$$\begin{aligned}\langle 0|\partial X^\mu(z) &= -i\sqrt{\frac{\alpha'}{2}}\langle 0|\sum_{m=1}^{\infty}\alpha^\mu_{-m}z^{-m-1},\ \text{letting } z\to\infty \\ \langle 0|\partial X^\mu(\infty) &= -i\sqrt{\frac{\alpha'}{2}}\langle 0|\alpha^\mu_{-1},\end{aligned}$$

then the product can be consider as

$$\underbrace{\langle 0|}_{\text{time at }+\infty}\ \text{time order stuffed}\ \underbrace{|0\rangle}_{\text{time at }-\infty}$$

From the equation of $\partial X^\mu,\ \alpha^\mu_1$ is

$$\alpha^\mu_{-1} = \oint \frac{dz}{2\pi i}\underbrace{z^{-1}\partial X^\mu(z)}_{\text{conservedcurrent}},$$

$$z \to e^{-iz}$$

$\alpha^\mu_{-2}|0\rangle$ also can be obtained from

$$\begin{aligned}\partial^2 X^\mu(z)|0\rangle &= -i\sqrt{\frac{\alpha'}{2}}(\alpha^\mu_{-2}+2\alpha^\mu_{-3}z+...)|0\rangle,\ \text{for any } \alpha^\mu_{-\text{m}} \\ \partial^m X^\mu(0) &= -i\sqrt{\frac{\alpha'}{2}}((m-1)!\alpha^\mu_{-m}+O(z))|0\rangle, \text{letting z} \to 0\end{aligned}$$

$$\alpha^\mu_{-m}(0) \simeq i\sqrt{\frac{2}{\alpha'}}\frac{1}{(m-1)!}\partial^m X^\mu(z).$$

$\alpha^\mu_{-m}\alpha^\nu_{-n}|0\rangle$ can be obtained from

$$:\partial^m X^\mu \partial^\nu X^\nu:|0\rangle = -\frac{\alpha'}{2}(m-1)!(n-1)!\alpha^\mu_{-m}\alpha^\nu_{-n} + \underbrace{...}_{\text{go to}} 0 \text{ in the infinite past}|0\rangle$$

Some examples of vertex operators are

$$\begin{aligned} e^{ik\cdot X}|0\rangle &= |0;k\rangle \\ \lim_{z\to 0} :e^{ik\cdot X(z)}:|0\rangle &= e^{ik\cdot X}|0\rangle, \text{ then} \\ :\partial X^{m_1}\partial X^{m_2}...e^{ik\cdot X}: &\simeq \alpha_{-m_1}\alpha_{-m_2}....|0;k\rangle \end{aligned}$$

2.10 Primary fields

A primary field $A,\ A\to|A\rangle, A(0)|0\rangle = |A\rangle$, with a state $|m_1+m_2+...;k>$ can be writen as

$$\begin{aligned} |A\rangle &= \alpha{-m_1}^{\mu_1}\alpha^{\mu_2}_{-m_2}...|0;k\rangle\text{where} \\ A(z) &= \frac{1}{(n_1-1)!}\frac{1}{(n_2-1)!}...:\partial^{n_1}X^{\mu_1}\partial^{n_2}X^{\mu_2}...e^{ik\cdot X}:. \end{aligned}$$

If we let $\lambda = 2$ for the bc theory:

$$\begin{aligned} b_{-m}|\psi\rangle &\longrightarrow \frac{1}{(m-2)!}\partial^{m-2}b,\ b_0|\psi\rangle = 0 \\ c_{-m}|\psi\rangle &\longrightarrow \frac{1}{(m+1)!}\partial^{m+1}c \\ T_{bc}(z) &= :(\partial b)c: -\lambda\partial:(bc):, \end{aligned}$$

where

$$\begin{aligned} b(z) &= \sum b_m z^{-m-\lambda} \\ c(z) &= \sum c_m z^{-m+\lambda} \end{aligned}$$

The Virasoro operator in this case is

$$\begin{aligned} L^{bc}_m &= \oint\frac{dz}{2\pi i}z^{m+1}T_{bc} \\ &= \sum_n -(n+\lambda)b_n c_{m-n} + \lambda(m+1)b_n c_{n-m} + a\delta_{n,0} \\ &= \sum_n (m\lambda - n):b_n c_{n-m}: + a\delta_{m,0} \\ T_{bc}(z)T_{bc}(z') &\sim \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2}T_{bc} + \frac{1}{z-z'}\partial T_{bc} \end{aligned}$$

where $c = 1-3(2\lambda-1)^2$, and the commutation relations for L are given by:

$$\left[L^{bc}_m, L^{bc}_n\right] = (n-m)L^{bc}_{n+m} + \frac{c}{12}(m^3-n)\delta_{n+m,0}$$

For $m=1$ and $n=-1$, the commutator is

$$\begin{aligned}
\left[L_1^{bc}, L_{-1}^{bc}\right]|\Psi\rangle &= 2L_0^{bc}|\Psi\rangle \\
L_1 L_{-1}|\Psi\rangle - L_{-1}L_1|\Psi\rangle &= \lambda b_0 c_1 (1-\lambda) b_{-1} c_0 |\Psi\rangle \\
&= \lambda(1-\lambda) c_1 b_{-1}|\Psi\rangle \\
&= \lambda(1-\lambda)\{c_1, b_{-1}\}|\Psi\rangle \\
&= \lambda(1-\lambda)|\Psi\rangle
\end{aligned}$$

2.11 Operator product expansion

Consider the commutator between L_m and A

$$\begin{aligned}
[L_m, A] &= z^{m+1}\partial A + h(m+1) z^m A, \quad \text{where} \\
A(z) &= \sum A_n z^{-n-h}
\end{aligned}$$

For $m=0$,

$$\begin{aligned}
[L_0, A] &= z\partial A + hA \\
[L_0, A_n] &= -nA_n \\
\text{as } z\to 0,\ [L_0, A(0)] &= hA(0) \\
L_0|A\rangle &= [L_0, A(0)]\,|0\rangle + A(0)L_0|0\rangle,\ L_0|0\rangle = 0, \\
&= hA(0)|0\rangle = h|A\rangle \\
\text{as } z\to 0,\ [L_m, A] &= \text{for } m>0,\ \text{then} \\
L_m|A\rangle &= 0, m>0,
\end{aligned}$$

where L_0 is bounded from below.

2.12 Unitary CFTs

Define an inner product $\langle ...|...\rangle$ such that $L_m^\dagger = L_{-m}$. We want an inner product of positive norm.

$$\langle\psi|L_m|\chi\rangle = \langle L_m^\dagger \psi|\chi\rangle$$

Example: For X^μ: $\langle 0;k|0;k'\rangle = 2\pi\delta(k-k')$.
$[\alpha_m^\mu, \alpha_n^\nu] = \eta^{\mu\nu}\delta_{m+n,0}, \quad \alpha_m^{\mu\dagger} = \alpha_{-m}^\mu$.
$\parallel \alpha_{-1}^\mu|0;k\rangle \parallel^2 < 0$ for $\mu=0$ and $\parallel \alpha_{-1}^\mu|0;k\rangle \parallel^2 > 0$ for $\mu = i \neq 0$.
This can be corrected by letting $\Phi \rightarrow X'$. This conformal field theory is unitary and its action is

$$S = \frac{1}{2\pi\alpha'}\int d^2z\,\partial\phi\bar{\partial}\phi.$$

Theorem: For highest weight A, $h \geq 0$,.
Proof: $2h\langle A|A\rangle = \langle A|\,[L_1, L_{-1}]\,|A\rangle = \parallel L_{-1}|A\rangle \parallel^2 \geq 0$.

Corollary: Any eigenstate of L_0 has $h \geq 0$ (it has energy $\geq h$ h.w.s. energy).
Theorem: $c > 0$: $\frac{c}{12}(m^3 - m) = \langle A| [L_m, L_{-m}] |A\rangle - 2m\langle A|L_0|A\rangle \geq 0$ for $L_0|A\rangle = 0$.
And we have gotten what we wanted; a positive norm for the highest weight state.