String Theory I

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UNIT 1

A first look at strings

Following *"String Theory"* by J. Polchinski, Vol.I.
Notes written by students (work still in progress).
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1.1 Units

First, we must explain the unit convention we are going to use. Take the following two results from Quantum Mechanics and Special Relativity:

$$E = \hbar\omega \tag{1.1.1}$$

$$E = mc^2 \tag{1.1.2}$$

These two equations link energy to frequency and mass through some constant of proportionality. The question is, are these constants fundamental in nature or created by man? The answer is that they are artificial creations, existing purely because of the units we have chosen to work in. We could easily choose units such that $\hbar = c = 1$. By doing this, the number of fundamental units in the universe is reduced to 1, e.g., energy, all others being related to it:

$$[\mathrm{energy}] = [1/\mathrm{time}] = [\mathrm{mass}] = [1/\mathrm{length}] \tag{1.1.3}$$

1.2 Why Strings?

Our motivation behind the developement of String Theory is our desire to find a unified theory of everything. One of the major obstacles that previous theories have been unable to overcome is the formation of a quantum theory of gravity, and it is in this respect that String Theory has had notable success

(in fact, at present, String theory is the only theory which includes gravitational interactions). This leads us to believe that, while String Theory may not be the final answer, it is certainly a step in the right direction.
Let us first discuss the problems one runs into when trying to create a quantum theory of gravity using Quantum Field Theory as our guide. Take the Hydrogen atom, whose energy levels are given by:

$$E_n = -\frac{E_1}{n^2} \ , \ E_1 = \frac{\hbar^2}{2m_e a_0} \ , \ a_0 = \frac{\hbar^2}{m_e e^2} \tag{1.2.1}$$

where E_1 is the ground state energy and n labels the energy levels.
Suppose we had only the most basic knowledge of physics: what would we guess the energy of the Hydrogen atom to be? The parameters of the system are the mass of the electron m_e, the mass of the proton m_p, $\hbar$ and the electron charge e. We may neglect m_p as we are interested in the energy levels of the electron. We might guess the energy to be

$$E_0 = m_e c^2 \tag{1.2.2}$$

Equations (1.2.1) and (1.2.2) are clearly not the same, but if we take the ratio we obtain

$$\frac{E_1}{E_0} = \frac{e^4}{\hbar^2 c^2} = \left(\frac{1}{137}\right)^2 = \alpha^2 \tag{1.2.3}$$

This is a ratio, so is independent of our choice of units, so α is a fundamental constant that exists in nature independent of our attempts to decribe the world, and indicates some fundamental physics underlying the situation. In fact α is the fine structure constant and describes the probability for an electromagnetic interaction, e.g. proton - electron scattering (of which hydrogen is a special case in which the scattering results in a bound state).
From the diagram of an e-p interaction,

INSERT FIGURE HERE

each vertex contributes a factor e to the amplitude for the interaction, so that

$$\mathcal{A}_1 \equiv \text{Amplitude} \sim \text{e}^2 \tag{1.2.4}$$

$$\text{Probability} \sim |\text{Amplitude}|^2 \sim \text{e}^4 \sim \alpha^2 \tag{1.2.5}$$

Now according to classical analysis, this is the only amplitude we would get for the interaction, but in quantum mechanics there can be intermediate scattering events that cannot be observed: e.g.,

INSERT FIGURE HERE

contributes $o(\alpha^2)$ to the overall amplitude for the interaction. Inserting a complete set of states, we obtain the amplitude of this second-order process in terms of $\mathcal{A}_1$,

$$\mathcal{A}_2 \sim \int \frac{dE'}{E'} |\mathcal{A}_1|^2 \sim \alpha^2 \int \frac{dE'}{E'} \tag{1.2.6}$$

This is logarithmically divergent. However, all higher-order amplitudes have the same divergence and when we sum the series in α:

$$\text{Amplitude} = () + \alpha() + \alpha^2() + ... \tag{1.2.7}$$

it yields finite expressions for physical quantities.

Let us try it for the gravitational interaction between two point masses, each of mass M, separated by a distance r. The potential energy is:

$$V = \frac{GM^2}{r} \tag{1.2.8}$$

which shows upon comparison with electromagnetism that the "charge" of gravity is $e_g \sim \sqrt{G}M$. A gravitational "Hydrogen atom" will have energy levels

$$E_n \sim \frac{E_1}{n^2} \ , \ E_1 \sim \frac{Me_g^4}{2\hbar^2} \sim G^2M^5 \tag{1.2.9}$$

Comparing with $E_0 = Mc^2$, we obtain the ratio

$$\frac{E_1}{E_0} \sim G^2M^4 \sim e_g^4 \tag{1.2.10}$$

Immediately we see problems with using this charge to describe the gravitational interaction, because e_g is energy (mass) dependent. The classical scattering amplitude is

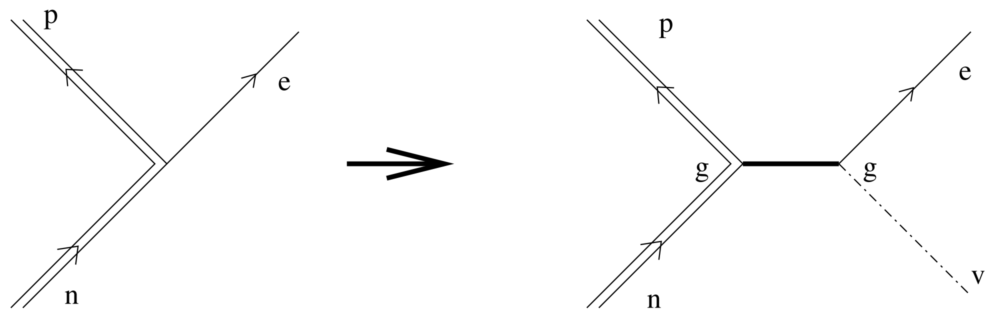
$$\mathcal{A}_1 \sim e_g^2 \sim GE^2 \tag{1.2.11}$$

where $E = Mc^2$ and the second-order contribution (exchange of two gravitons) is

$$\mathcal{A}_2 \sim \int \frac{dE'}{E'} |\mathcal{A}_1|^2 \sim G^2 \int dE'(E')^3 \tag{1.2.12}$$

which has a quartic divergence. Worse yet, higher-order amplitudes have worse divergences, making it impossible to make any sense of the perturbative expansion (1.2.7) (*non-renormalizability* of gravity).

We may see our way to a possible solution by considering the problem of beta decay: Initially it was treated as a three body problem with the proton - neutron - electron interaction occuring at one vertex. When the energies of the resultant electrons did not match experiment, the theory was modified to include a fourth particle, the neutrino, and the interaction was 'smeared' out: the proton and neutron interacted at one vertex, where a W boson was created, which traveled a short distance before reaching the electron - neutrino vertex.



So maybe we can solve our problems with quantum gravity by smearing out the interactions, so that the objects mediating the force are no longer point particles but extended one dimensional objects - strings.

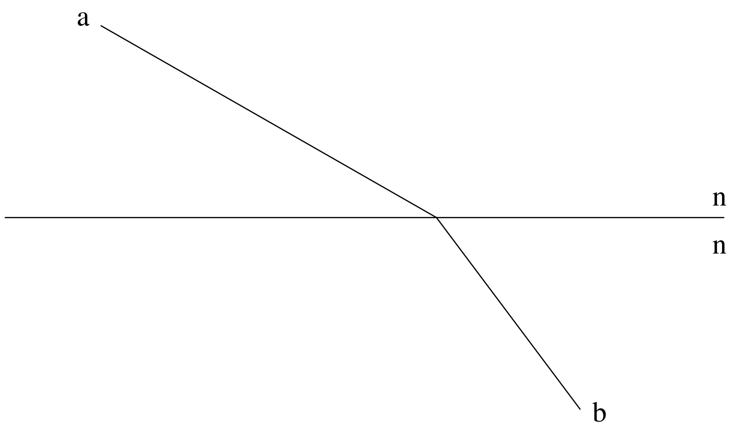
This is the general concept from which we will proceed. It is a difficult task - the gravitational interaction must obey a much larger symmetry than Lorentz invariance, it must be invariant under completely general co-ordinate transformations, and we must of course still be able to describe the weak, strong and electromagnetic interactions.

In this chapter we will take a first look at strings. Initially we examine the completely general equations of motion for a point particle using the method of least action, and then apply that method to the case of a general string moving in D dimensions. We will obtain the equations of motion for the string, and then attempt to quantize it and obtain its energy spectrum. This will highlight some basic results of string theory, as well as some fundamental difficulties.

1.3 Point particle

We begin by examining the case of a point particle, illustrating the method we will use for strings. The trajectory of a point particle in D-dimensional space is decribed by coordinates $X^\mu(\tau)$, where τ is a parameter of the particle's trajectory. For a massive particle, τ is its proper time. X^0 will be a timelike cordinate, the remaining $\vec{X}$ spanning space. Infinitesimal distances in spacetime:

$$-dT^2 = ds^2 = -(dX^0)^2 + (d\vec{X})^2 \tag{1.3.1}$$



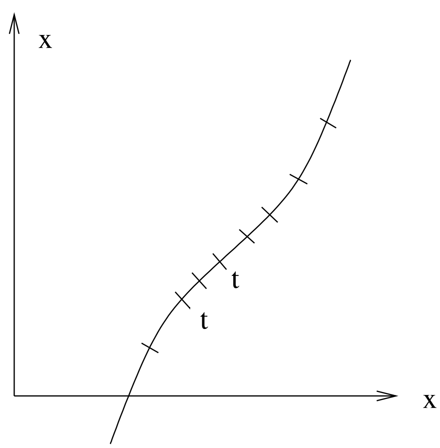
We wish to derive the equation of motion from an action principle. This is similar to Fermat's principle of minimizing time along a light ray. For the light ray joining points A and B, this yields Snell's Law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{1.3.2}$$

Along the trajectory of a free relativistic particle, proper time is maximized. Therefore, trajectories are obtained as extrema of the action

$$S = m \int_a^b dT \tag{1.3.3}$$

where we multiplied by the mass to obtain a dimensionless quantity (in units where $\hbar = 1$).



a and b are the fixed start and end points on the trajectory. There is a problem that when the mass $m = 0$, since the action is zero. This problem will be cleared up a little later.

We can write the action as

$$S = m\int_a^b d\tau \frac{dT}{d\tau} = m\int_a^b d\tau \sqrt{\left(\frac{dX^0}{d\tau}\right)^2 - \left(\frac{d\vec{X}}{d\tau}\right)^2} \qquad (1.3.4)$$

(invariant under reparametrizations $\tau \to \tau'(\tau)$) from which we can define the Lagrangian for the system:

$$L = m\sqrt{\left(\frac{dX^0}{d\tau}\right)^2 - \left(\frac{d\vec{X}}{d\tau}\right)^2} \qquad (1.3.5)$$

Using the convention $\dot{X} = dX/d\tau$, we write Lagrange's equations:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) = \frac{\partial L}{\partial X^\mu} \qquad (1.3.6)$$

For this Lagrangian,

$$\frac{\partial L}{\partial X^\mu} = 0 \qquad (1.3.7)$$

and we obtain the equation of motion for the point particle:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) = \frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{L}\right) = 0 \qquad (1.3.8)$$

To see the physical meaning of this equation, switch from τ to X^0 (or set $\tau = X^0$ to "fix the gauge"). Then the 3-velocity is

$$v^i = \frac{dX^i}{dX^0} \qquad (1.3.9)$$

and the equation of motion (1.3.8) reads

$$\ddot{u}^\mu = 0 \ , \ u^\mu = \gamma(1,\vec{v}) \ , \ \gamma = \frac{1}{L} = \frac{1}{\sqrt{1-\vec{v}^2}} \qquad (1.3.10)$$

i.e., that the acceleration is constant, as expected.
We will now look at a better expression for the action: defining an extra field $\eta(\tau)$, which at the moment is arbitrary, we write a new Lagrangian:

$$L = \frac{1}{2\eta}\dot{X}^\mu \dot{X}_\mu - \frac{1}{2}\eta m^2 \qquad (1.3.11)$$

This has the nice feature that it is still valid for $m = 0$. Lagrange's equation for η is

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\eta}}\right) = 0 = \frac{\partial L}{\partial \eta} \qquad (1.3.12)$$

which implies

$$-\frac{1}{2\eta^2}\dot{X}^\mu\dot{X}_\mu - \tfrac{1}{2}m^2 = 0 \;\;\Rightarrow\;\; \eta = \sqrt{\frac{-\dot{X}^\mu\dot{X}_\mu}{m^2}} \tag{1.3.13}$$

Using this equation to eliminate η from the Lagrangian (1.3.11), we get back the original Lagrangian (1.3.5). Varying X^μ ,we obtain from (1.3.11)

$$\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\eta}\right) = 0 \tag{1.3.14}$$

which agrees with the previous eq. (1.3.8).
We will now examine the meaning of the field $\eta(\tau)$. The trajectory is parameterized by some co-ordinate of the system in terms of which an infinitesimal distance along the trajectory, ds, can be expressed. Let us suppose our trajectory is along the y axis. Then it would be easiest to parameterize the system with the co-ordinate y, and then $ds = dy$. We could, however, choose the parameter to be θ, the the angle between a line drawn from a fixed point on the x axis at a distance ℓ to a position on the y axis. Then our new distance would be

$$ds = \ell\left(\frac{1}{\cos^2\theta}\right)d\theta \tag{1.3.15}$$

The factor $\ell/cos^2\theta$ is our η^2. It represents the geometry of the system due to our choice of co-ordinates. In Minkowski space,

$$ds^2 = -\gamma_{\tau\tau}d\tau^2 \;\;,\;\; \gamma_{\tau\tau} = \eta^2 \tag{1.3.16}$$

where $\gamma_{\tau\tau}$ is the (single component) metric tensor. Under a reparametrization,

$$\tau \to \tau'(\tau) \;\;,\;\; \gamma_{\tau\tau} \to \left(\frac{d\tau}{d\tau'}\right)^2\gamma_{\tau\tau} \tag{1.3.17}$$

i.e. $\gamma_{\tau\tau}$ transforms as a tensor (this follows from the invariance of ds^2). We can see that the action is invariant under this transformation: we have

$$\eta' = \frac{d\tau}{d\tau'}\eta \tag{1.3.18}$$

and $\dot{X}^\mu$ transforms as

$$\dot{X}^{\mu\prime} = \frac{dX^\mu}{d\tau'} = \frac{d\tau}{d\tau'}\frac{dX^\mu}{d\tau} \tag{1.3.19}$$

Thus the Lagrangian transforms as

$$L' = \frac{d\tau'}{d\tau}L \tag{1.3.20}$$

and the action transforms as

$$S = m\int_a^b d\tau L = m\int_a^b d\tau' L' \tag{1.3.21}$$

thus proving its invariance.
Now let us form the Hamiltonian for the system: the conjugate momenta to co-ordinate X^μ and the parameter η are

$$P_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{\dot{X}^\mu}{\eta} \ , \ P_\eta = \frac{\partial L}{\partial \dot{\eta}} = 0 \tag{1.3.22}$$

Then the Hamiltonian is:

$$H = P_\mu \dot{X}^\mu + P_\eta \dot{\eta} - L = \tfrac{1}{2}\eta(P^\mu P_\mu + m^2) \tag{1.3.23}$$

Here the role of η is that of a Lagrange multiplier; it is not a dynamical variable. From Hamilton's equation:

$$\frac{\partial H}{\partial \eta} = \dot{P}_\eta = 0 = P^\mu P_\mu + m^2 \tag{1.3.24}$$

which is Einstein's equation for the relativistic energy of a paricle of mass m. Define χ as:

$$\chi = \frac{1}{2m} P^\mu P_\mu + \tfrac{1}{2}m = \frac{H}{\eta m} \tag{1.3.25}$$

Then $\chi = 0$ is a constraint which generates reparametrizations through Poisson brackets:

$$\delta X^\mu \sim \{X^\mu \, , \, \chi\} = \frac{P^\mu}{m} \ , \ \delta P^\mu \sim \{P^\mu \, , \, \chi\} = 0 \tag{1.3.26}$$

We may identify X^0 with time and solve for its conjugate momentum

$$P_0 = \sqrt{\vec{P}^2 + m^2} \tag{1.3.27}$$

This is the true Hamiltonian of the system. Equations of motion:

$$\dot{X}^i = \frac{\partial P_0}{\partial P_i} = \frac{P^i}{P_0} \ , \ \dot{P}_i = 0 \tag{1.3.28}$$

same equation as before, if we note $v^i = \dot{X}^i$, $1 - \vec{v}^2 = m^2/P_0^2$ and therefore,

$$\frac{\dot{v}^i}{\sqrt{1-\vec{v}^2}} = \frac{P^i}{m} \tag{1.3.29}$$

This system may be quantized by

$$[P_i \, , \, X^j] = -i\delta_i^j \tag{1.3.30}$$

Eigenstates of the Hamiltonian:

$$H|\vec{k}\rangle = \omega|\vec{k}\rangle \ , \ \omega = \sqrt{\vec{k}^2 + m^2} \tag{1.3.31}$$

Alternatively, we may define light-cone coordinates in spacetime:

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^1) \qquad \vec{X}_T = (X^2, \ldots, X^{D-1}) \tag{1.3.32}$$

The Lagrangian reads

$$L = \frac{1}{2\eta}\dot{X}^\mu \dot{X}_\mu - \frac{1}{2}\eta m^2 = -\frac{1}{\eta}\dot{X}^+\dot{X}^- + \frac{1}{2\eta}\dot{\vec{X}}_T^2 - \frac{1}{2}\eta m^2 \tag{1.3.33}$$

Let $X^+ = \tau$ play the role of time; then P_+ is the Hamiltonian. X^- and $\vec{X}_T$ are the coordinates and P_- and $\vec{P}_T$ are their conjugate momenta. The Lagrangian becomes:

$$L = -\frac{1}{\eta}\dot{X}^- + \frac{1}{2\eta}\dot{X}_i^2 - \frac{1}{2}\eta m^2 \tag{1.3.34}$$

yielding

$$P_- = \frac{\partial L}{\partial \dot{X}^-} = -\frac{1}{\eta} \tag{1.3.35}$$

$$P_i = \frac{\partial L}{\partial \dot{X}^i} = \frac{1}{\eta}\dot{X}_i \tag{1.3.36}$$

The Hamiltonian is

$$P_+ = \dot{X}^- P_- + \dot{X}^i P_i - L = \frac{\vec{P}_T^2 + m^2}{2P_-} \tag{1.3.37}$$

Note that there is no term $P_+\dot{X}^+$ because X^+ is not a dynamical variable in the gauge-fixed theory.
Quantization:

$$[P_i\,,\, X^j] = -i\delta_i^j \ , \ [P_-\,,\, X^-] = -i \tag{1.3.38}$$

Eigenstates of the Hamiltonian:

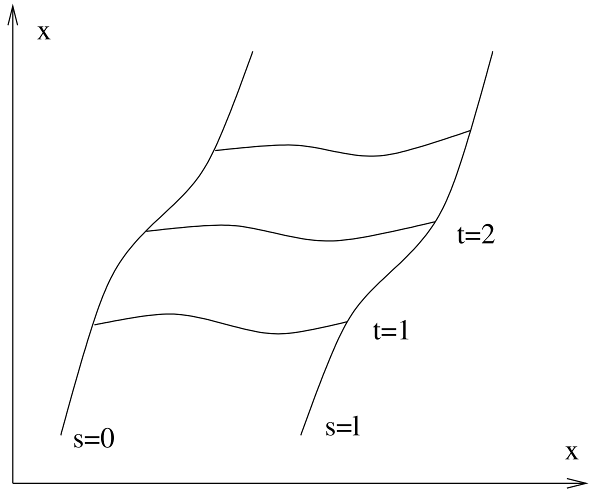
$$P_+|k_-, \vec{k}_T\rangle = \omega_+|k_-, \vec{k}_T\rangle \ , \ \omega_+ = \frac{\vec{k}_T^2 + m^2}{2k_-} \tag{1.3.39}$$

To compare with our earlier result, define ω and k_1 by

$$\omega_+ = \frac{1}{\sqrt{2}}(\omega + k_1) \ , \ k_- = \frac{1}{\sqrt{2}}(\omega - k_1) \tag{1.3.40}$$

Then $\omega^2 - \vec{k}^2 = \omega^2 - k_1^2 - \vec{k}_T^2 = 2\omega_+ k_- - \vec{k}_T^2 = m^2$.

1.4 Strings



Parametrize the string with $\sigma \in [0, \ell]$. In analogy to Fermat's minimization of time, we will minimize the area of the world-sheet mapped out by the string. To find an expression for the area, pick a point on the worldsheet and draw the tangent vectors

$$\vec{t}_\tau = \frac{\partial \vec{X}}{\partial \tau} = \dot{\vec{X}} \ , \ \vec{t}_\sigma = \frac{\partial \vec{X}}{\partial \sigma} = \vec{X}' \tag{1.4.1}$$

The infinitesimal parallelepiped with sides $\vec{t}_\tau d\tau$ and $\vec{t}_\sigma d\sigma$ has area

$$dA = |\vec{t}_\tau \times \vec{t}_\sigma| d\tau d\sigma \tag{1.4.2}$$

We obtain the total area by integrating over the worldsheet coordinates (τ, σ). The action to be minimized is the Nambu-Goto action

$$S_{NG} = T \int dA \tag{1.4.3}$$

where T is a constant that makes the action dimensionless (tension of the string). Using

$$(\vec{a} \times \vec{b})^2 = \vec{a}^2 \vec{b}^2 - (\vec{a} \cdot \vec{b})^2 = \begin{vmatrix} \vec{a}^2 & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b}^2 \end{vmatrix} \tag{1.4.4}$$

we deduce

$$S_{NG} = T \iint d\tau d\sigma \sqrt{-\det\ h_{ab}} \quad , \quad h_{ab} = \partial_a X^\mu \partial_b X_\mu \tag{1.4.5}$$

where the minus sign in the square root is because we are working in Minkowski space. h_{ab} is the two-dimensional metric induced on the worldsheet,

$$h_{ab} = \begin{pmatrix} \dot{\vec{X}}^2 & \dot{\vec{X}} \cdot \vec{X}' \\ \dot{\vec{X}} \cdot \vec{X}' & \vec{X}'^2 \end{pmatrix} \tag{1.4.6}$$

and the Lagrangian density is

$$L = \sqrt{-\det h_{ab}} = \sqrt{(\dot{\vec{X}} \cdot \vec{X}')^2 - \dot{\vec{X}}^2 \vec{X}'^2} \tag{1.4.7}$$

where we ignored T (or set $T = 1$).

$$\sigma^a = (\tau, \sigma) \qquad a = 0, 1 \tag{1.4.8}$$

Lagrange Equation:

$$\partial_a \frac{\partial L}{\partial(\partial_a X^\mu)} = 0 \ \Rightarrow \ \left(\frac{\partial L}{\partial \dot{X}^\mu}\right)^{\cdot} + \left(\frac{\partial L}{\partial X^{\mu\prime}}\right)' = 0 \tag{1.4.9}$$

We have

$$\frac{\partial L}{\partial \dot{X}^\mu} = \frac{\dot{X}_\mu \vec{X}'^2 - X'_\mu \dot{\vec{X}} \cdot \vec{X}'}{L} \ , \ \frac{\partial L}{\partial X^{\mu\prime}} = \frac{X'_\mu \dot{\vec{X}}^2 - \dot{X}_\mu \dot{\vec{X}} \cdot \vec{X}'}{L} \tag{1.4.10}$$

Let us choose the worldsheet coordinates (τ, σ) so that the metric h_{ab} becomes proportional to the two-dimensional Mikowski metric,

$$h_{ab} \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.4.11}$$

Upon comparison with (1.4.6), we deduce the constraints

$$\dot{\vec{X}}^2 + \vec{X}'^2 = 0 \ , \ \dot{\vec{X}} \cdot \vec{X}' = 0 \tag{1.4.12}$$

which may also be cast into the form

$$(\dot{\vec{X}} \pm \vec{X}')^2 = 0 \tag{1.4.13}$$

Using the constraints, the Lagrange eq. (1.4.9) reduces to

$$\frac{\partial^2 X_\mu}{\partial \tau^2} = \frac{\partial^2 X_\mu}{\partial \sigma^2} \tag{1.4.14}$$

which is the wave equation. Introducing coordinates

$$\sigma^\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma) \tag{1.4.15}$$

so that

$$\partial_\pm = \frac{1}{\sqrt{2}}(\partial_\tau \pm \partial_\sigma) \tag{1.4.16}$$

the constraints and the wave equation become, respectively,

$$(\partial_\pm \vec{X})^2 = 0 \ , \ \partial_+ \partial_- X^\mu = 0 \tag{1.4.17}$$

The general solution to the wave equation is

$$X^\mu = f(\sigma^+) + g(\sigma^-) \tag{1.4.18}$$

where f and g are arbitrary functions. Let us write out the action explicitly with the simplifications made above. The parameter τ takes values from -∞ to +∞ and σ takes values between 0 and l, the length of the string. Anticipating future results, we write the constant T

$$T = \frac{1}{2\pi\alpha'} \tag{1.4.19}$$

where α' is called the Regge slope. Then, from equation (81),

$$S_{NG} = \frac{1}{4\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma(\dot{X}^2 - \acute{X}^2) \tag{1.4.20}$$

$$= \frac{1}{4\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma(\partial_a X^\mu \partial^a X_\mu) \tag{1.4.21}$$

We now examine the effects of boundary conditions which have yet to be taken into account in the equations of motion. We start off by varying the co-ordinates:

$$X^\mu \to X^\mu + \delta X^\mu \tag{1.4.22}$$

Starting from equation (93), the variation in the action is

$$\delta S_{NG} = \frac{1}{4\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma\,(\partial_a \delta X^\mu \partial^a X_\mu + \partial_a X^\mu \partial^a \delta X_\mu) \tag{1.4.23}$$

$$= \frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma\,(\partial_a \delta X^\mu \partial^a X_\mu) \tag{1.4.24}$$

And noting the total derivative

$$\partial_a(\delta X^\mu \partial^a X_\mu) = \partial_a \delta X^\mu \partial^a X_\mu + \delta X^\mu \partial_a \partial^a X_\mu \tag{1.4.25}$$

The last term is just the wave equation, which equals zero, so we are left with:

$$\delta S_{NG} = \frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma \partial_a(\delta X^\mu \partial^a X_\mu) \tag{1.4.26}$$

$$= \frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty} d\tau \left[\delta X^\mu \acute{X}_\mu\right]_0^l \tag{1.4.27}$$

We will now introduce two sets of boundary conditions that will get rid of this term and leave the equations of motion unchanged at the boundary:
Open string (Neumann) boundary conditions, which correspond to there being no forces at the boundary:

$$\acute{X}_\mu(\sigma = 0) = \acute{X}_\mu(\sigma = l) = 0 \tag{1.4.28}$$

Closed string boundary conditions, which means there is no boundary and the string co-ordinates are periodic:

$$X_\mu(\sigma = 0) = X_\mu(\sigma = l) \tag{1.4.29}$$

$$\acute{X}_\mu(\sigma = 0) = \acute{X}_\mu(\sigma = l) \tag{1.4.30}$$

We shall now look at deriving invariant quantities in the theory from symmetries using Noether's theorem. We will start with Poincare invariance, which is invariant under the transformation

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + Y^\mu \tag{1.4.31}$$

where Λ and Y are constant quantities. We construct the Noether current by applying this symmetry to the action. Taking the second term, we write the change in X^μ as

$$\delta X^\mu = Y^\mu \tag{1.4.32}$$

The change in the action is found by inserting this into equation (96):

$$\delta S_{NG} = \frac{1}{2\pi\alpha'} \int_{-\infty}^{+\infty} \int_0^l d\tau d\sigma \partial_a Y^\mu \partial^a X_\mu \tag{1.4.33}$$

The Noether current P^a_μ is defined by

$$\delta S = \int \int d\tau d\sigma \partial_a Y^\mu P^a_\mu \tag{1.4.34}$$

So in this case,

$$P^a_\mu = T\partial^a X_\mu \tag{1.4.35}$$

and

$$\partial_a P^a_\mu = T\partial_a \partial^a X_\mu = 0 \tag{1.4.36}$$

i.e. the Poincare symmetry has led to a conserved quantity in the Noether current.
Now let's do the same for the variation

$$\delta X^\mu = \epsilon \Lambda^\mu{}_\nu X^\nu \tag{1.4.37}$$

where ϵ is a small quantity. The variation in the action is now

$$\delta S_{NG} = \frac{1}{2\pi\alpha'} \int_{-\infty}^{+\infty} \int_0^l d\tau d\sigma \left(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu \right) \Lambda_{\mu\nu} \partial^a \epsilon \tag{1.4.38}$$

From this we can define the current as

$$J^{\mu\nu}_a = T(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu) \tag{1.4.39}$$

which is conserved:

$$\partial^a J_a^{\mu\nu} = T(\partial^a X^\mu \partial_a X^\nu - \partial^a X^\nu \partial_a X^\mu + X^\mu \partial^a \partial_a X^\nu - X^\nu \partial^a \partial_a X^\mu) = 0 \quad (1.4.40)$$

since the first two terms cancel and the last two terms are the wave equation, which equals zero.

Analogous to electromagnetism, we can define a charge. In EM, the charge is the integral over a volume of the zeroth (time) component of the current 4-vector j^μ, so here we define the charge for the current $P_\mu^a = (P_\mu^\tau, P_\mu^\sigma)$ as

$$P_\mu = \int_0^l d\sigma P_\mu^\tau \quad (1.4.41)$$

P_μ^τ can be seen to be the momentum of the string at a certain point, so $P_\mu, \mu > 0$ is the total momentum of the string, and P_0 is the total energy of the string.

Differentiating P_μ with respect to time:

$$\frac{dP_\mu}{d\tau} = \int_0^l d\sigma \dot{P}_\mu^\tau = \int_0^l d\sigma P_\mu^{\prime\sigma} = P_\mu^\sigma \Big|_0^l = 0 \quad (1.4.42)$$

where the second equality follows from the wave equation and the last equality form the boundary conditions at 0 and l. This is simply conservation of momentum.

Similarly, for the current $J_{\mu\nu}^a$, which we interpret as the angular momentum of the string, we define the charge

$$J_{\mu\nu} = \int_0^l d\sigma J_{\mu\nu}^a \quad (1.4.43)$$

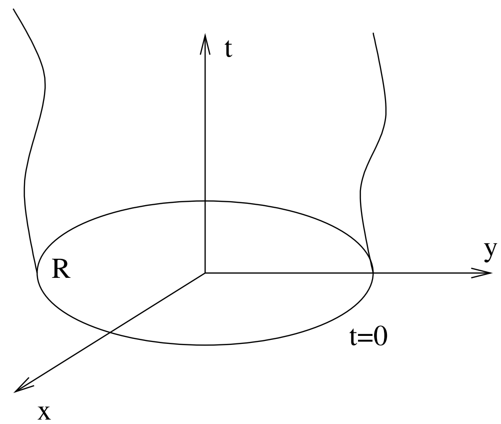
and we find

$$\frac{dJ_{\mu\nu}}{d\tau} = 0 \quad (1.4.44)$$

So from Poincare invariance we obtain conservation of momentum and angular momentum.

We now give two examples to demonstrate the above concepts:

Example 1



We take a closed string whose initial configuration is a circle centered on the x-y origin with radius R, and whose initial velocity is $\vec{v} = 0$. Then $X^\mu = (t, x, y)$. The solution to the wave equation satisfying these boundary conditions is:

$$x = R\cos\frac{2\pi\tau}{l}\cos\frac{2\pi\sigma}{l} \tag{1.4.45}$$

$$y = R\cos\frac{2\pi\tau}{l}\sin\frac{2\pi\sigma}{l} \tag{1.4.46}$$

$$t = \frac{2\pi R}{l}\tau \tag{1.4.47}$$

Let's check the constraints $\dot{X}^2 + \acute{X}^2 = 0$:

$$-\dot{t}^2 + \dot{x}^2 + \dot{y}^2 - \acute{t}^2 + \acute{x}^2 + \acute{y}^2 \tag{1.4.48}$$

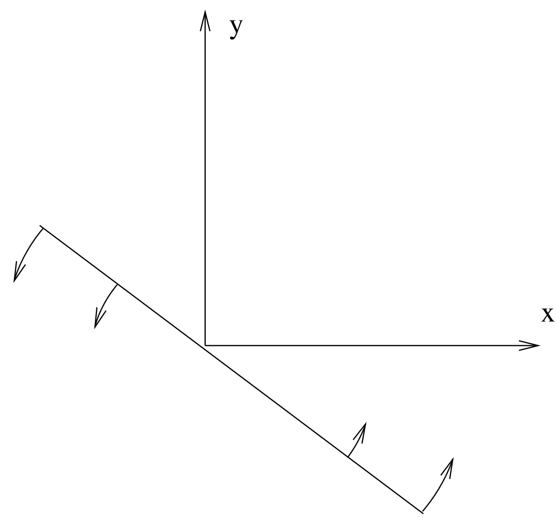
$$= -\left(\frac{2\pi R}{l}\right)^2 + \left(\frac{2\pi R}{l}\right)^2 \sin^2\frac{2\pi\tau}{l} + \left(\frac{2\pi R}{l}\right)^2 \cos^2\frac{2\pi\tau}{l} = 0 \tag{1.4.49}$$

The total energy of the string is given by

$$P_0 = \int_0^l d\sigma P_0^\tau = T\int_0^l d\sigma \partial_\tau X_0 = T\int_0^l d\sigma \partial_\tau t = \frac{2\pi RT}{l}\int_0^l d\sigma = 2\pi RT = E \tag{1.4.50}$$

The length of the string is $2\pi R$, so T can be identified as energy per unit length of the string, i.e. the tension.

Example 2



Now we consider an open string rotating in the x-y plane. The solution to the wave equation is

$$x = R\cos\frac{\pi\tau}{l}\cos\frac{\pi\sigma}{l} \tag{1.4.51}$$

$$y = R\cos\frac{\pi\tau}{l}\sin\frac{\pi\sigma}{l} \tag{1.4.52}$$

$$t = \frac{\pi R}{l}\tau \tag{1.4.53}$$

The speed of each point on the string is given by

$$\vec{v} = \left(\frac{dx}{dt},\frac{dy}{dt}\right) = \frac{l}{\pi R}\left(\frac{dx}{d\tau},\frac{dy}{d\tau}\right) = \cos\frac{\pi\sigma}{l}\left(-\sin\frac{\pi\tau}{l},\cos\frac{\pi\tau}{l}\right) \tag{1.4.54}$$

From which we see that

$$\vec{v}^2 = \cos^2\frac{\pi\sigma}{l} \tag{1.4.55}$$

Thus at the ends of the string $\sigma = 0, l$ we see that $\vec{v}^2 = 1$, i.e. the ends of the string travel at the speed of light. This is to be expected, since the string is massless and there are no forces on the ends of the string (Neumann boundary conditions). The intermediate points on the string don't travel at the speed of light because they experience the tension of the string.
The energy of the string is worked out exactly as before, and is found to be:

$$P_0 = TR\pi \tag{1.4.56}$$

Let us now work out the z component of the angular momentum of the string:

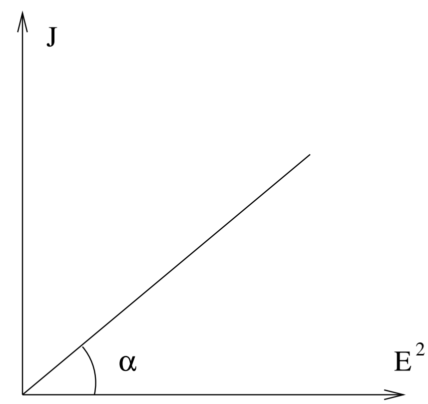
$$J_xy = T\int_0^l d\sigma(x\partial_a y - y\partial_a x) \tag{1.4.57}$$

$$= T\int_0^l d\sigma\frac{\pi R^2}{l}\left(\cos^2\frac{\pi\tau}{l}\cos^2\frac{\pi\sigma}{l} + \sin^2\frac{\pi\tau}{l}\cos^2\frac{\pi\sigma}{l}\right) \tag{1.4.58}$$

$$= \frac{T\pi R^2}{l}\int_0^l d\sigma\cos^2\frac{\pi\sigma}{l} = \tfrac{1}{2}T\pi R^2 \tag{1.4.59}$$

From the total energy and angular momentum we can form the quantity

$$\frac{J_{xy}}{E^2} = \frac{1}{2\pi T} = \alpha' \tag{1.4.60}$$



Regge Slope

Now if we want our strings to correspond to fundamental particles, their angular momenta correspond to the spins of the particles, which are either integer or half integer. Thus the above relation suggests that if we plot the spins of particles against their energies squared, we would observe a straight line. This was indeed observed for strongly interacting particles. In fact, string theory started out being a theory of the strong interaction, but then QCD came along. Now string theory become a theory of everything!

1.5 The Mode Expansion

We must first introduce the Hamiltonian formalism for future reference: we have the Lagrangian

$$L = \tfrac{1}{2}T(\dot{X}^2 - \acute{X}^2) \tag{1.5.1}$$

The conjugate momentum to the variable X is

$$\Pi_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = T\dot{X}_\mu \tag{1.5.2}$$

Then the Hamiltonian is given by

$$H = \int_0^l d\sigma(\Pi_\mu \dot{X}^\mu - L) \tag{1.5.3}$$

$$= \tfrac{1}{2}T\int_0^l d\sigma\left(\frac{\Pi^2}{T^2} + \acute{X}^2\right) \tag{1.5.4}$$

$$= \tfrac{1}{2}T\int_0^l d\sigma(\dot{X}^2 + \acute{X}^2) = 0 \tag{1.5.5}$$

So this is not a good Hamiltonian. We will find a good one later.
Now let us examine the mode expansion for open strings, which obey the Neumann boundary conditions. The general solution to the wave equation is a fourier expansion. Here we write such an expansion as follows:

$$X^\mu = x^\mu + 2\alpha'\frac{\pi}{l}p^\mu\tau + i\sqrt{2\alpha'}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\frac{1}{n}\alpha_n^\mu e^{\frac{-\pi i n\tau}{l}} cos\left(\frac{n\pi\sigma}{l}\right) \tag{1.5.6}$$

where x^μ is a constant and the first mode p^μ has been written out explicitly. The constants have been chosen on dimensional grounds
The fact that X^μ must be a real number yields the condition:

$$\alpha_n = (\alpha_{-n})^* \tag{1.5.7}$$

The conjugate momentum to X^μ is

$$\Pi^\mu = T\dot{X}^\mu = 2\alpha' T\frac{\pi}{l}p^\mu + \frac{\alpha' T}{l}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\alpha_n^\mu e^{\frac{-\pi i n\tau}{l}} cos\left(\frac{n\pi\sigma}{l}\right) \tag{1.5.8}$$

The centre of mass position of the string is given by

$$\bar{X}^\mu = \frac{1}{l}\int_0^l d\sigma X^\mu = x^\mu + 2\alpha'\frac{\pi}{l}p^\mu\tau \tag{1.5.9}$$

so the centre of mass moves in a straight line. The total momentum is

$$P^\mu = \int_0^l d\sigma\Pi^\mu = 2\alpha' l\frac{\pi}{l}p^\mu T = p^\mu \tag{1.5.10}$$

In both cases all the harmonic terms vanish on integration.
We now move to the light cone gauge mentioned before: we defined the transverse co-ordinates X^+ and X^- and fix the gauge by imposing the condition:

$$X^+ = x^+ + 2\alpha'\frac{\pi}{l}p^+\tau \qquad \alpha_n^+ = 0 \quad \forall n \tag{1.5.11}$$

and set

$$2\alpha'\frac{\pi}{l}p^{+} = 1 \tag{1.5.12}$$

so that

$$X^{+} = x^{+} + \tau \tag{1.5.13}$$

which is the light cone gauge condition from before with an arbitrary constant x^{+}.
The centre of mass position can now be written

$$\bar{X}^{\mu} = x^{\mu} + \frac{p^{\mu}}{p^{+}}\tau \tag{1.5.14}$$

The constraint from before:

$$(\dot{X} \pm \acute{X})^2 = 0 \tag{1.5.15}$$

$$-2(\dot{X}^{+} \pm \acute{X}^{+})(\dot{X}^{-} \pm \acute{X}^{-}) + (\dot{X}^{i} \pm \acute{X}^{i})^2$$

$$\begin{aligned}
&= -((\dot{X}^0 + \dot{X}^1) \pm (\acute{X}^0 + \acute{X}^1))((\dot{X}^0 - \dot{X}^1) \pm (\acute{X}^0 - \acute{X}^1)) + (\dot{X}^i \pm \acute{X}^i)^2 \\
&= -((\dot{X}^0 \pm \acute{X}^0) + (\dot{X}^1 \pm \acute{X}^1))((\dot{X}^0 \pm \acute{X}^1) - (\dot{X}^1 \pm \acute{X}^1)) + (\dot{X}^i \pm \acute{X}^i)^2 \\
&= -(\dot{X}^0 \pm \acute{X}^0)^2 + (\dot{X}^1 \pm \acute{X}^1)^2 + (\dot{X}^i \pm \acute{X}^i)^2 = (\dot{X}^{\mu} \pm \acute{X}^{\mu})^2 = 0
\end{aligned} \tag{1.5.16}$$

$$\therefore 2(\dot{X}^{+} \pm \acute{X}^{+})(\dot{X}^{-} \pm \acute{X}^{-}) = (\dot{X}^{i} \pm \acute{X}^{i})^2 \tag{1.5.17}$$

$$\therefore 2(\dot{X}^{-} \pm \acute{X}^{-}) = (\dot{X}^{i} \pm \acute{X}^{i})^2 \tag{1.5.18}$$

The mode expansion of the world-sheet fields:

$$X^{\mu} = x^{\mu} + 2\alpha'\frac{\pi}{l}p^{\mu}\tau + i\sqrt{2\alpha'}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\frac{1}{n}\alpha_n^{\mu}e^{\frac{-\pi i n \tau}{l}}\cos\left(\frac{n\pi\sigma}{l}\right) \tag{1.5.19}$$

$$\dot{x}^{-} \pm \acute{x}^{-} == 2\alpha'\frac{\pi}{l}p^{-} + \sqrt{2\alpha'}\frac{\pi}{l}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\alpha_n^{-}e^{\frac{-\pi i n}{l}(\tau\pm\sigma)} \tag{1.5.20}$$

$$\therefore (\dot{x}^{i} \pm \acute{x}^{i})^2 = \left[2\alpha'\frac{\pi}{l}p^{i} + \sqrt{2\alpha'}\frac{\pi}{l}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\alpha_n^{i}e^{\frac{-\pi i n}{l}(\tau\pm\sigma)}\right]^2 \tag{1.5.21}$$

You can write α_n^- in terms of α_n^i:

$$(\dot{X}^i \pm \acute{X}^i)^2 = \frac{p^i p^i}{p^+ p^+} + \frac{\pi^2}{l^2} 2\alpha' \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i + 2\frac{\pi^2}{l^2}(2\alpha')^{3/2} p^i f(\sigma,\tau) \tag{1.5.22}$$

Concentrating on the zeroth mode:

$$2\frac{p^-}{p^+} = \frac{p^i p^i}{(p^+)^2} + \frac{1}{\alpha'(p^+)^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i \tag{1.5.23}$$

$$\therefore p^- = \frac{1}{2p^+}\left[p^i p^i + \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i\right] = H \tag{1.5.24}$$

From Einstein's equation:

$$p^\mu p_\mu + m^2 = 0 \tag{1.5.25}$$

$$m^2 = -p^\mu p_\mu = 2p^+ p^- - p^i p^i = \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i \tag{1.5.26}$$

Now quantizing the system:

$$\left[p^i, x^j\right] = -i\delta^{ij} \qquad \left[\Pi^i(\sigma), X^j(\sigma')\right] = -i\delta^{ij}\delta(\sigma - \sigma') \tag{1.5.27}$$

$$\Pi_\mu = T\dot{X}_\mu = \frac{p^\mu}{l} + \sqrt{2\alpha'}\frac{\pi}{l}T \sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty} \alpha_n^\mu e^{\frac{-\pi i n\tau}{l}} \cos\left(\frac{n\pi\sigma}{l}\right) \tag{1.5.28}$$

$$X^\mu = x^\mu + 2\alpha'\frac{\pi}{l}p^\mu\tau + i\sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty} \frac{1}{n}\alpha_n^\mu e^{\frac{-\pi i n\tau}{l}} \cos\left(\frac{n\pi\sigma}{l}\right) \tag{1.5.29}$$

$$\left[\Pi^i(\sigma), X^j(\sigma')\right] = \frac{1}{l}\left[p^i, x^j\right] + \frac{i}{l} \sum_{m,n\neq 0} \frac{1}{n}\left[\alpha_m^i, \alpha_n^j\right] e^{\frac{-\pi i\tau}{l}(n+m)} \cos\left(\frac{n\pi\sigma}{l}\right) \cos\left(\frac{m\pi\sigma'}{l}\right) \tag{1.5.30}$$

$$\left[\alpha_m^i, \alpha_n^j\right] = m\delta^{ij}\delta_{m+n,0} \tag{1.5.31}$$

$$\therefore \quad \left[\Pi^i(\sigma), X^j(\sigma')\right] = -i\delta^{ij}\delta(\sigma - \sigma') \tag{1.5.32}$$

This is just the commutation relation for the harmonic oscillator operators with nonstandard normalization

$$a = \frac{1}{\sqrt{m}}\alpha_m^i \qquad a^\dagger = \frac{1}{\sqrt{m}}\alpha_- m^i \tag{1.5.33}$$

$$\therefore \quad \left[a^\dagger, a\right] = 1 \tag{1.5.34}$$

The state $|0,k\rangle$ is defined to be annihilated by the lowering operators and to be an eigenstate of the center-of-mass momenta

$$a|0\rangle = a|0,k\rangle = 0 \tag{1.5.35}$$

$$\Pi_{m,i}|0,k\rangle = |0,k\rangle \tag{1.5.36}$$

$$M^2 = \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i = \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty} m a^\dagger a = \frac{1}{\alpha'} N \tag{1.5.37}$$

$$M^2|0,k\rangle = \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(D-2)n}{2}|0,k\rangle = \frac{(D-2)}{2\alpha'} \left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} n \right) |0,k\rangle \tag{1.5.38}$$

We have to perform the sum:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} n \tag{1.5.39}$$

To perform this sum, we will multiply by the sum by $e^{\frac{-2\pi n\epsilon}{l}}$ and then take the limit of $\epsilon \to 0$

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} n e^{\frac{-2\pi n\epsilon}{l}} = \frac{\partial}{\partial C} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} n e^{-nC} = \frac{\partial}{\partial C} \left(\frac{1}{1-e^{-C}} \right) = \frac{e^{-C}}{(1-e^{-C})^2} = \frac{1}{C^2} - \frac{1}{12} \tag{1.5.40}$$

$$\text{where } \; C = \frac{2\pi\epsilon}{l} \tag{1.5.41}$$

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} n e^{-nC} = \frac{1}{c^2} - \frac{1}{12} \tag{1.5.42}$$

$$M^2 = \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i = \frac{(D-2)}{2\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} n = 2p^+p^- - p^i p^i \tag{1.5.43}$$

The last equality was obtained using(64)

$$M^2 = \frac{(D-2)}{2\alpha'} \left(\frac{1}{c^2} - \frac{1}{12} \right) = \frac{(D-2)}{2\alpha'} \frac{l^2}{(2\pi\epsilon)^2} - \frac{(D-2)}{24\alpha'} = \frac{(D-2)}{(2\pi\epsilon)^2} \pi p^+ l - \frac{(D-2)}{24\alpha'} = 2p^+p^- - p^i p^i \tag{1.5.44}$$

The mass of each state is thus determined in terms of the level of excitation.

$$M^2 = \frac{1}{\alpha'} \left(N - \frac{D-2}{24} \right) \qquad N|0\rangle = 0 \tag{1.5.45}$$

This operator acting on the 0 ket yields:

$$M^2|0\rangle = -\frac{(D-2)}{24\alpha'}|0\rangle \tag{1.5.46}$$

The mass-squared is negative for $D > 2$. The state is a tachyon

The lowest excited states of the string are obtained by exciting one of the $n = 1$ modes once:

$$M^2(\alpha^i_{-1}|0\rangle) = \frac{1}{\alpha'}\left(1 - \frac{D-2}{24}\right)(\alpha^i_{-1})|0\rangle = \frac{1}{\alpha'}\left(\frac{26-D}{24}\right)|0\rangle \tag{1.5.47}$$

Lorentz invariance now requires that this state be massless, so the number of spacetime dimensions is $D = 26$

1.6 Closed strings

We must now look at the mode expansion and quantization os closed strings, which are required when looking at string interactions. The procedure is very similar to that of open strings, except now we have Dirichlet boundary conditions, i.e. X^μ is periodic. The mode expansion is now made up of left and right moving parts:

$$X^\mu = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma) \tag{1.6.1}$$

such that

$$X^\mu_R = \tfrac{1}{2}x^\mu + \alpha' p^\mu(\tau - \sigma) + i\sqrt{\tfrac{1}{2}\alpha'}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\frac{1}{n}\alpha^\mu_n e^{\frac{-2\pi i n(\tau-\sigma)}{l}} \tag{1.6.2}$$

$$X^\mu_L = \tfrac{1}{2}x^\mu + \alpha' p^\mu(\tau + \sigma) + i\sqrt{\tfrac{1}{2}\alpha'}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\frac{1}{n}\tilde{\alpha}^\mu_n e^{\frac{-2\pi i n(\tau+\sigma)}{l}} \tag{1.6.3}$$

The sum of these is periodic. In the sum the integer n is in effect 2n so there are now twice as many modes as for the open string. Once again, the fact that X^μ is real means that

$$\alpha^\mu_n = (\alpha^\mu_{-n})^* \qquad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^* \tag{1.6.4}$$

Quantizing as before:

$$\left[\alpha^i_m, \alpha^j_n\right] = \left[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n\right] = m\delta^{ij}\delta_{m+n,0} \tag{1.6.5}$$

$$\left[\alpha^i_m, \tilde{\alpha}^j_n\right] = 0 \tag{1.6.6}$$

The mass operator is now

$$M^2 = 2p^+p^- - p^ip^i = \frac{2}{\alpha'}\left(N_R + N_L + \frac{2(D-2)}{24}\right) = \frac{2}{\alpha'}\left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - \frac{2(D-2)}{24}\right) \tag{1.6.7}$$

There are two symmetries in these expansions: the transformations

$$\tau \to \tau + \text{constant} \tag{1.6.8}$$

$$\sigma \to \sigma + \text{constant} \tag{1.6.9}$$

don't change the physics of the string. The first symmetry is shared with the open string, but the spatial translational symmetry is new.
The generator of the time translations is H, which we saw before to be

$$H = \int_0^l d\sigma(\dot{X}^2 + \acute{X}^2) \tag{1.6.10}$$

and the fact that H = 0 leads to the invariance under time translations. In the case of closed strings,

$$H = \int_0^l d\sigma[(\dot{X}_R + \dot{X}_L)^2 + (\acute{X}_R - \acute{X}_L)^2] \tag{1.6.11}$$

$$\sim \int_0^l d\sigma(\dot{X}_R^2 + \dot{X}_L^2) = H_R + H_L \tag{1.6.12}$$

where we have used the fact that we can write

$$\partial_\sigma \sim \partial_\tau \tag{1.6.13}$$

since τ and σ are interchangeable in the expansions of X_R and X_L to within a minus sign.
This generation of time translations comes from the constraint $\dot{X}^2 + \acute{X}^2 = 0$, so it is reasonable to suppose that spatial translations come from the other constraint $\dot{X} \cdot \acute{X} = 0$. Defining the operator D:

$$D = \int_0^l d\sigma \dot{X} \cdot \acute{X} \sim \int_0^l d\sigma \dot{X}^2 = \int_0^l d\sigma(\dot{X}_R^2 - \dot{X}_L^2) \sim N_R - N_L = 0 \tag{1.6.14}$$

So

$$N_R = N_L \tag{1.6.15}$$

which common sense tells us must be the case.

Now let us determinant the lowest states in the mass spectrum. For the vacuum state

$$M^2|0\rangle = -\frac{4(D-2)}{24\alpha'}|0\rangle \tag{1.6.16}$$

so the mass squared is negative for the vacuum state, as for the open string. The first excited state is $|\Omega^{ij}\rangle = \tilde{\alpha}^i_{-1}\alpha^j_{-1}|0\rangle$, where we must remember to keep $N_R = N_L$. We obtain

$$M^2|\Omega^{ij}\rangle = \frac{2}{\alpha'}\left(\frac{2-2(D-2)}{24}\right)|\Omega^{ij}\rangle \tag{1.6.17}$$

As before, we wish $M^2 = 0$ for the first excited state, so again we get $D = 26$. The situation is a bit more complicated than for the open string case, so let's look in a bit more detail.
The state $|\Omega^{ij}\rangle$ can be split into three parts: a symmetric, traceless part; an antisymmetric part and a scalar part:

$$|\Omega^{ij}\rangle = \left[\tfrac{1}{2}(|\Omega^{ij}\rangle + |\Omega^{ij}\rangle) - \frac{2}{D-2}\delta^{ij}|\Omega^{kk}\rangle\right] + \tfrac{1}{2}(|\Omega^{ij}\rangle - |\Omega^{ij}\rangle) + \frac{1}{D-2}\delta^{ij}|\Omega^{kk}\rangle \tag{1.6.18}$$

We call the three states $|G^{ij}\rangle, |B^{ij}\rangle, |\Phi\rangle$. The symmetric, traceless, spin 2 state $|G^{ij}\rangle$ can now be identified with the graviton, which didn't exist in the open string theory, so it seems some progress has been made. The spin 0 scalar state is called the dilaton.
Now we impose a further symmetry: invariance under the transformation

$$\sigma \to -\sigma \tag{1.6.19}$$

which is the condition for unoriented strings. This means

$$X_R \leftrightarrow X_L \tag{1.6.20}$$

$$\alpha_n \leftrightarrow \tilde{\alpha}_n \tag{1.6.21}$$

This condition immediately disallows the antisymmetric state, since under the transformation,

$$|B^{ij}\rangle \to -|B^{ij}\rangle \tag{1.6.22}$$

while $|G^{ij}\rangle$ and $|\Phi\rangle$ remain unchanged.
Next let us turn our attention to the fact that we have been working in light cone gauge, which is not Lorentz invariant. We would like to reassert lorentz invariance. We can do this by generalizing the commutation identity:

$$\left[\alpha^i_m, \alpha^j_n\right] = m\delta^{ij}\delta_{m+n,0} \qquad \to \qquad [\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{1.6.23}$$

This gives

$$\left[\alpha_m^0, \alpha_n^0\right] = -m\delta_{m+n,0} \tag{1.6.24}$$

Now let us define a state

$$|\phi\rangle = \alpha_{-1}^0|0\rangle \tag{1.6.25}$$

The norm of this state is

$$\langle\phi|\phi\rangle = \langle 0|\alpha_1^0\alpha_{-1}^0|0\rangle = \langle 0|[\alpha_1^0, \alpha_{-1}^0]|0\rangle + \langle|\alpha_{-1}^0\alpha_1^0|0\rangle = -\langle 0|0\rangle = -1 \tag{1.6.26}$$

Thus we have a negative norm state, which is physically meaningless. This tells us there is something wrong with our theory. We will come back to this in the next couple of chapters, and solve this problem.

1.7 Gauge invariance

Let us examine the open string state

$$|\phi\rangle = A_\mu(k)\alpha_{-1}^\mu|0, k\rangle \tag{1.7.1}$$

under the transformation

$$A_\mu \to A_\mu + k_\mu\omega(k) \tag{1.7.2}$$

where $\omega(k)$ is an arbitrary function. This is analogous to a gauge transformation in electromagnetism $\vec{A} \to \vec{A} + \nabla\omega$. The change in $|\phi\rangle$ is

$$|\delta\phi\rangle = k_\mu\omega\alpha_{-1}^\mu|0, k\rangle \tag{1.7.3}$$

and the norm is

$$\langle\delta\phi|\delta\phi\rangle = k_\mu k_\nu\omega^2\langle 0, k|\alpha_1^\mu\alpha_{-1}^\nu|0, k\rangle = k^2\omega^2 = 0 \tag{1.7.4}$$

since k = 0 (the mass is zero, and $m^2 = k^\mu k_\nu$). Thus the theory has produced gauge invariance! The equivalent state for the closed string is

$$|\phi\rangle = g_{\mu\nu}\tilde{\alpha}_{-1}^\mu\alpha_1^\nu|0, k\rangle \tag{1.7.5}$$

The gauge transformation is

$$g_{\mu\nu} \to g_{\mu\nu} + k_\mu\omega_\nu + k_\nu\omega_\mu \tag{1.7.6}$$

This time we find the norm state to be

$$\langle\delta\phi|\delta\phi\rangle \sim k^2 = 0 \tag{1.7.7}$$

Thus gauge invariance holds for the closed string state too. What's more, we can identify $g_{\mu\nu}$ with the gravitational potential, a sign that general relativity might be included in the theory.