String Theory II

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UNIT 6

Compactification and Duality

6.1 The Kaluza-Klein Mechanism

Before we introduce the Kaluza-Klein mechanism, let us briefly review electromagnetism and gauge symmetry. The field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and the action

$$S = \int d^4x \, \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right).$$

leads to the Maxwell equations

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

Gauge invariance in the theory implies $A_\mu \to A_\mu - \partial_\mu \lambda$ provided the current is conserved ($\partial_\mu J^\mu = 0$). The conserved charge is given by

$$Q = \int d^3x J^0, \quad \frac{dQ}{dt} = 0.$$

Suppose J^μ is due to a scalar field Φ which has mass m. If we forget about the charge for the moment, let p^μ be the momentum of the particle Φ represents (Φ represents no particle that anybody has observed). Einstein tells us

$$p^\mu p_\mu = -m^2$$

Quantize the system: $p_\mu \to i\partial_\mu$, so $\partial_\mu \partial^\mu \Phi = m^2 \Phi$ (Klein-Gordon equation). This is obtained from the action

$$S = \frac{1}{2} \int d^4x \, \partial_\mu \Phi^* \partial^\mu \Phi + m^2 |\Phi|^2$$

The conserved current: $j^\mu = \Phi^*\partial^\mu\Phi-$ c.c. , $\partial_\mu J^\mu = 0$. If Φ has charge, we need to couple J^μ to A^μ. This is done by $p_\mu \to p_\mu - qA_\mu$, or $\partial_\mu \to \partial_\mu + iqA_\mu$, where q is the charge. Therefore the action for a charged scalar field, Φ is

$$S = \frac{1}{2}\int d^2x \, \left((\partial_\mu - iqA_\mu)\Phi^*(\partial^\mu + iqA^\mu)\Phi + m^2|\Phi|^2\right)$$

comparing with $\int d^4x \, A_\mu J^\mu$, we obtain

$$J_\mu = -\frac{iq}{2}(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*).$$

Gauge invariance: $\Phi \to e^{iq\lambda}\Phi, \ A_\mu \to A_\mu - \partial_\mu\lambda$. We have $(\partial_\mu + igA_\mu)\Phi\tilde{\psi}e^{ig\lambda}(\partial_\mu + igA_\mu)\Phi$ So the action and the currents are gauge invariant. Since λ is real, $|\Phi|$ is invariant, so Φ moves on a circle in the complex plane as λ changes. λ represents an angle in this picture.
Kaluza-Kleins suggestion was to take this picture literally and assume the the e&m is nothing but the effect of an extra (fifth) dimension. How? Let us see... Imagine a five-dimensional manifold in which the fifth dimension is a circle of radius R. Choose the coordinates

$$x^\mu = (t, x, y, z, u), \qquad u \equiv u + 2\pi R.$$

The line element is

$$ds^2 = G_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + 2G_{\mu u}dx^\mu du + G_{uu}du^2.$$

Suppose G_{MN} is independent of u (invariance under u translation). Parametrize G_{MN} as follows: $A_\mu = G_{\mu u}/G_{uu}$, $g_{\mu\nu} = G_{\mu\nu} - G_{uu}A_\mu A_\nu$. Then

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + G_{uu}(du + A_\mu dx^\mu)^2$$

Reparametrizaions $u \to u + \lambda(x^\mu)$ implies $A_\mu \to A_\mu - \partial_\mu\lambda$, i.e., gauge transformations!
Let p_μ be the momentum conjugate to u. Then $e^{iap_\mu}f(u) = f(u+a)$ (p_μ generates translations). Since $f(u+2\pi R) = f(u)$, we need $e^{2\pi i R p_\mu} = 1$, so $p_\mu = n/R$ (momentum is quantized). A general $f(u)$ may be expanded in momentum eigenstates ($e^{inu/pR}$).

$$f(u) = \sum_{n=-\infty}^{\infty} a_n e^{inu/R}.$$

For a field $\phi(x^\mu) = \phi(x^\mu, u)$, we have

$$\phi(x^\mu) = \sum_{n=-\infty}^{\infty} a_n(x^\mu)e^{inu/R}.$$

The wave equation $\partial_\mu\partial^\mu\phi = 0$ (massless ϕ) becomes

$$\partial_\mu\partial^\mu\Phi_n - \frac{n^2}{R^2}\Phi_n = 0$$

i.e., $a_n(x^\mu)$ represents a **massive** field of mass $m = n/R$ from a four-dimensional point of view. Einstein's equation correspondingly reads $p_\mu p^\mu = -n^2/R^2$.
At energies $E \ll 1/R$, we only see the $n = 0$ mode of $\alpha_0(x^\mu)$. At high energies (early universe), we see more modes.
What about charge? To isolate the effects of A_μ, let $G_{uu} = 1$ and $g_{\mu\nu} = \eta_{\mu\nu}$.
From $\phi = \sum a_n(x^\mu)e^{inu/R}$ and $u \to u + \lambda(x^\mu)$ we obtain $a_n(x^\mu) \to e^{in\lambda/R}a_n$.
Comparing with $\phi \to e^{iq\lambda}\phi$ for a field ϕ of charge q, we see an indication that $q = u/R$ for the mode a_n. So, $q = m$; the mass of a_n.
To see the gauge invariance in full swing, go back to the wave equation for a_n and put back in the curvature of space-time:

$$D^\mu \partial_\mu (a_n e^{inu/R}) = 0$$

where

$$D_M v_N = \partial_M v_N - \Gamma^L_{MN} v_L, \quad D^M v_M = \partial^M v_M - G^{MN}\Gamma^L_{MN} v_L,$$

and the Christophel symbol expressed in terms of the metric may be written as

$$\Gamma^L_{MN} = \frac{1}{2} G^{LP} \left(\partial_M G_{PN} + \partial_N G_{MP} - \partial_P G_{MN} \right).$$

A short calculation reveals

$$\begin{aligned} (\partial_\mu + A_\mu \partial_u)^2 (a_n e^{inu/R}) &= 0 \\ \partial_\mu + i\frac{n}{R}A_\mu)a_n &= \frac{n^2}{R^2}a_n \end{aligned}$$

which confirms that a_n has charge $q = n/R$. Moreover, the Maxwell equations come from the Einstein action

$$S \sim \int d^5x \, \sqrt{-G} R^{(5)},$$

where

$$G = detG_{MN} = -1 \qquad\qquad R^{(5)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

So the Einstein action reduces to electromagnetism! If we put back four-dimensional curvature ($g_{\mu\nu} \neq \eta_{\mu\nu}$), then we get $G = detg_{\mu\nu}$ and $R^{(5)} = R^{(4)} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ which is four-dimensional gravity and electromagnetism!

6.2 Strings

Consider a closed string moving along u (as well as $x, y, z, \ldots$), described by the function $V(\sigma, \tau)$. The action is (concentrate on U)

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, \partial U \bar{\partial} U$$

and we still demand $U \equiv U + 2\pi R$. As before, center-of-mass momentum is quantized: $p = n/R,\ n \in \mathbb{Z}$. Recall the mode expansion without compactification:

$$X^\mu(\sigma,\tau) = x^\mu + 2\pi\alpha' p^\mu \frac{\tau}{\ell} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m}\left(\alpha_m^\mu e^{-2\pi i m(\sigma+\tau)/\ell} + \tilde{\alpha}_m^\mu e^{2\pi i m(\sigma-\tau)/R}\right),$$

which can also be written in terms of $z = e^{2\pi i(\sigma+\tau)/\ell},\ \bar{z} = e^{-2\pi i(\sigma-\tau)/\ell}$ as

$$X^\mu(z,\bar{z}) = x^\mu + 2\pi\alpha' p^\mu \frac{\tau}{\ell} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m}\left(\alpha_m^\mu z^{-m} + \tilde{\alpha}_m^\mu \bar{z}^{-m}\right),$$

with derivatives

$$\begin{aligned} \partial X^\mu(z,\bar{z}) &= -i\frac{\alpha'}{2}p^\mu z^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\alpha_m^\mu z^{-m-1}, \\ \bar{\partial} X^\mu(z,\bar{z}) &= -i\frac{\alpha'}{2}p^\mu z^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\tilde{\alpha}_m^\mu \bar{z}^{-m-1}. \end{aligned}$$

Notice that the momentum is $p = \frac{1}{2\pi\alpha'}\oint(dz\partial X - d\bar{z}\bar{\partial}X)$ and if we go around the string once, we obtain

$$\oint(dz\partial X + d\bar{z}\bar{\partial}X) = 0. \tag{6.2.1}$$

With $u \equiv u + 2\pi R$ (compactified), (6.2.1) is no longer necessarily true. When we go around the string, u can change by a multiple of $2\pi R$, i.e.,

$$U(\sigma+\ell) = U(\sigma) + 2\pi R w, \qquad w \in \mathbb{Z}.$$

This allows a solution of the form $U(\sigma,\tau) = 2\pi w R\frac{\sigma}{\ell}$. Since $z/\bar{z} = e^{4\pi i\sigma/\ell} \Rightarrow 2\pi\sigma/\ell = -i/2\ln(z/\bar{z})$, so

$$U(\sigma,\tau) = -\frac{i}{2}wR\ln\left(\frac{z}{\bar{z}}\right)$$

has to be added. We obtain

$$\begin{aligned} U(z,\bar{z}) &= u - i\frac{\alpha'}{2}\frac{n}{R}\ln|z|^2 - \frac{i}{2}wR\ln\left(\frac{z}{\bar{z}}\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}(\alpha_m z^{-m} + \tilde{\alpha}_m\bar{z}^{-m}), \\ &= u - i\frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z - i\frac{\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}(\alpha_m z^{-m} + \tilde{\alpha}_m\bar{z}^{-m}), \end{aligned}$$

with the derivatives

$$\begin{aligned} \partial U(z,\bar{z}) &= -i\frac{\alpha'}{2}p_L z^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\alpha_m z^{-m-1}, \\ \bar{\partial} U(z,\bar{z}) &= -i\frac{\alpha'}{2}p_R \bar{z}^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\tilde{\alpha}_m \bar{z}^{-m-1}, \end{aligned}$$

where $P_{\frac{L}{R}} = \frac{n}{R} \pm \frac{wR}{\alpha'}$. Notice that

$$\frac{1}{2\pi\alpha'}\oint(dz\partial X - d\bar{z}\bar{\partial}X) = \frac{1}{2}(p_L + p_R) = p = \frac{n}{R},$$

and

$$\frac{1}{2\pi\alpha'}\oint(dz\partial X + d\bar{z}\bar{\partial}X) = \frac{1}{2}(p_L - p_R) = \frac{wR}{\alpha'} \neq 0.$$

We may express the Virasoro generators in terms of the left and right-handed momenta.

$$L_0 = \frac{\alpha' p_L^2}{4} + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n, \qquad \tilde{L}_0 = \frac{\alpha' p_R^2}{4} + \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\tilde{\alpha}_n$$

These are the same as in the uncompactified case, except now $p_L \neq p_R$.

6.3 Partition Function

First, let us recall the uncompactified case

$$\begin{aligned}
Z &= \text{Tr}\,(q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24}), \qquad q = e^{2\pi i\tau}, \\
&= \text{Tr}\,(q\bar{q})^{\alpha' p^2/4-1/24}\prod_n\left(\sum_{N_n=0}^{\infty} q^{nN_n}\right)\left(\sum_{\tilde{N}_n=0}^{\infty}\bar{q}^{n\tilde{N}_n}\right), \\
&= \left(\text{Tr}\,e^{-\pi\tau_2\alpha' p^2}\right)\left|\prod_n(1-q^n)^{-1}\right|^2(q\bar{q})^{-1/24}, \\
&= |\eta(\tau)|^{-2}V\int\frac{dp}{2\pi}e^{-\pi\tau_2\alpha' p^2}, \\
&= 6|\eta(\tau)|^{-2}\frac{V}{2\pi}\frac{1}{\sqrt{\alpha'\tau_2}},
\end{aligned}$$

which is invariant under modular transformations ($\tau \to \tau+1,\ \tau \to -1/\tau$). Notice that for $X^\mu = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2$, which is a solution of the wave equation,we have $\partial X^\mu = -i\frac{\alpha'}{2}p^\mu z^{-1},\ \bar{\partial}X^\mu = -i\frac{\alpha'}{2}p^\mu\bar{z}^{-1}$ and the action is

$$\begin{aligned}
S &= \frac{1}{2\pi\alpha'}\int d^2\,z\partial X_\mu\bar{\partial}X^\mu = \frac{\alpha'}{8\pi}p^2\int\frac{d^2z}{|z|^2}, \quad z = e^{i(\sigma^1+i\sigma^2)}, \\
&= \frac{\alpha'}{8\pi}p^2 2V_{\text{Torus}}, \qquad (V_{\text{Torus}} = 2\pi(2\pi\tau_2)) \\
&= \alpha' p^2\tau_2\pi.
\end{aligned}$$

Therefore

$$e^{-S} = e^{-\pi\tau_2\alpha' p^2},$$

which is the factor whose trace contributes to the partition function.

Partition Function for Compactified Space

$$\begin{aligned} Z &= \mathrm{Tr}\,(q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24}), \\ &= |\eta(\tau)|^{-2}\sum_{n,w} q^{\alpha' p_L^2/4}\bar{q}^{\alpha' p_R^2/4}, \qquad p_{\substack{L\\R}} = \frac{n}{R} \pm \frac{wR}{\alpha'} \\ &= |\eta(\tau)|^{-2}\sum_{n,w} \exp\left[-\pi\tau_2\alpha'\left(\frac{n^2}{R^2}+\frac{w^2R^2}{\alpha'^2}\right)+2\pi i\tau_1 nw\right]. \end{aligned}$$

Use the Poisson resummation formula:

$$\sum_{n=-\infty}^{\infty} e^{-\pi a n^2 + 2\pi i b n} = \frac{1}{\sqrt{a}}\sum_{m=-\infty}^{\infty} e^{-\pi(m-b)^2/a}.$$

Therefore the partition function for the compactified case is

$$\begin{aligned} Z &= |\eta(\tau)|^{-2}\frac{R}{\sqrt{\tau_2\alpha'}}\sum_{m,w}\exp\left[-\frac{\pi R^2}{\alpha'\tau_2}\left((m-\tau_1 w)^2+\tau_2^2 w^2\right)\right], \\ &= |\eta(\tau)|^{-2}\frac{V}{2\pi\sqrt{\alpha'\tau_2}}\sum_{m,w}\exp\left(-\frac{\pi R^2}{\alpha'\tau_2}|m-\tau w|^2\right), \end{aligned}$$

which is the same as the uncompactified case. Modular invariance implies

$$\begin{aligned} \tau \to \tau+1 \quad &\Leftrightarrow \quad m \to m+w, \\ \tau \to -\frac{1}{\tau} \quad &\Leftrightarrow \quad m \to -w,\ w \to m. \end{aligned}$$

Notice that the solution with the correct boundary conditions satisfying

$$\begin{aligned} U(\sigma^1+2\pi,\sigma^2) &= U(\sigma^1,\sigma^2)+2\pi wR, \\ U(\sigma^1+2\pi\tau_1,\sigma^2+2\pi\tau_2) &= U(\sigma^1,\sigma^2)+2\pi mR. \end{aligned}$$

can be written as

$$U(\sigma^1,\sigma^2) = \frac{wR}{\tau_2}(\tau_2\sigma^1-\tau_1\sigma^2)+\frac{mR}{\tau_2}\sigma^2$$

where

$$\partial_1 U = wR, \qquad \partial_2 U = \frac{R}{\tau_2}(m-w\tau_1).$$

The action is given by

$$S = \frac{1}{4\pi\alpha'}\int d\sigma_1 d\sigma_2\left[(\partial_1 U)^2+(\partial_2 U)^2\right] = \frac{1}{4\pi\alpha'}2\pi 2\pi\tau_2\frac{R^2}{\tau_2^2}|m-w\tau|^2+\ldots$$

Thermodynamics

From thermodynamics we know the partition function is

$$Z = \sum_E e^{-\beta E}, \qquad \beta = \frac{1}{T}$$

and T is the temperature.
Let $|E\rangle$ be the eigenstate of the Hammiltonian with eigenvalue E. Then

$$Z = \sum_E \langle E|e^{-\beta H}|E\rangle = \mathrm{Tr}\; e^{-\beta H}.$$

This is a special case of the string partition function ($\beta \in \mathbb{R}$). To calculate this, insert the complete sets

$$Z = \sum_{E,x,y} \langle E|x\rangle\langle x|e^{-\beta H}|y\rangle\langle y|E\rangle$$

Let $\beta \to it\hbar$, then $\langle x|e^{-\beta H}|y\rangle \to \langle x|e^{-iHt/\hbar}|y\rangle = \langle x(t)|y(0)\rangle$ a correlator (Greens function) of the Schrodinger equation. Suppose t is small. Then insert $1 = \sum_p |p\rangle\langle p|$, where $|p\rangle$ is an eigenstate of the momentum ($H = H(p,q)$).

$$\begin{aligned}\langle x(t)|y(0)\rangle &= \sum_p \langle x|p\rangle e^{-iHt/\hbar}\langle p|y\rangle, \\ &= \int dp\; e^{-ipx}e^{-iHt/\hbar}e^{ipy}.\end{aligned}$$

For $x - y \simeq -\dot{q}t$, so

$$\langle x(t)|y(0)\rangle = \int dp\; e^{i(\dot{q}p - H)t} = \int dp\; e^{iS}, \quad S = \int_0^t dt'\;(\dot{q} - H)$$

The dominant contribution is from the stationary point, $\frac{\partial S}{\partial p} = 0$. This is at $\dot{q} = \frac{\partial H}{\partial p}$, which is the Hamilton-Jacobi equation. Then

$$\langle x(t)|y(0)\rangle = e^{iS}.$$

This integrates for finite t. Then

$$Z = \sum_{e,x,y} e^{iS(x,y)}\psi_E^*(x)\psi_E(y) = \sum_x e^{iS(x,x)}$$

(sum over all possible closed paths) and we used the orthogonality of $|x\rangle$ such that $\langle x|y\rangle = \delta(x-y)$ and assumed the energy eigenstates formed a complete set.

6.4 Vertex Operators

Recall

$$U(z,\bar z) = u - i\frac{\alpha'}{2}\left(\frac{n}{r}+\frac{wR}{\alpha'}\right)\ln z - i\frac{\alpha'}{2}\left(\frac{n}{r}-\frac{wR}{\alpha'}\right)\ln \bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\alpha_m z^{-m}+\tilde\alpha_m \bar z^{-1}\right)$$

Momenta $p_L,\ p_R$ are different. Their eigenvalues are

$$p_L = \frac{n}{r}+\frac{wR}{\alpha'}, \qquad p_R = \frac{n}{r}-\frac{wR}{\alpha'}.$$

In the uncompactified case, u commutes with $p=\frac{1}{2}(p_L+p_R),\ [u,p]=i$. Here $p_L,\ p_R$ are independent operators, so u should alsp consist of two independent operators, $u=u_L+u_R$, such that

$$[u_L,p_L]=[u_R,p_R]=i.$$

Thus U may be broken into holomorphic (U_L) and antiholomorphic (U_R) pieces as

$$\begin{aligned} U_L(z) &= u_L - i\frac{\alpha'}{2}p_L\ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha_m z^{-m}, \\ U_R(\bar z) &= u_R - i\frac{\alpha'}{2}p_R\ln \bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\tilde\alpha_m \bar z^{-m}. \end{aligned}$$

The operator product expansions are

$$U_L(z)U_L(0)\sim -\frac{\alpha'}{2}\ln z, \qquad U_R(\bar z)U_R(0)\sim -\frac{\alpha'}{2}\ln \bar z, \qquad U_L(z)U_R(0)\sim 0.$$

The vertex operator also splits into a holomorphic and antiholomorphic pieces

$$V_L(z) =: e^{i\left(\frac{n}{r}+\frac{wR}{\alpha'}\right)U_L(z)}:, \qquad V_R(\bar z) =: e^{i\left(\frac{n}{r}-\frac{wR}{\alpha'}\right)U_R(\bar z)}:$$

c.f., the uncompactified case,

$$V_L(z) =: e^{ik\cdot X_L(z)}:, \qquad V_R(\bar z) =: e^{ik\cdot X_R(\bar z)}:$$

The OPE for these vertex operators is

$$\begin{aligned} V_L^k(z)V_L^{k'}(0) &= :e^{ik\cdot X_L(z)}::e^{ik'\cdot X_L(0)}:, \\ &\sim e^{ik\cdot k'\frac{\alpha'}{2}\ln z}:e^{i(k+k')X_L(0)}:, \\ &= z^{\frac{\alpha'}{2}k\cdot k'}:V_L^{k+k'}(0):. \end{aligned}$$

This has a branch cut. If you let z go around the circle $\mathcal{C}$ once, it picks up a factor $e^{\pi i\alpha' k\cdot k'}$ $(z^{\frac{\alpha'}{2}k\cdot k'} = e^{\frac{\alpha'}{2}k\cdot k'\ln z}\to e^{\frac{\alpha'}{2}k\cdot k'(\ln|z|+iArg z)})$. On the other hand,

$z^{-\frac{\alpha'}{2}k\cdot k'}$ picks up a factor $e^{-\pi i\alpha' k\cdot k'}$. The two factors cancel each other, so the OPE of the *full* vertex operator is single valued: $(V = V_L V_R)$

$$V^k(z,\bar{z})V^{k'}(0,0) \sim |z|^{\alpha' k\cdot k'/2} V^{k+k'}(0,0)$$

In the compactified case, let $k_L = \frac{n}{r} + \frac{wR}{\alpha'},\ k_R = \frac{n}{r} - \frac{wR}{\alpha'}$. Then, similar to the uncompactified case, we find

$$V^{k_L}(z)V^{k'_L}(0) \sim |z|^{\alpha' k_L\cdot k'_L/2} V_L^{k_L+k'_L}(0) \qquad V_R^k(\bar{z})V^{k'_R}(0) \sim |\bar{z}|^{\alpha' k_R\cdot k'_R/2} V^{k_R+k'_R}(0)$$

As z goes around a circle $\mathcal{C}$, we obtain factors $e^{-\pi i\alpha' k_L\cdot k'_L},\ e^{-\pi i\alpha' k_R\cdot k'_R}$. The total factor for the full vertex is $e^{i\pi\alpha'(k_L\cdot k'_L - k_R\cdot k'_R)} = e^{2\pi i(nw'+n'w)} = 1$, so the full vertex is ok.
Subtlety: If instead of going around, consider points $z_1,\ z_2$ and interchange them and let $(k_L, k_R) \leftrightarrow (k'_L, k'_R)$ i.e., consider the commutator of two vertices. This is equivalent to letting $z \leftrightarrow -z$, which introduces a factor $(-1)^{\frac{\alpha'}{2}k_L\cdot k'_L} = e^{i\pi\frac{\alpha'}{2}k_L\cdot k'_L}$ in the left part and $e^{i\pi\frac{\alpha'}{2}k_R\cdot k'_R}$ in the right part. Overall,

$$e^{i\pi\alpha'(k_L\cdot k'_L - k_R\cdot k'_R)} = e^{2\pi i(nw'+n'w)} = \pm 1$$

So if $nw'+n'w$ is odd, the vertices anticommute! To remedy this, we will define the vertex as

$$V^{k_L,k_R} = C_{k_L,k_r}(p) : e^{i(k_L U_l + k_R U_R)} :$$

where C_{k_L,k_R} is known as a *cocyle*. One possible choice is all choices are equivalent.

$$C_{k_L,k_r}(p) = e^{i\pi\frac{\alpha'}{2}(k_L - k_R)r}, \qquad p = /2(p_L + p_R)$$

It satisfies

$$C_k(p)C_{k'}(p) = C_{k+k'}(p), \qquad k = (k_L, k_R).$$

When we commute two vertices, we pick up the factor

$$e^{i\pi\frac{\alpha'}{2}(k_L-k_R)\frac{k'}{2}} e^{-i\pi\frac{\alpha'}{2}(k'_L-k'_R)\frac{k}{2}} = e^{\pi i(nw'-n'w)}$$

Thus the overall factor is now

$$e^{\pi i(nw'-n'w)} e^{\pi i(nw'+n'w)} = e^{2\pi i n w'} = 1.$$

Thus the overall factor is one, so the vertices always commute.

6.5 Amplitudes

Recall in the uncompactified case,

$$A = \langle V_1(z_1)V_2(z_2) + ... + V_N(z_N)\rangle \sim 2\pi\delta(k_1 + k_2 + ... + k_N)\prod_{i<j}|z_i - z_j|^{\alpha' k_i\cdot k_j} \tag{6.5.1}$$

where the vertex operators are given by

$$V_i(z_i) =: e^{ik_i \cdot X(z_i)} : .$$

The delta function in (6.5.1) comes from the zero mode, $e^{ik_i\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}$. Explicitly

$$\begin{aligned}
\langle e^{ik_1\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}...e^{ik_N\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}\rangle &= \langle e^{ik_1\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}...e^{ik_{N-1}\cdot(x-i\frac{\alpha'}{2}p\ln|z|^2)}|k_N\rangle \\
&\sim \langle 0|k_1+k_2+...+k_N\rangle \\
&\sim \delta(k_1+k_2+...+k_N)
\end{aligned}$$

Each $e^{ik_i\cdot x}$ shifts the state $|k\rangle \to |k+k_i\rangle$. In the compactified case, we get the same holomorphic and antiholomorphic except momenta are not different:

$$\prod_{i<j}|z_i-z_j|^{\alpha' k_i\cdot k_j} \to \prod_{i<j}(z_i-z_j)^{\alpha' k_{Li}k_{Lj}/2}(\bar{z}_i-\bar{z}_j)^{\alpha' k_{Ri}k_{Rj}/2}.$$

The zero modes contribute

$$\delta(k_{L1}+k_{L2}+...+k_{LN})\delta(k_{R1}+k_{R2}+...+k_{RN}) \sim \delta(n_1+n_2+...+n_N)\delta(w_1+w_2+...+w_N).$$

Cocycles give additional $\pm$ signs.

6.6 Spectrum

Recall in the uncompactified case:

$$L_0 = \frac{\alpha' p^2}{4} + N, \qquad N = \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n.$$

In the compactified case, $p \to p_L$ in L_0 and $p \to p_R$ in $\tilde{L}_0$. A string will travel in $(t, x, y, z, ..., u)$ i.e., the 26 dimensional space-time with one-dimension compactified. Then

$$L_0 = \frac{\alpha'(p^2+p_L^2)}{4} + N, \qquad p^2 = p_\mu p^\mu$$

and

$$N = \sum_{n=1}^{\infty}\alpha_{-n}^{\mu}\alpha_{n\mu} + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n,$$

and similarly for $\tilde{L}_0$. The Mass-shell condition states

$$(L_0-1)|\text{phys}\rangle = (\tilde{L}_0-1)|\text{phys}\rangle = (L_0-\tilde{L}_0)|\text{phys}\rangle.$$

The Hamiltonian is a constraint. We define the mass by $m^2 = k_\mu k^\mu$, where k^μ is the eigenvalue of p^μ. Then

$$L_0 - 1 = 0 \Rightarrow \frac{\alpha'}{4}(-m^2+k_L^2) + N - 1 = 0 \Rightarrow m^2 = k_L^2 + \frac{4}{\alpha'}(N-1)$$

From the antiholomorphic piece we get

$$m^2 = k_R^2 + \frac{4}{\alpha'}(\tilde{N} - 1)$$

Subtract the two and we find

$$\tilde{N} - N = \frac{\alpha'}{4}(k_L^2 = k_R^2) = nw$$

If we add the two conditions we find

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2) + \frac{k_L^2 + k_R^2}{2} = \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

We now explicitly see the Kaluza-Klein mass term and the winding potential energy.

Massless States

$n = w = 0,\ N = \tilde{N} = 1$, same as in the noncompactified theory. They are:

$$\alpha^\mu_{-1}\alpha^\nu_{-1}|0;k\rangle, \qquad \alpha^\mu_{-1}\tilde{\alpha}_{-1}|0;k\rangle, \qquad \alpha_{-1}\tilde{\alpha}^\mu_{-1}|0;k\rangle, \qquad \alpha_{-1}\alpha_{-1}|0;k\rangle.$$

The states are represented by the (graviton and antisymmetric tensor), vectors, and scalar particles respectively. Recall in Kaluza-Klein field theory, we had $g_{\mu\nu},\ A_\mu,\ G_{uu}$. Our gravition corresponds to $g_{\mu\nu}$, the scalar to G_{uu}, but we have **two** vectors instead of one! Which vector corresponds to A_μ? To answer this question consider three-point amplitudes of two tachyons and a vector. The tachyons have momenta $k_1,\ k_2$ and the vector has momentum k. A tachyon is described by the vertex

$$: e^{ik_L U_l + ik_R U_R + ik\cdot X} :\Leftrightarrow \text{state} \quad |0, k, k_L, k_R\rangle$$

The two vectors are

$$|B\rangle = B_\mu(k)\alpha^\mu_{-1}\tilde{\alpha}_{-1}|0;k\rangle, \qquad |C\rangle = C_\mu(k)\tilde{\alpha}^\mu_{-1}\alpha_{-1}|0;k\rangle$$

The amplitude for $|B\rangle$ is

$$A \sim B_\mu\langle 0; k_1, k_{1L}, k_{1R}| : e^{ik_{2L}U_L + ik_{2R}U_R + ik_2\cdot X(1)} : \alpha^\mu_{-1}\tilde{\alpha}_{-1}|0;k\rangle$$

The relevent parts of $U,\ X^\mu$ are:

$$\begin{aligned} U_L &= u_L - i\frac{\alpha'}{2}p_L \ln z + ... \\ U_R &= r_R - i\frac{\alpha'}{2}p_R \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_1\bar{z}^{-1} + ... \\ X^\mu &= x^\mu - i\frac{\alpha'}{2}p^\mu \ln |z|^2 + i\sqrt{\frac{\alpha'}{2}}\alpha^\mu_1 z^{-1} + ... \qquad \text{where } z = 1 \end{aligned}$$

So

$$\begin{aligned} A &\sim B_\mu \frac{\alpha'}{2} \langle 0; k_1 + k_2, k_{1L} + k_{2L}, k_{1R} + k_{2R} | \tilde{\alpha}_1 k_{2R} \alpha_1^\nu k_{2\nu} \alpha_{-1}^\mu \tilde{\alpha}_{-1} | 0; k \rangle \\ &\sim \alpha' k_2 \cdot B k_{2R} \delta(k_1 + k_2 + k) \delta(k_{1L} + k_{2L}) \delta(k_{1R} + k_{2R}) \\ &\sim \alpha' (k_2 - k_3) \cdot B k_{2R} \delta(k_1 + k_2 + k) \delta(n_1 + n_2) \delta(w_1 + w_2) \end{aligned}$$

where we used $k \cdot B = 0$ (gauge invariance coming from $Q|B\rangle = 0$ or observe that if $B \propto k$, $|B\rangle$ is a null state. Notice that the two tachyons have *opposite* quantum numbers n, w which makes sense because they annihilate each other. Alternatively, if k_2 is outgoing, the two tachyons have the same quantum numbers (scattering of a single tachyon). The diagram for the other vector C is

$$A' \sim (k_2 - k_3) \cdot B k_{2L} \delta(k_1 + k_2 + k) \delta(n_1 + n_2) \delta(w_1 + w_2).$$

i.e., $k_{2R} \to k_{2L},\ B_\mu = C_\mu$. Notice that if the sum (corresponding to $|B\rangle + |C\rangle$) is

$$A + A' \sim \alpha'(k_2 - k_3) \cdot B \frac{n}{R} \delta(k_1 + k_2 + k) \delta(n_1 + n_2) \delta(w_1 + w_2)$$

i.e., the strength of the interaction is proportional to the change. Therefore, the photon is

$$A_\mu(k)(\alpha_{-1}^\mu \tilde{\alpha}_{-1} + \tilde{\alpha}_{-1}^\mu \alpha_{-1})|0; k\rangle.$$

The other vector $A'_\mu(k)(\alpha_{-1}^\mu \tilde{\alpha}_{-1} - \tilde{\alpha}_{-1}^\mu \alpha_{-1})|0; k\rangle$ leads to the amplitude

$$A - A' \sim (k_2 - k_3) \cdot B \frac{wR}{\alpha'} \delta(k_1 + k_2 + k) \delta(n_1 + n_2) \delta(w_1 + w_2)$$

so it couples to a different charge: the winding number (magnetic?) This is absent in particle theory and is only a string effect.

6.7 $R = \sqrt{\alpha'}$

When $R = \sqrt{\alpha'}$, $m^2 = 0$ implies $\frac{n^2}{\alpha'} + \frac{w^2}{\alpha'} + \frac{2}{a'}(N + \tilde{N} - 2) = 0,\ \tilde{N} - N = nw$. Apart from $n = w = 0,\ N = \tilde{N} = 1$, we have the following possibilities

- $n = w = \pm 1$, in which case we have $N + \tilde{N} = 1,\ \tilde{N} - N = +1$, so $N = 0,\ \tilde{N} = 1$.
- $n = -w = \pm 1$, in which case we have $N + \tilde{N} = 1,\ \tilde{N} - N = -1$, so $N = 1,\ \tilde{N} = 0$.
- $n = \pm 2,\ w = 0$, in which case we have $N + \tilde{N} = 0,\ \tilde{N} - N = 0$, so $N = \tilde{N} = 0$.
- $n = 0,\ w = \pm 2$, in which case we have $N + \tilde{N} = 0,\ \tilde{N} - N = 0$, so $N = \tilde{N} = 0$.

The first and second possibilities are new vectors and they are charged! This is reminiscent of the Weak interactions where we have $W^{\pm}$ (charged vectors). Together with a mixture of γ and Z^0, they form a triplet, which is a representation of SU(2). Similarly, matter comes in doublets $(e, \nu_e),\ (u, d)$ which also transform under SU(2), which is a gauge symmetry, much like electromagnetism, where the photon and matter transform under U(1). ($W^{\pm}$ also have mass, but that is only because they ate a Higgs). Recall in E&M (matter represented by a scalar, e.g., tachyon - both fiction)

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \Phi|^2 + m^2 |\Phi|^2 \right],$$

where

$$D_\mu = \partial_\mu + iqA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Gauge invariance: $\Phi \to e^{iq\lambda}\Phi,\ \ A_\mu \to A_\mu - \partial_\mu \lambda$ based on the gauge group U(1).

To extend this to weak interactions, where we have three vectors, $A^1_\mu,\ A^2_\mu,\ A^3_\mu$, we view them as a vector in an abstract space (three-dimensional). Rotations in this space should be independent of the physics. In other words, the action, S, must be invariant under such rotations. We might guess that we should have

$$-\frac{1}{4} \sum_{i=1}^{3} F^i_{\mu\nu} F^{i\mu\nu}$$

(a Weak field for each vector). We also need $\int d^4x \sum_i A^i_\mu J^{i\mu}$, where $J^{i\mu}$ is made of $A^{i\mu}$. These requirements **severly** restrict the form of the action. It turns out that the fields must be defined by

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \epsilon^{ijk} A^j_\mu A^k_\nu$$

and the action is

$$S = -\frac{1}{4} \int d^4x \sum_i F^i_{\mu\nu} F^{i\mu\nu}$$

and leads to nonlinear Maxwell equations given by

$$\partial_\mu F^{i\mu\nu} - \epsilon^{ijk} A^j_\mu F^{k\mu\nu} = 0.$$

Gauge transformations: $A^i_\mu \to A^i_\mu - \partial_\mu \lambda^i - \epsilon^{ijk} A^j_\mu \lambda^k$ (infinitesimal). Finite transformations: Introduce Pauli matrices, σ^i. Define the matrix field

$$A_\mu = \frac{1}{2} A^i_\mu \sigma^i$$

where σ^i represents the Pauli spins matrices which obey the algebra

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

The field strength is

$$F_{\mu\nu} = \frac{1}{2} F^i_{\mu\nu} \sigma^i$$

$$S = e^{\frac{i}{2}\lambda^i\sigma^i} \in SU(2), \qquad SS^\dagger = \mathbb{I}$$

For $\lambda \ll 1$ we may expand S

$$S \simeq 1 + \frac{i}{2}\lambda^j\sigma^j$$

Let us see how $A_\mu = \frac{1}{2}A^i_\mu\sigma^i$ transforms under a finite transformation

$$\begin{aligned}
A_\mu &\rightarrow S\left(A_\mu - i\partial_\mu\right)S^\dagger \\
&\rightarrow (1 + \frac{i}{2}\lambda^i\sigma^i)(A_\mu - i\partial_\mu)(1 - \frac{i}{2}\lambda^j\sigma^j) \\
&\rightarrow A_\mu - \frac{1}{2}\sigma^j\partial_\mu\lambda^j + \frac{i}{4}\lambda^i[\sigma^i, \sigma^j]A^j_\mu + O(\lambda^2) \\
&\rightarrow A_\mu - \frac{1}{2}\sigma^j\partial_\mu\lambda^j - \frac{1}{2}\epsilon^{ijk}\sigma^i\lambda^jA^k_\mu \\
A^i_\mu &\rightarrow A^i_\mu - \partial_\mu\lambda^i - \epsilon^{ijk}\lambda^jA^k_\mu
\end{aligned} \tag{6.7.1}$$

For the matter fields, we need to define $D_\mu\Phi$. Φ is a doublet, (u, d) or (e, ν_e), etc. (not quite, because of the spin), so define

$$D_\mu = \partial_\mu + iA_\mu.$$

where A_μ is a matrix. Notice that we have no degree of freedom in introducing a charge q, because

$$\Phi \rightarrow S\Phi, \qquad A_\mu \rightarrow S(A_\mu - i\partial_\mu)S^\dagger,$$

so

$$\begin{aligned}
D_\mu\Phi &\rightarrow \partial_\mu(S\Phi) + iA'_\mu S\Phi, \\
&= S(\partial_\mu\Phi + S^\dagger\partial_\mu S\Phi + iS^\dagger A'_\mu S\Phi), \\
&= SD_\mu\Phi.
\end{aligned} \tag{6.7.2}$$

This would have worked nicely had we chosen

$$\Phi \rightarrow S'\Phi, \qquad S' = e^{\frac{i}{2}q\lambda^i\sigma^i},$$

because $iS'^\dagger A'_\mu S' \neq A_\mu - S^\dagger\partial_\mu S$. This is only true in the *Abelian* case, $S = e^{iq\lambda}$, because all "matrices" commute.
We found two vectors that coupled to charges p_L and p_R (or n and w). The charge operator (which measures the charge of a state) is then the momentum, and there are two of them. The corresponding (conserved) currents generate gauge symmetries, both being $U(1)$, so the gauge group is $U(1) \times U(1)$.

The two currents are ∂U and $\bar{\partial} U$. To see their action, consider a state of charge (n, w) or (k_L, k_r), e.g., $V =: e^{ik_L U + ik_R U}$:. The OPEs are given by

$$\begin{aligned} \partial U(z) V(0,0) &\sim \partial U_L(z) : e^{ik_L U_L} : \\ &\sim ik_L \partial \left(-\frac{\alpha'}{2} \ln z \right) : e^{ik_L U_L}(0) : \\ &\sim -ik_L \frac{\alpha'}{2} \cdot \frac{1}{z} V(0,0) \\ \bar{\partial} U(\bar{z}) V(0,0) &\sim -ik_R \frac{\alpha'}{2} \cdot \frac{1}{\bar{z}} V(0,0) \end{aligned}$$

The charges are

$$Q_L = \frac{1}{2\pi i} \oint_C dz \partial U(z) \qquad Q_R = -\frac{1}{2\pi i} \oint_C d\bar{z} \partial U(\bar{z}).$$

so

$$[Q_L, V] = -i\frac{\alpha'}{2} k_L V \qquad [Q_R, V] = -i\frac{\alpha'}{2} k_R V$$

The electric charge is $Q \sim Q_L + Q_R$. These are generators of gauge transformations, indeed

$$\delta V = -\lambda [Q_L, V] = -i\frac{\alpha'}{2} k_L \lambda V$$

so $V \to (1 + i\frac{\alpha'}{2} k_L \lambda) V$ which is the infinitesimal of $V \to e^{i\frac{\alpha'}{2} k_L \lambda} V$, or for the state $|V\rangle = V(0)$, $|V\rangle \to e^{i\frac{\alpha'}{2} k_L \lambda} |V\rangle$ (similarly for Q_R).
At the special radius ($R = \sqrt{\alpha'}$), we have vectors which are charged. They **must** combine with the two photons, just like the $X^{\pm}$ combine with the neutral vector in weak interactions (**must** because we know of no other consistent theory that has charged vectors).
The vectors are

$$① = \tilde{\alpha}^{\mu}_{-1} |0; k; \pm 1, \pm 1\rangle, \qquad ② = \alpha^{\mu}_{-1} |0; k; \pm 1, \pm 1\rangle$$

① corresponds to the vertex : $\tilde{\alpha}^{\mu}_{-1} |0, k, -1, -1\rangle \quad \sim \; : \partial X^{\mu} e^{ik \cdot X} e^{-i\frac{2}{\sqrt{\alpha'}} U_L(z)} :$
(since $n = w$, $k_R = 0$)
② corresponds to the vertex : $\alpha^{\mu}_{-1} |0, k, -1, +1\rangle \quad \sim \; : \partial X^{\mu} e^{ik \cdot X} e^{-i\frac{2}{\sqrt{\alpha'}} U_R(\bar{z})} :$
(since $n = -w$, $k_L = 0$)
Just like with the two photons, they lead to conserved currents.

$$j^{\pm}(z) =: e^{ik_L U_L}(z), \qquad \tilde{j}^{\pm}(\bar{z}) =: e^{ik_R U_R}(\bar{z})$$

where

$$k_L = \frac{n}{R} + \frac{wR}{\alpha'} = \pm \frac{2}{\sqrt{\alpha'}}, \qquad k_R = \pm \frac{2}{\sqrt{\alpha'}}.$$

The OPEs are given by

$$\begin{aligned} j^+(z)j^+(0) &\sim\ :e^{i\frac{2}{\sqrt{\alpha'}}U_L}(z)::e^{i\frac{2}{\sqrt{\alpha'}}U_L}(0): \\ &\sim\ e^{\frac{-4}{\alpha'}ln(z)\frac{-\alpha'}{2}}:e^{i\frac{4}{\sqrt{\alpha'}}U_L}(0) \\ &\sim\ z^2:e^{i\frac{4}{\sqrt{\alpha'}}U_L}(0):\sim\ 0 \\ j^-(z)j^-(0) &\sim\ 0 \end{aligned}$$

$$\begin{aligned} j^+(z)j^-(0) &\sim\ z^{-2}:e^{i\frac{2}{\sqrt{\alpha'}}U_L}(z)e^{-i\frac{2}{\sqrt{\alpha'}}U_L}(0): \\ &=\ z^{-2}\left[1+z\,i\frac{2}{\sqrt{\alpha'}}\partial U_L+z^2(\ \ )+...\right] \\ &\sim\ \frac{1}{z^2}+i\frac{2}{\sqrt{\alpha'}}\partial U_L\frac{1}{z} \end{aligned}$$

Define $j^3=\frac{i}{\sqrt{\alpha'}}\partial U$ (the photon!). The OPEs may be expressed in terms of the photon.

$$\begin{aligned} j^+(z)j^-(0) &\sim\ \frac{1}{z^2}+\frac{2}{z}j^3(z)+... \\ j^3(z)j^+(0) &\sim\ -\frac{2}{\alpha'}\left(-\frac{\alpha'}{2}\partial\ln z\right)j^+(0)\sim\frac{1}{z}j^+(0) \\ j^3(z)j^-(0) &\sim\ -\frac{1}{z}j^-(0) \end{aligned}$$

Be defining $j^1=\frac{1}{2}(j^++j^-),\ j^2=\frac{1}{2}(j^+-j^-)$, we may write all the OPEs in the form

$$j^a(z)j^b(0)\sim\frac{1}{2z^2}\delta_{ab}+i\epsilon^{abc}j^c(0).$$

The corresponding charges

$$Q^a=\oint\frac{dz}{2\pi i}j^a(z)$$

satisfy an SU(2) algebra

$$[Q^a,Q^b]=i\epsilon^{abc}Q^c$$

so the gauge group is $SU(2)\times SU(2)$ (which is enlarged from $U(1)\times U(1)$). You can think of this as an abstract three-dimensional space (in fact, two) in which particles are free to rotate. U(1) is then a subgroup of SU(2) corresponding to rotations around the z-axis. In fact, the symmetry of the theory has an infinite number of generators (much like $T(z)$ generated an infinite number of symmetries through its modes, L_n). Expand

$$j^a(z)=\sum j^a_m z^{-m-1}$$

so that the charges Q^a are the zero modes, $Q^a = j_0^a$. Then the current algebra is an affine Lie algebra (Kac-Moody)

$$[j^a_\mu, j^b_\nu] = \frac{m}{2}\delta_{m+n,0}\delta^{ab} + i\epsilon^{abc}j^c_{m+n}.$$

The constant tells us that j^a is **not** a tensor. It can be shown that a general algebra has $\frac{km}{2}\delta_{m+n,0}\delta^{ab}$ constant term, $k \in \mathbb{N}$ (level), so in our case $k = 1$.

6.8 Away from $R = \sqrt{\alpha'}$

Once we realize there is a symmetric at the special radius $R = \sqrt{\alpha'}$, you can not ignore it when you move away from $R = \sqrt{\alpha'}$. This is because R is dynamical and we have already seen that there is a string mode, $\Phi(k)\tilde{\alpha}_{-1}\alpha_{-1}|0;k\rangle$ which mixes with G_{uu} and changes R, much like the graviton $g_{\mu\nu}(k)(\tilde{\alpha}^\mu_{-1}\alpha^\nu_{-1} + \tilde{\alpha}^\nu_{-1}\alpha^\mu_{-1})|0;k\rangle$ changes the background metric (in the uncompactified case).
So what happens to the SU(2) symmetry as we move away from $R = \sqrt{\alpha'}$? Recall weak interactions...
A scalar called the Higgs moves from the unstable symmetric point to a stable minimum of the potential (Mexican hat) So $|\Phi|$ goes from 0 to a value $\langle|\Phi|\rangle \sim \nu$. Φ can settle into any minimum and all positions are equivalent. However each position breaks the symmetry. It is similar to the SUN-EARTH system. The underlying physics (Newton's law) is rotationally invariant, but the orbit of the Earth is not (it is an ellipse, even as a circle there is an axis that breaks the symmetry).
Recall the action for a scalar.

$$S = \int d^4x \left(|D_\mu\Phi|^2 + V(\Phi)\right).$$

It contains a term quadratic in the vectors A^i_μ, which gives rise to a mass term after the shift $|\Phi| \to |\Phi| + \nu$.
Back to strings...
Massless scalars $\forall R:\ \eta_{\mu\nu}\tilde{\alpha}^\mu_{-1}\alpha^\nu_{-1}|0;k\rangle$ (dilaton) $\tilde{\alpha}_{-1}\alpha_{-1}|0;k\rangle$ ($G_{uu|}$). The latter changes R and corresponds to the vertex

$$:\partial U(z)\bar{\partial}U(\bar{z})e^{ik\cdot X}:,$$

or

$$:j^3(z)\tilde{j}^3(\bar{z})e^{ik\cdot X}:.$$

At $R = \sqrt{\alpha'}$, we have additional scalars

① : $\bar{\partial}Ue^{ik_LU_L}e^{ik\cdot X}$: (c.f. vector $:\bar{\partial}X^\mu e^{ik\cdot X}e^{ik_LU_L}:$) or $:j^\pm\tilde{j}^3e^{ik\cdot X}:$

② : $\bar{\partial}Ue^{ik_RU_R}e^{ik\cdot X}$: (c.f. vector $:\partial X^\mu e^{ik\cdot X}e^{ik_RU_R}:$) or $:j^3\tilde{j}^\pm e^{ik\cdot X}:$

③ : $n = \pm 2,\ w = 0 \Rightarrow k_L = k_R = \dfrac{n}{R} = \pm\dfrac{2}{\sqrt{\alpha'}}$, vertex : $e^{ik_lU_L}e^{ik_RU_R}e^{k\cdot X} = j^\pm\tilde{j}^\pm e^{ik\cdot X}$

④ : $n = 0,\ w = \pm 2 \Rightarrow k_L = -k_R = \dfrac{wR}{\alpha'} = \pm\dfrac{2}{\sqrt{\alpha'}}$, vertex : $e^{ik_lU_L}e^{ik_RU_R}e^{k\cdot X} = j^\pm\tilde{j}^\pm e^{ik\cdot X}$

Putting everything together, we have

$$: j^a(z)\tilde{j}^b(\bar{z})e^{ik\cdot X} :$$

transforming as a $(\mathbf{3},\mathbf{3})$ of $SU(2)\times SU(2)$. This is the Higgs (**not** a doublet, unlike weak interactions). At the symmetric point, all scalars are massless. Away from the symmetric point, all except $j^3\tilde{j}^3$ get masses

$$m^2 = \left(\frac{R^2-\alpha'}{R\alpha'}\right)^2,$$

using

$$m^2 = \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N+\tilde{N}-2)$$

breaking the $SU(2)\times SU(2)$ symmetry, down to $U(1)\times U(1)$ (with one (two) massless vectors).
Unlike with weak interactions, we can get arbitrarily close to the symmetric point by varying $R\to\sqrt{\alpha'}$. This shows that the potential contains a *flat* direction (it costs nothing to move along this direction) - not a Mexican hat.

Conclusion

There is a underlying gauge symmetry $SU(2)\times SU(2)$ in string theory which is not present in particle theory (KK). Strings see space-time in an unusual way-not yet understood.

6.9 T-duality

Recall

$$m^2 = \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N+\tilde{N}-2).$$

As $R\to\infty$, $n=0$ states becomes infinitely massive and decouple. $w=0$ states go to a continuum of small masses so we get ordinary particle theory at an uncompactified extra dimension. As $R\to 0,\ w=0$ states becomes infinitely massive and decouples. $n=0$ states so to a continuum, so this is similar to the$R\to\infty$ limit, i.e., we still have an *extra* uncompactified dimension, even though it has been shrunk to 0! This behavior generalizes to a symmetry under

$$R\to R' = \frac{\alpha'}{R}.$$

Spectra are identical if we just interchange $(n\leftrightarrow w)$. THM: The theories at R and $R'=\frac{\alpha'}{R}$ are identical.

Proof: Let us start with the R theory. Recall $U = U_L + U_R$. Define $Z = U_L - U_R$. Z has the *same* OPE's as U, because they all come in pairs, so the signs cancel (e.g., $Z_R Z_R = (-U_R)(-U(R))$. However,

$$\begin{aligned} U_L &= u_L - i\frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z \ldots \\ U_R &= u_R - i\frac{\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln \bar{z} \ldots \end{aligned}$$

so

$$Z = u_L - u_R - i\frac{\alpha'}{2}\left[\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z - \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln \bar{z}\right] + \ldots \quad (6.9.1)$$

and the R' theory has

$$Z = u_L + u_R - i\frac{\alpha'}{2}\left[\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z + \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln \bar{z}\right] + \ldots$$

so

$$U = u_L + u_R - i\frac{\alpha'}{2}\left[\left(\frac{w}{R} + \frac{nR}{\alpha'}\right)\ln z - \left(\frac{w}{R} - \frac{nR}{\alpha'}\right)\ln \bar{z}\right] + \ldots$$

which has the same momentum as Z if we interchange $(w \leftrightarrow n)$. QED
The self-dual point is $R = R'$, which is the special point we discussed before. The set of inequivalent theories lies in the internal $[\alpha', \infty)$, so there is a "min" $R = \sqrt{\alpha'}$ from the string point of view.
T-duality, $R \leftrightarrow \frac{\alpha'}{R}$ is a $\mathbb{Z}_2$ symmetry. It is part of $SU(2) \times SU(2)$. Indeed, note that, if $\delta R = R - R_{\min}$, then for small δR, $\delta R' = R' + \sqrt{\alpha'} = \frac{\alpha'}{R} - \sqrt{\alpha'} = \frac{\sqrt{\alpha'}}{R}(\sqrt{\alpha'} - R) \simeq -\delta R$ since $(R \simeq \sqrt{\alpha'})$. $\delta R = j^3 \tilde{j}^3$, so reversing its sign means, e.g., in terms of the first $SU(2)$, a reflection in the 12-plane.
However, to get $\delta R \to -\delta R$, we may also rotate (generated by j^1) around the 1-axis by π. This is an $SU(2)$ transformation. Thus the points $\sqrt{\alpha'} + \delta R$ and $\sqrt{\alpha'} - \delta R$ are *gauge* equivalent.

$$R = \frac{\sqrt{\alpha'}}{k} \cong R = k\sqrt{\alpha'}$$

Recall

$$m^2 = \frac{n^2}{R^2} + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2).$$

For $R = k\sqrt{\alpha'}$, we have massless scalars with $N = \tilde{N} = 0$,

$$m^2 = \frac{n^2}{k^2\alpha'} + \left(\frac{wk^2}{\alpha'}\right)^2 - \frac{4}{\alpha'} = 0 \Rightarrow n = \pm 2k,\ w = 0.$$

In the T-dual theory, $R = \sqrt{\alpha'}/k$, these scalars have $n = 0,\ w = \pm 2k\ (w \leftrightarrow n)$. Therefore, $k_L = \frac{wR}{\alpha'} = \pm\frac{2}{\alpha'} = -k_R$, and the vertex operators are

$$: e^{ik_L U_L} e^{ik_R U_R} e^{ik\cdot X} :=: e^{\pm i\frac{2}{\alpha'}U_L} e^{\mp i\frac{2}{\alpha'}U_R} e^{ik\cdot X} :=: j^{\pm}\tilde{j}^{\mp} e^{ik\cdot X} :$$

which is part of the set

$$j^a \tilde{j}^b e^{ik\cdot X} :$$

at the symmetric point $R = \sqrt{\alpha'}$! How come? (Total of three: $j^3\tilde{j}^3,\ j^+\tilde{j}^-,\ j^i\tilde{j}^+$) To see the connection you must think of twists. Why? Because you can! Let us start with $k = 2$ for simplicity. We wish to compare $r = \sqrt{\alpha'}$ and $R = \sqrt{\alpha'}/2$ (equivalent to $R = 2\sqrt{\alpha'}$). Consider the expansions

$$\begin{aligned} U_L &= u_L - i\frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha_m z^{-m}, \\ U_R &= u_R - i\frac{\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\ln \bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\tilde{\alpha}_m \bar{z}^{-m}. \end{aligned}$$

For

$$\begin{aligned} R = \sqrt{\alpha'} \quad &: \quad U_L = u_L - i\sqrt{\frac{\alpha'}{2}}(n + w)\ln z + \ldots \\ R = \frac{\sqrt{\alpha'}}{2} \quad &: \quad U_L = u_L - i\sqrt{\frac{\alpha'}{2}}(n + \frac{w}{2})\ln z + \ldots \end{aligned}$$

To mimic $R = \sqrt{\alpha'}/2$ at $R = \sqrt{\alpha'}$, we need to (a) $n \to 2n$, (b) $w \to w/2$. (a) is easier so let us try it first. We need to restrict n to even numbers. This is a restriction on the Hilbert space.
c.f. Harmonic Oscillator: $H = p^2/2m + 1/2\ m\omega^2 x^2$. H has eigenvalues $(n + 1/2)\hbar\omega$. We can restrict wavefunctions to evenfunctions. This is consistent, because H is even (H commutes with the parity operator). Then H has the eigenvalues $(2n + 1/2)\hbar\omega$. Given $\Psi(x)$, we can construct the even function

$$\psi_{\text{even}} = \frac{1}{2}(1 + P)\psi(x) = \frac{1}{2}(\psi(x) + \psi(-x)),$$

where $1/2(1 + P)$ is a projection operator.
For strings, the restriction on the Hilbert space (even n) is consistent. Indeen, consider two even-n states,

$$V =: e^{ik_L U_L} e^{ik_R U_R} :, \qquad V' =: e^{ik'_L U_L} e^{ik'_R U_R} :$$

The OPE gives

$$V(z, \bar{z})V(z', \bar{z}') \sim z^{\frac{\alpha'}{2}k_L k'_L} e^{\frac{\alpha'}{2}k_R k'_R} V_{kL+kL',k_R+k'_R}.$$

The operator similar to parity is $(-1)^n$ where $n \sim$ charge $\sim$ momentum, so it acts just like parity, $U \to -U$.

Having completed (a), we turn to (b). This is harder. We need to allow half-integer winding numbers. How can a closed string wind half-way? **Answer**: Fold the circle $U \equiv U + 2\pi R$ by identifying point $U \equiv -U$. There are two fixed points: 0 and $\pi R = -\pi R$. This creates a singular "manifold" known as an *orbifold*. Now the string can wind half-way, say from $-\pi R/2$ to $\pi R/2$ (these two points are identified, so the string is closed).
In general, the ends of the string can be at opposite points, i.e., $U(\sigma + 2\pi) = -U(\sigma)$. We are allowed to impose anti-periodic boundary conditions! Does the theory make sense? There is no a priori guarantee that it will, but alas, it does (also note, $p = 0$, string cannot move away from the fixed point, so $n = 0$). Of course, we need to restrict the Hilbert space again to "even parity" states. Again, this is a consistent truncation and the resulting theory is **identical** to the theory on a $R = \sqrt{\alpha'}/2$ circle.
The truncated theory is called "twisted". Now let us compare the massless scalars. In the original $R = \sqrt{\alpha'}$ theory we had $9\ (3 \times 3)$ scalars, $: j^a \tilde{j}^b e^{i\kappa \cdot X} :$.
These are indeed the massless scalars at $R = \sqrt{\alpha'}/2$. The above generalizes to $\forall k \in \mathbb{N}$.

Open Strings

Just like closed strings, open strings have a quantized momentum in the compact dimensions, $p = \frac{n}{R}$. However, there is no *winding* for open strings, so $w = 0$ (just like KK particles). The mass formula is

$$m^2 = \frac{n^2}{R^2} + \frac{1}{\alpha'}(N-1).$$

As $R \to 0$, $n \neq 0$ states become infinitely massive and decouple. Thus the compact dimensions disappears. That would be considered normal behavior, were it not for the fact that open strings cannot help but create and interact with closed strings. The latter exhibit weird behavior (the compact dimension does **not** disappear as $R \to 0$). So how does one reconcile the two pictures? It is easier to think in terms of the $R \to \infty$ limit and we have already seen that this is possible with closed strings because of T-duality. Recall that the theory at $R' = \alpha'/R$ is equivalent to the theory at R if written in terms of $Z = U_L - U_R$ instead of $U_L + U_R$.
Recall the expansion

$$U(z,\bar{z}) = u - i\alpha' p \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{1}{m}\alpha_m (z^{-m} + \bar{z}^{-m}).$$

For compact U, $p = n/R$.

$$z = e^{\pi i(\sigma+\tau)/\ell}, \qquad \bar{z} = e^{-\pi i(\sigma-\tau)/\ell}$$

We will set $\ell = \pi$ for simplicity. So

$$\begin{aligned} \partial_\sigma U_L = \partial_\sigma z \partial U = iz\partial U_L, &\quad \partial_\tau U_L = iz\partial U_L, \\ \partial_\sigma U_R = \partial_\sigma \bar{z}\bar{\partial} U_R = -i\bar{z}\bar{\partial} U_R, &\quad \partial_\tau U_R = i\bar{z}\bar{\partial} U_R, \end{aligned}$$

so

$$\partial_\sigma Z = iz\partial U_+ i\bar{z}\bar{\partial}U, \quad \partial_\tau U = iz\partial U + i\bar{z}\bar{\partial}U = \partial_\sigma Z$$

so

$$\begin{aligned} Z(\sigma=\pi) - Z(\sigma=0) &= \int_0^\pi d\sigma \partial_\sigma Z = \int_0^\pi d\sigma \partial_\tau U = \int_0^\pi d\sigma \partial_\tau (2\alpha' p\tau) \\ &= 2\alpha' p\pi = 2\alpha'\pi \frac{n}{R} = 2\pi n R' \end{aligned}$$

In other words, the ends of the string lie at the same point in the compact dimension (in terms of the dual coordinate). Including the noncompact dimensions, this implies that end-points lie on a hyperplane (D-brane).
Notice that translation invariance is broken, or equivalently, the momentum in the compact dimension is not conserved. Since $p = \frac{n}{R}$ and in the dula theory $n \leftrightarrow w$, this is equivalent to the non-conservation of winding number in the small R theory. That is obvious. A wound closed string can break into two open strings.
Consider massless modes. These are as in the uncompactified case, i.e., the photon: $\alpha_{-1}^\mu|0;k\rangle$. We need to split it into uncompactified $\alpha_{-1}^\mu|0;k\rangle$ and compactified, $\alpha_{-1}|0;k\rangle$ components. Corresponding vertices

$$: \partial_\tau X^\mu e^{ik\cdot X} e^{inU/R} : (\sigma = 0) \qquad : \partial_\tau U e^{ik\cdot X} e^{inU/R} :$$

but $n=0$ for the massless and $k^2 = 0$. The former is a photon *tangent* to the D-brane. The letter can be written as

$$: \partial_\sigma Z e^{ik\cdot X} : .$$

Just like the gravition $g_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}$: contributes to the background and curves it, the vertex $A\partial_\sigma Z e^{ik\cdot X}$ shifts the position of the D-brane $z \to z + A\partial_\sigma Z$ ($\partial_\sigma z$ is perpendicular to the D-brane). Therefore, the D-brane is a dynamical object. Its fluctuations are described by open strings attached to it.
The D-brane is *our* Universe! Notice that the photon (and other particles) are confined to the D-brane. No wonder we never wander off into the extra dimension(s). On the other hand, gravity has to be present in the extra dimension, because gravity *creates* space.
This D-brane fills space. We can imagine more compact dimensions and have p non-compact dimensions. Then we have a Dp-brane.

Scattering

Let us compare open and closed strings. We will need to mix them in order to describe scattering by D-branes. Recall the operator product expansions for closed strings

$$X(z,\bar{z})X(0,0) \sim -\frac{\alpha'}{2}\ln|z|^2 + \ldots$$

We also have

$$\partial\bar{\partial}(X(z,\bar{z})X(0,0)) \sim -\pi\alpha'\delta^2(z,\bar{z})$$

so $G(z,\bar{z}) = X(z,\bar{z})X(0,0)$ is a Green function. $\ln|z|^2$ satisfies the boundary condition of periodicity in $\sigma : z = e^{i(\sigma+\tau)}$, so $\sigma \to \sigma + 2\pi \Rightarrow z \to z$.
It is the electrostatic potential of the uniformly charged straight line. Open strings, $z = e^{i(\sigma+\tau)}$, but not $0 \le \sigma \le \pi$, so z is on the upper-half plane. The boundary is the real axis and that is where the vertex operators are. The Green function (and the OPE) is found in two steps. First we need to satisfy Neumann boundary conditions, $\partial_\sigma X = 0$.
This translates into $\partial_n X = 0$ (normal to boundary vanishes), which can be satisfied by adding an image charge at $\bar{z}$. This is different from electrostatics, where the tanget needs to vanish, requiring an *opposite* charge for the image. Thus,

$$G(z,\bar{z};z',\bar{z}') = -\frac{\alpha'}{2}\ln|z-z'|^2 - \frac{\alpha'}{2}\ln|\bar{z}-\bar{z}'|^2 \tag{6.9.2}$$

When both z and z' approach the boundary (i.e., they become real), we obtain

$$G(z,\bar{z};z'\bar{z}') = -\alpha'\ln|z-z'|^2 = -2\alpha'\ln|z-z'|.$$

This shows that the OPE for open strings ought to be

$$X(z)X(0) \sim -2\alpha'\ln|z|.$$

Recall the amplitudes:
Closed strings:

$$A_n = \langle : e^{ik_1\cdot X}(z_1,\bar{z}_1)e^{ik_2\cdot X}(z_2,\bar{z}_2)...e^{ik_n\cdot X}(z_n,\bar{z}_n) :\rangle.$$

View this as a function of z_1 and differentiate. We obtain

$$\partial_{z_1}A_n = \langle : ik_1\partial X e^{ik_1\cdot X}(z_1,\bar{z}_1)e^{ik_2\cdot X}(z_2,\bar{z}_2)...e^{ik_n\cdot X}(z_n,\bar{z}_n) :\rangle.$$

By making use of the OPE $\partial X(z) : e^{ik\cdot X}(0) :\sim -ik\frac{\alpha'}{2}\frac{1}{z}e^{ik\cdot X}(0)$ we obtain

$$\partial_{z_1}A_n = \frac{\alpha'}{2}A_n\sum_i \frac{k_1 k_i}{z_1 - z_i}.$$

Integrating ...

$$A_n \propto \prod_{i<j}|z_i - z_j|^{\alpha' k_i k_j}$$

which is the holomorphic and antiholomorphic pieces multiplied together.
For open strings, a similar argument yields $A_n \propto \prod_{i<j}|z_i - z_j|^{2\alpha' k_i k_j}$. For closed string emission from a D-brane, we need to use (6.9.2). (See (6.2.33) Polchinski)

UNIT 7

Superstrings

7.1 Bosons and fermions

Bosonic strings have the action

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \partial X^\mu \bar{\partial} X_\mu.$$

We wish to build a theory that has supersymmetry (SUSY). Why? It turns out that this is the only (known) way of obtaining a consistent theory.
For SUSY, each boson (commuting field), must have a fermionic (anticommuting) counterpart. We have already seen anticommuting fields. We called them b, c. Recall the b, c action

$$S_{bc} = \frac{1}{2\pi} \int d^2 z b \bar{\partial} c.$$

and their OPEs are

$$b(z)c(0) \sim \frac{1}{z}.$$

The wave equation was given by $\bar{\partial} b = \bar{\partial} c = 0$, i.e., b and c are purely holomorphic. The energy-momentum tensor is

$$T =: (\partial b)c : -\lambda \partial(: bc :)$$

where we assume the weights $h_b = \lambda,\ h_c = 1 - \lambda$. The OPE for the energy-momentum tensor is

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2} T(0) + \frac{1}{z} \partial T(0)$$

where $c = -3(2\lambda - 1)^2 + 1$. Earlier we required $\lambda = 2$, so $c = -26$ (hence $D = 26$ for the bosonic string) in order to do BRST quantization properly

($Q^2_{BRST} = 0$). A more symmetric choice is $\lambda = \frac{1}{2}$. Then $h_b = h_c = \frac{1}{2}$ and $c = 1$. Define

$$b = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \qquad c = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2).$$

Then the action is

$$S = \frac{1}{2\pi}\int d^2 z b\bar{\partial}c = \frac{1}{4\pi}\int d^2 z(\psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2).$$

The stress-energy tensor written in terms if the new fields may be expressed as

$$T(z) = -\frac{1}{2}\psi_1\partial\psi_1 - \frac{1}{2}\psi_2\partial\psi_2.$$

The system splits into two identical copies. Since $c = 1$, for the two together, each system has $c = \frac{1}{2}$.
Pick one such system, $\psi = \psi_1$, say. Make D copies of it, $\psi \to \psi^\mu$ ($\mu = 0, 1, ..., D-1$) and let us try ψ^μ as a SUSY partner of X^μ.
The stress-energ tensor is given by

$$T = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu.$$

The TT OPE becomes

$$T(z)T(0) \sim \frac{(3D/2)}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0)$$

where we used

$$X^\mu(z,\bar{z})X^\nu(0,0) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z|^2, \qquad \psi^\mu(z)\psi^\nu(0) \sim \frac{1}{z}\eta^{\mu\nu}.$$

$T(z)$ is a conserved current that generates conformal transformations which are symmetries of the theory (in fact $v(z)T(z)$ is conserved for arbitrary $v(z)$, leading to an infinite number of symmetries). The new theory (X^μ, ψ^μ) has even more symmetries! Let us define a supercurrent as

$$T_F = i\sqrt{\frac{\alpha'}{2}}\psi^\mu(z)\partial X_\mu(z)$$

Any $\eta(z)T_F(z)$ is conserved and generates a symmetry mixing X^μ and ψ^μ (superconformal transformation) - η must be anticommuting so that ηT_F is commuting. To see this consider

$$\begin{aligned} T_F(z)X^\mu(0,0) &\sim -i\sqrt{\frac{\alpha'}{2}}\frac{\alpha'}{2}\frac{1}{z}\psi^\mu(0) = -i\sqrt{\frac{\alpha'}{2}}\frac{1}{z}\psi^\mu(0) \\ T_F(z)\psi^\mu(0) &\sim i\sqrt{\frac{2}{\alpha'}}\frac{1}{z}\partial X^\mu(0,0) \end{aligned}$$

So

$$\begin{aligned}\delta X^\mu &= -i\epsilon\oint\frac{dz}{2\pi i}\eta(z)T_F(z)X^\mu(0,0) = -\sqrt{\frac{\alpha'}{2}}\epsilon\eta\psi^\mu(0)\\ \delta\psi^\mu &= -i\epsilon\oint\frac{dz}{2\pi i}\eta(z)T_F(z)\psi^\mu(0) = -\sqrt{\frac{2}{\alpha'}}\epsilon\eta\partial X^\mu(0,0)\end{aligned}$$

The other OPEs are given by

$$\begin{aligned}T(z)T_F(0) &\sim 2\left(-\frac{1}{a'}\right)\left(i\sqrt{\frac{\alpha'}{2}}\right)\left(\frac{\alpha'}{2}\partial^2\ln|z|\right)\partial X^\mu(z)\psi_\mu(0) + \left(-\frac{1}{2}\right)\left(i\sqrt{\frac{\alpha'}{2}}\right)\left(\partial\frac{1}{z}\right)\psi^\mu(z)\partial X_\mu(0)\\ &\quad + \left(-\frac{1}{2}\right)\left(i\sqrt{\frac{\alpha'}{2}}\right)\frac{1}{z}\partial\psi_\mu(z)\partial X^\mu(0)\\ T(z)T_F(0) &\sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0)\\ T_F(z)T_F(0) &\sim \left(i\sqrt{\frac{\alpha'}{2}}\right)^2\frac{1}{z}\left(\frac{\alpha'}{2}\partial^2\ln|z|\right)D + \left(i\sqrt{\frac{\alpha'}{2}}\right)^2\frac{1}{z}\partial X^\mu\partial X_\mu\\ &\quad + \left(i\sqrt{\frac{\alpha'}{2}}\right)^2\left(\frac{\alpha'}{2}\partial^2\ln|z|\right)\psi^\mu(z)\psi_\mu(0)\\ &\sim \frac{D}{z^3} + \frac{2}{z}T(0).\end{aligned}$$

The first OPE shows that T_F has weight $h = 3/2$. There is a corresponding construction for the anti-holomorphic operators. Since ψ^μ is holomorphic, we need to add a *new* anti-holomorphic fermionic field $\tilde\psi^\mu(\bar z)$ with the action

$$\tilde S = \frac{1}{4\pi}\int d^2z\tilde\psi^\mu\partial\tilde\psi_\mu.$$

The wave equation is given by

$$\partial\tilde\psi_\mu = 0,$$

so, indeed $\tilde\psi^\mu$ is anti-holomorphic. They OPE is

$$\tilde\psi^\mu(\bar z)\tilde\psi^\nu(0) \sim \frac{1}{\bar z}\eta^{\mu\nu}.$$

The stress-energy tensors are

$$\tilde T = -\frac{1}{2}\tilde\psi^\mu\bar\partial\tilde\psi_\mu, \qquad \tilde T_F = i\sqrt{\frac{\alpha'}{2}}\tilde\psi^\mu\bar\partial X_\mu.$$

The OPEs are similar to the OPEs of their holomorphic counterparts. Notice that the central charge for this theory is $c = 3D/2$. This is now a superconformal theory ($N = 1,\ \tilde N = 1$ where $N, \tilde N$ counts the number of $T_F, \tilde T_F$'s). Other examples

7.2 The ghosts

Recall,

$$S_{bc} = \frac{1}{2\pi}\int d^2z b\bar{\partial}c, \qquad T = (\partial b)c - \lambda\partial(bc), \qquad b(z)c(0) \sim \frac{1}{z}.$$

The weights and central charge for the bc system are

$$h_b = \lambda, \quad h_c = 1-\lambda, \quad c_{bc} = -3(2\lambda-1)^2 + 1.$$

Since (b, c) are anti-commuting fields, their partners will have to be commuting. We have already met them. They are the (β, γ) fields with action

$$S_{\beta\gamma} = \frac{1}{2\pi}\int d^2z \beta\bar{\partial}\gamma,$$

which is the same action as the bc action. Let $h_\beta = \lambda'$, $h_\gamma = 1-\lambda'$. The combined system will have SUSY if we can find a T_F that mixes b, c with β, γ. Such a T_F will most likely contain a $(\partial\beta)c$ (c.f. $(\partial b)c$ in T_{bc} and $(\partial\beta)\gamma$ in $T_{\beta\gamma}$,

$$T_{\beta\gamma} = (\partial\beta)\gamma - \lambda'\partial(\beta\gamma).$$

Since $h = 3/2$ for T_F, we need $1 + \lambda' + 1 - \lambda = 3/2$, i.e., $\lambda' = \lambda - 1/2$. The central charge is

$$c_{\beta\gamma} = 3(2\lambda'-1)^2 - 1 = 3(2\lambda-2)^2 - 1.$$

The central charge for the combination of the two systems becomes

$$c_{\text{total}} = c_{bc} + c_{\beta\gamma} = -3(2\lambda-1)^2 + 3(2\lambda'-2)^2 = 3(3-4\lambda).$$

For the special (interesting) case $\lambda = 2$, in which $c_{bc} = -26$ (hence $d = 26$ for bosonic strings), we have $c_{\text{total}} = 3(3-4\times 2) = -15$. If we combine this system with the $(X^\mu,\ \psi^\mu,\ \tilde{\psi}^\mu)$, for which $c = 3D/2$ and demand $c_{\text{total}} = 0$, we need $3D/2 - 15 = 0 \Rightarrow D = 10$. Therefore superstrings must live in 10-dimensions.

Linear Dilaton

Recall

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu + V_\mu\partial^2 X^\mu,$$

where V_μ is a fixed vector (breaking translational invariance). The central charge for this theory is

$$c = D + 6\alpha' V^\mu V_\mu.$$

By adding the fermion ψ^μ, with $T = -\frac{1}{2}\psi^\mu\partial\psi_\mu$ and $c = D/2$, we obtain

$$c = \frac{3D}{2} + 6\alpha' V^\mu V_\mu,$$

and

$$T_F = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X_\mu - i\sqrt{2\alpha'}V_\mu\partial\psi^\mu.$$

7.3 Mode Expansions

Let us do closed strings first. Recall the expansion

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu z^{-m-1},$$

where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$ and $[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$. X^μ obeys periodic boundary conditions. We could have imposed anti-periodic boundary conditions on X^μ, and we did so with U (the compactified coordinate) and got an orbifold, but this breaks translational invariance. That is ok for dimensions we cannot see (e.g., compactified), but not for the four dimensions that describe our space-time. ψ^μ and $\tilde{\psi}^\mu$ on the other hand have no such concerns (also note the absence of a spin-statistics theorem in two-dimensions), so we have two possibilities.

- anti-periodic boundary conditions (Neveu-Schwarz (NS)): $\psi^\mu(\sigma+2\pi) = -\psi^\mu(\sigma)$.
- periodic boundary conditions (Ramond (R)): $\psi^\mu(\sigma+2\pi) = \psi^\mu(\sigma)$.

These have two distinct Hilbert spaces (sectors). There are also two Hilbert spaces for $\tilde{\psi}^\mu$, so in all there are four Hilbert spaces (sectors): NS-NS, R-NS, NS-R, R-R.
Let us first describe $\psi\mu$ in NS. ψ^μ is a function of $\sigma+\tau$. When expanding in Fourier modes, because of anti-periodicity, only the terms $e^{-i(2m+1)(\sigma+\tau)/2}$ contribute (since $\sigma\to\sigma+2\pi \Rightarrow e^{-i(2m+1)(\sigma+\tau)/2} \to e^{-\pi i(2m+1)}e^{-i(2m+1)(\sigma+\tau)/2}$)
Define $r = m+1/2 \in Z+1/2$, then

$$\psi^\mu(\sigma+\tau) = \sqrt{i}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu e^{-ir(\sigma+\tau)}$$

where the factor of $\sqrt{i}$ was introduced for convenience. Transforming to the z-picture, $z = e^{i(\sigma+\tau)}$, we have

$$\begin{aligned}\psi^\mu(z) &= \left(\frac{\partial w}{\partial z}\right)^h \psi^\mu(\sigma+\tau)\\ &= \frac{1}{\sqrt{iz}}\psi^\mu(\sigma+\tau)\\ &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu z^{-r-\frac{1}{2}}\end{aligned}$$

which is a Laurent expansion. We saw the same in terms of the X^μ field. We obtain anti-commutation relations of the ψ^μ fields by analyzing the OPE

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{1}{z}\eta^{\mu\nu}.$$

The anti-commutation relations are

$$\{\psi_r^\mu, \psi_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}.$$

We find similar results for the right-moving sector

$$\tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}_r^\mu \bar{z}^{-r-\frac{1}{2}}, \qquad \bar{\partial}X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}} \tilde{\alpha}_m^\mu \bar{z}^{-m-1},$$

and the anti-commutation relations are

$$\{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}.$$

The stress-energy tensor is

$$T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}, \qquad h = 2.$$

The OPE gives the Virasoro algebra with central extension

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m-1)(m+1)\delta_{m+n,0}.$$

In terms of the OPEs we find

$$T_F(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0).$$

We may expand $T_F(z)$ in terms of modes

$$T_F(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r z^{-r-\frac{3}{2}}.$$

Recall

$$[L_m, G_r] = ((h-1)m - r)G_{r+m} = (\frac{1}{2}m - r)G_{r+m}.$$

Finally

$$T_F(z)T_F(z') \sim \frac{D}{(z-z')^3} + \frac{2}{z-z'}T(z'), \quad \frac{3D}{2} = c, \text{ so } D = \frac{2c}{3}.$$

Find the anit-commutator $\{G_r, G_s\}$ in two steps. First

$$G_r = \oint \frac{dz}{2\pi i} r^{r+\frac{1}{2}} T_F(z),$$

and

$$\oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z)T_F(z') = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} \frac{D}{(z-z')^3} + 2z'^{r+\frac{1}{2}}T(z')$$

$$f(z) = z^{r+\frac{1}{2}}, \quad f'(z) = \left(r + \frac{1}{2}\right) r^{r-\frac{1}{2}}, \quad f''(z) = \left(r^2 - \frac{1}{4}\right) z^{r-\frac{3}{2}},$$

so

$$\oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} \frac{D}{(z-z')^3} = \frac{D}{2}\left(r^2 - \frac{1}{4}\right) z'^{r-\frac{3}{2}}$$

Second step: apply $\oint \frac{dz'}{2\pi i} z'^{s+\frac{i}{2}}$ to isolate G_s:

$$\begin{aligned}
\{G_r, G_s\} &= \frac{D}{2}\left(r^2 - \frac{1}{4}\right) \oint \frac{dz'}{2\pi i} z'^{r+s-1} + 2 \oint \frac{dz'}{2\pi i} z'^{r+s+1} T(z') \\
&= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right) \delta_{r+s,0} \\
&= 2L_{r+s} + \frac{c}{12}\left(4r^2 - 1\right) \delta_{r+s,0}
\end{aligned}$$

The algebra of $(L_m,\, G_r)$ closes, as expected: NS algebra. Next, let us study the mode expansion: using

$$\partial X^\mu = -i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \alpha_m^\mu z^{-m-1}, \quad \psi^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu z^{-r-\frac{1}{2}},$$

and

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{4}(\psi^\mu \partial\psi_\mu - (\partial\psi^\mu)\psi_\mu)$$

we have

$$\begin{aligned}
L_m &= \oint \frac{dz}{2\pi i} z^{m+1} T(z) = \frac{1}{2}\sum_{n,n'} \oint \frac{dz}{2\pi i} \alpha_n^\mu \alpha_{n'\mu} z^{-n-n'-m-1} + \frac{1}{4}\sum_{r,r'} \oint \frac{dz}{2\pi i} \psi_r^\mu \psi_{r'\mu}(r-r') z^{-r-r'+m-1} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}^\mu \alpha_{n\mu} + \frac{1}{4}\sum_{r\in\mathbb{Z}+\frac{1}{2}} (2r-m)\psi_{m-r}^\mu \psi_{r\mu}.
\end{aligned}$$

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu \partial X_\mu \Rightarrow G_r = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z) = \sum_{n,r'} \oint \frac{dz}{2\pi i} \alpha_n^\mu \alpha_{r'\mu} z^{-n+r+r'-1} = \sum_{n\in\mathbb{Z}} \alpha_n^\mu \psi_{r-n\,\mu}.$$

Normal ordering: No question in G_r, $\forall\, r$ and $L_m, \forall\, m \neq 0$. Potential problem with L_0. After normal ordering, we get $L_0 + a$ where a is a constant to be determined. To determine a, look at $[L_+, L_{-1}] = 2L_0$. We have $L_1|0\rangle = 0$, so $\langle 0|[L_+, L_{-1}]|0\rangle = \langle 0[L_{+1}L_{-1}]|0\rangle = ||L_{-1}|0\rangle||^2$, because $L_{-1}^+ = L_1$.
Now $L_{-1}|0\rangle = \frac{1}{2}\sum \alpha_{-1-n}^\mu \alpha_{n\mu}|0\rangle + \frac{1}{4}\sum(2r+1)\psi_{-1-r}^\mu \psi_{r\mu}|0\rangle$ There are non-vanishing terms only if $-n-1 < 0\; n < 0$, i.e., $0 < n < -1$ which is impossible! Also $0 < r < -1$, which implies $r = 1/2$, but then $2r+1 = 0$, so it also vanishes. Therefore

$$L_{-1}|0\rangle = 0, \qquad ||L_{-1}|0\rangle||^2 = 0,$$

so

$$\langle 0|2L_0|0\rangle = 2a = 0 \;\Rightarrow a = 0.$$

In the above, we used $\alpha_n^\mu|0\rangle = \psi_r^\mu|0\rangle,\; n, r > 0$, and the hermicity property, $(\alpha_{-n}^\mu)^\dagger = \alpha_n^\mu,\; (\psi_{-r}^\mu)^\dagger = \psi_r^\mu$.

The ghosts

The ghost system $(b,c;\beta,\gamma)$ is a superconformal system on its own right. It is opposite to (X^μ,ψ^μ) in that the role of X^μ is played by the fermionic (b,c). So $b,c,$ obey periodic boundary conditions (necessary due to definition of $Q_{\rm BRST}$). Then (β,γ) may obey periodic (R) or anti-periodic (NS) boundary conditions. Let us do NS first. Recall

$$h_b=\lambda,\quad h_c=1-\lambda,\quad h_\beta=\lambda',\quad h_\gamma=1-\lambda',\quad \lambda'=\lambda-\frac{1}{2}.$$

We are interested in the $\lambda=2$ case, in order to couple this system to the (X^μ,ψ^μ) system. Then

$$h_b=2,\quad h_c=1-1,\quad h_\beta=\frac{3}{2},\quad h_\gamma=-\frac{1}{2},$$

and the expansions are

$$b=\sum_{m\in\mathbb{Z}}b_m z^{-m-2},\;\; c=\sum_{m\in\mathbb{Z}}c_m z^{-m+1},\;\; \beta=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\beta_r z^{-r-\frac{3}{2}},\;\; \gamma=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\gamma_r z^{-r+\frac{1}{2}}.$$

From the operator product expansions, we get standard (anti) commutators

$$\{b_m,c_n\}=\delta_{m+n,0},\qquad\qquad [\gamma_r,\beta_s]=\delta_{r+s,0}.$$

$b_m,\ c_m,\ \beta_r,\ \gamma_r$ are all annihilation operators for $r,m>0$. Recall the subtlety with the zero modes b_0,c_0, satisfying $\{b_0,c_0\}=1$. We have two choices for the vacuum. Choose $c_0|0\rangle=0$. The conformal generators are

$$L_m=\oint\frac{dz}{2\pi i}z^{m+1}T(z),$$

$$\begin{aligned}
T(z) &= (\partial b)c-\lambda\partial(bc)+(\partial\beta)\gamma-\lambda'\partial(\beta\gamma)\\
&= (\partial b)c-2\partial(bc)+(\partial\beta)\gamma-\frac{3}{2}\partial(\beta\gamma)\\
&= \sum_{n,n'}(-n'-2)b_{n'}z^{-n'-3}c_n z^{-n+1}-2(-n-n'-1)b_{n'}c_n z^{-n-n'-2}\\
&\quad+\sum_{r,r'}\left(-r'-\frac{3}{2}\right)\beta_{r'}z^{-r'-\frac{5}{2}}\gamma_r z^{-r+\frac{1}{2}}-\frac{3}{2}(-r-r'-1)\beta_{r'}\gamma_r z^{-r-r'-2}\\
&= \sum_{n,n'}(n'+2n)b_{n'}c_c z^{-n-n'-2}+\frac{1}{2}\sum_{r,r'}(3r+r')\beta_{r'}\gamma_r z^{-r-r'-2}
\end{aligned}$$

So

$$L_m=\oint\frac{dz}{2\pi i}z^{m+1}T(z)=\sum_n(m+n)b_{m-n}c_n+\frac{1}{2}\sum_r(m+2r)\beta_{m-r}\gamma_r$$

The SUSY generators are

$$G_r = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z), \qquad T_F(z) = -\frac{1}{2}(\partial\beta)c + \lambda'\partial(\beta c) - 2b\gamma$$

where

$$\begin{aligned} T_F(z) &= \sum_{s,n} -\frac{1}{2}\left(-s-\frac{3}{2}\right)\beta_s z^{-s-\frac{5}{2}} c_n z^{-n+1} + \frac{3}{2}\left(-s-n-\frac{1}{2}\right)\beta_s c_n z^{-s-n-\frac{3}{2}} - 2b_n\gamma_s z^{-n-s-\frac{3}{2}} \\ &= \sum_{s,n} -\frac{1}{2}(2s+3n)\beta_s c_n z^{-n-s-\frac{3}{2}} - 3b_n\gamma_s z^{-n-s-\frac{3}{2}}. \end{aligned}$$

So

$$G_r = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z) = -\sum_n \frac{1}{2}(2r+n)\beta_{r-n}c_n + 2b_n\gamma_{r-n}.$$

Normal ordering: again, only L_0 has a problem; should be $L_0 + a$. To find a, consider $[L_1, L_{-1}] = 2L_0$.

$$L_{-1} = \sum_n (n-1)b_{-1-n}c_n + \frac{1}{2}\sum_r (2r-1)\beta_{-1-r}\gamma_r.$$

When applied to the ground state, $|0\rangle$, only the terms $n = -1$, $r = -1/2$ contribute (recall $c_0|0\rangle = 0$), so $L_{-1}|0\rangle = -2b_0c_{-1}|0\rangle - \beta_{-1/2}\gamma_{-1/2}|0\rangle$. Similarly, we obtain $\langle 0|c_{-1} = \langle 0|(2b_1c_0 + \beta_{1/2}\gamma_{1/2})$. So

$$\langle 0|[L_1, L_{-1}]|0\rangle = \langle 0|L_1L_{-1}|0\rangle = -2\langle 0|(b_1c_0b_0c_{-1} - \beta_{\frac{1}{2}}\gamma_{\frac{1}{2}}\beta_{-\frac{1}{2}}\gamma_{-\frac{1}{2}}|0\rangle = -2+1 = -1.$$

So $2a = \langle 0|2L_0|0\rangle = -1$, so $a = -1/2$ (-1 from the bc and $1/2$ from the $\beta\gamma$).

7.4 Open Strings

Open strings do not have independent ocsillators α_n^μ, $\tilde{\alpha}_n^\mu$. Instead, $\alpha_n^\mu = \tilde{\alpha}_n^\mu$. Thus,

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu z^{-m-1}, \qquad \bar{\partial} X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu \bar{z}^{-m-1}.$$

where $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ (c.f. $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$ for closed strings). Similarly for the ψ^μ's:

$$\psi^\mu(z) = \sum_r \psi_r^\mu z^{-r-\frac{1}{2}}, \qquad \tilde{\psi}^\mu(\bar{z}) = \sum_r \psi_r^\mu \bar{z}^{-r-\frac{1}{2}}.$$

The spectrum

For a physical state, $|\psi\rangle$, we demand

$$L_n|\psi\rangle = G_r|\psi\rangle = 0, \quad \text{for } r, n > 0.$$

Also, $L_{-n}|\psi\rangle,\ G_r|\psi\rangle$ are orthogonal to all physical states $|\psi'\rangle$: $\langle\psi'|L_{-n}|\psi\rangle = \langle\psi|L_n|\psi'\rangle = 0$, and similarly for $G_r|\psi\rangle$. They are in the equivalence class of zero. Check also $L_{-n}|\psi\rangle$ is null: $||L_{-n}|\psi\rangle||^2 = 0$. Physical states also obey the constraint

$$\left(L_0 - \frac{1}{2}\right)|\psi\rangle = 0$$

i.e., the Hamiltonian $H = L_0 - 1/2 = 0$ (vanishes).
We build the Hilbert space by applying $\alpha^\mu_{-n},\ \psi^\mu_{-r}$ oscillators only (no ghost modes- the lead to states in the same equivalence classes as above) to the ground state.

$$H = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha^\mu_{-n}\alpha_{n\mu} : + \frac{1}{2}\sum_r r : \psi^\mu_{-r}\psi_{r\mu} : - \frac{1}{2}$$

plus the ghost oscillators, but they do not contribute. Since $\alpha_0 = \sqrt{2\alpha'}p^\mu$ for open strings, we have

$$H = \alpha' p^2 + N - \frac{1}{2}, \qquad N = \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha_{n\mu} + \sum_{r=\frac{1}{2}}^{\infty} r\psi^\mu_{-r}\psi^{r\mu}.$$

The lowest state: $|0;k\rangle$ for which $N = 0$, so $\alpha' k^2 - 1/2 = 0$, so $m^2 = -k^2 = -1/2\alpha'$, a tachyon!
So we still have a tachyon. This was to be expected, because we took the bosonic theory and enlarged it therefore we should expect the new SUSY theory to contain all the states of the bosonic theory and more.
The next state: $|1\rangle$

$$A_\mu(k)\psi^\mu_{-\frac{1}{2}}|0;k\rangle$$

has $N = \frac{1}{2}$. We see that this is a massless state since

$$\alpha' k^2 + \frac{1}{2} - \frac{1}{2} = 0, \quad \Rightarrow \quad m^2 = -k^2 = 0.$$

Also, $G_r = \sum_n \alpha^\mu_n \psi_{r-n\mu}$, so when $G_{1/2}$ acts on our state, only the $n = 0$ term contributes. So

$$G_{\frac{1}{2}}|1\rangle = \alpha^\mu_0 A_\mu(k)|0;k\rangle = \sqrt{2\alpha'}k\cdot A|0;k\rangle = 0, \ \Rightarrow \ k\cdot A = 0,$$

i.e., transverse polarization. Also note that this is a null state:

$$G_{-\frac{1}{2}}|0;k\rangle = \alpha^\mu_0\psi_{-\frac{1}{2}\mu}|0;k\rangle = \sqrt{2\alpha'}k\cdot\psi_{-\frac{1}{2}}|0;k\rangle$$

i.e., the state with longitudinal polarization is null (and orthogonal to all physical states).
Thus the massless state is a $D-2=8$ dimensional vector. It transforms under the group $SO(8)$. For **closed strings**, the situation is similar.
The $H=0$ constraint translates into $L_0=\tilde{L}_0=0$, i.e.,

$$\frac{\alpha'}{4}p^2+N-\frac{1}{2}=\frac{\alpha'}{4}p^2+\tilde{N}-\frac{1}{2}=0$$

Notice the difference in $\alpha' p^2 \to \frac{\alpha'}{4}p^2$, which is due to the different definitions of α_0^μ between closed and open string.
The lowest state: $|0;k\rangle$ with $\frac{\alpha'}{4}k^2=\frac{1}{2}$, so $m^2=-k^2=-\frac{2}{\alpha'}$ which is a tachyon!
The next level: $A_{\mu\nu}\psi^\mu_{-1/2}\psi^\nu_{-1/2}|0;k\rangle$, with $\frac{\alpha'}{4}k^2=0$, i.e., $m^2=0$. This deomposes into a scalar, an antisymmetric tensor, and a traceless symmetric tensor:

$$A_{\mu\nu}=\frac{1}{D-2}A^\rho_\rho\eta_{\mu\nu}+\frac{1}{2}(A_{\mu\nu}-A_{\nu\mu})+\frac{1}{2}(A_{\mu\nu}+A_{\nu\mu}+\frac{2}{D-2}A^\rho_\rho\eta_{\mu\nu}.$$

SectionGetting rid of the tachyon Comparing the tachyon with the massless states, there is a clear difference: the tachyon has one less fermionic excitation then the massless states. If we select the states with as *odd* number of fermionic excitations, that will get rid of the tachyon. This is similar to the harmonic oscillator, where we could select, e.g., all the odd states and still have a perfectly well defined physical system.
The operator that did the trick there was P (parity) which commuted with the Hamiltonian and could therefore be simultaneously diagonalized with it. Here we need to find an operator that has two eigenvalues and commutes with all generators of space-time symmetries (not just the Hamiltonian). The space-time symmetries from the Lorentz group (Poincare group rather, but Lorentz suffices). Let us review briefly. The angular momentum $\vec{L}=\vec{r}\times\vec{p}$. In terms of components we have

$$L_x=yp_z-zp_y,\quad L_y=zp_x-xp_z,\quad L_z=xp_y-yp_x.$$

where x and p obey the commutation relations $[x_i,p_j]=\delta_{ij}$.
Define the antisymmetric tensor $L_{ij}=x_ip_j-x_jp_i$, then $L_i=\frac{1}{2}\epsilon_{ijk}L_{jk}$. An antisymmetric tensor is a vector in three-dimensions. **Not** so in four-dimensions. So generalize $L_{ij}\to L_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu$, $[x_\mu,p_\nu]=i\eta_{\mu\nu}$ which includes *time*. $\vec{L}$ generates rotations:

$$\delta x_i=-\frac{i}{2}\omega^{kl}[L_{kl},x_i]=\omega_{ij}x_j$$

where ω_{ij} is an anti-symmetric tensor. In terms of the vector $\vec{\omega}$ we have $\delta\vec{x}=\vec{\omega}\times\vec{x}$. This generalizes to $L_{\mu\nu}\ :\ \delta x_\mu=\omega_{\mu\nu}x^\nu$. For e.g., L_{01}, we have $\delta t=\omega_{01}x,\ \delta x=-\omega_{01}t$, a boost! L_{0i} is a boost in the x_i-direction. The algebra of these Lorentz generators is

$$\begin{aligned}[L_{\mu\nu}, L_{\rho\sigma}] &= [x_\mu p_\nu - x_\nu p_\mu, x_\rho p_\sigma - x_\sigma p_\rho] \\ &= i(\eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho})\end{aligned}$$

Lie algebra of $SO(3,1)$, or in D-dimensions, $SO(D-1,1)$. Introduce spinors: we need to add a piece to $L_{\mu\nu}$ that will rotate the spinor (or boost it). Call this piece $\Sigma_{\mu\nu}$. It needs to satisfy the same $SO(D-1,1)$ algebra and will commute with $L_{\mu\nu}$ by constuction (since $L_{\mu\nu}$ involves space-time and $\Sigma_{\mu\nu}$ involves fermionic operators).
Guess:

$$\Sigma^{\mu\nu} = -i \sum_r \psi_r^\mu \psi_{-r}^\nu = -\frac{i}{2} \sum_r [\psi_r^\mu, \psi_{-r}^\nu].$$

Then the algebra is

$$\begin{aligned}[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] &= -\frac{1}{4}\left(\sum_r [\psi_r^\mu, \psi_{-r}^\nu], \sum_s [\psi_s^\rho, \psi_{-s}^\sigma]\right) \\ &= -\left(\sum_r \psi_r^\mu \psi_{-r}^\nu, \sum_s \psi_s^\rho, \psi_{-s}^\sigma\right) \\ &= i(\eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\sigma}\Sigma^{\nu\rho})\end{aligned}$$

where we used $\{\psi_r^\mu, \psi_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}$.
$\Sigma^{\mu\nu}$ generates Lorentz transformations on the fermionic fields $\psi^\mu(z)$. Notice that in $D=10$, there are five operators that commute with each other: Σ^{01}, Σ^{23}, Σ^{45}, Σ^{67}, Σ^{89} (trivial - they contain different ψ_r^μ modes). They can be simultaneously diagonalized. How do they act? Let us be specific and consider Σ^{23}. It acts on ψ_r^2, ψ_r^3 as follows:

$$\begin{aligned}\left[\Sigma^{23}, \psi_r^2\right] &= -i\sum_s [\psi_s^2 \psi_{-s}^3, \psi_r^2] = -\sum_s \{\psi_s^2, \psi_r^2\}\psi_{-s}^3 = i\psi_r^3 \\ \left[\Sigma^{23}, \psi_r^3\right] &= -[\Sigma^{32}, \psi_r^3] = -i\psi_r^2\end{aligned}$$

Eigenstates: $\psi_r^2 + i\psi_r^3$, $\psi_r^2 - i\psi_r^3$.

$$\begin{aligned}\left[\Sigma^{23}, \psi_r^2 + i\psi_r^3\right] &= \psi_r^2 + i\psi_r^3 \quad \text{eigenvalue} : +1 \\ \left[\Sigma^{23}, \psi_r^3 - i\psi_r^3\right] &= -\psi_r^2 + i\psi_r^3 \quad \text{eigenvalue} : -1\end{aligned}$$

Consider a finite transformation (rotation) $U(\theta) = e^{i\theta\Sigma^{23}}$. Then $U(\theta)(\psi_r^2 + i\psi_r^3)U^\dagger(\theta) = e^{i\theta}(\psi_r^2 + i\psi_r^3)$.
Proof:

$$\Sigma^{23}(\psi_r^2 + i\psi_r^3) = (\psi_r^2 + i\psi_r^3)(1+\Sigma^{23}) \Rightarrow (\Sigma^{23})^n(\psi_r^2 + i\psi_r^3 = (\psi_r^2 + i\psi_r^3)(1+\Sigma^{23})^n$$

$$\Rightarrow U(\theta)(\psi_r^2 + i\psi_r^3)U^\dagger(\theta) = (\psi_r^2 + i\psi_r^3)e^{i\theta(1+\Sigma^{23})} = e^{i\theta}(\psi_r^2 + i\psi_r^3)U(\theta).$$

Similarly, $U(\theta)(\psi_r^2 + i\psi_r^3)U^\dagger(\theta) = e^{-i\theta}(\psi_r^2 + i\psi_r^3)$. In particular, for $\theta = \pi$, the action of $U(\pi)$ on **both** $\psi_r^2 \pm i\psi_r^3$ is the same. Therefore $U(\pi)\psi_r^{2,3}U^\dagger(\pi) = e^{i\pi}\psi_r^{2,3} = -\psi_r^{2,3}$, i.e., $U(\pi)$ and $\psi_r^{2,3}$ anti-commute!
On the other hand $U(\pi)$ commutes with all other ψ_r^μ, $\mu \neq 2,3$. Thus $U(\pi)$ only has two eigenvalues, ± 1, like parity! If a state has an even number of $\psi_{-r}^2, 3$'s $(r > 0)$, then it belongs to eigenvalue $+1$ - with an odd number of $\psi_{-r}^{2,3}$'s, it has $U(\pi) = -1$. E.g.:

$$\psi_{-r}^2|0\rangle \ : \ U(\pi)\psi_{-r}^2|0\rangle = -\psi_{-r}^2 U(\pi)|0\rangle = -\psi_{-r}^2|0\rangle : \ (-1)$$

$$U(\pi)\psi_{-r_1}^2\psi_{-r_2}^3|0\rangle = -\psi_{-r_1}^2 U(\pi)\psi_{-r_2}^3|0\rangle = \psi_{-r_1}^2\psi_{-r_3}^3|0\rangle \ (+1)$$

etc.
We can do the same with all other Σ's. Thus we have

$$U_1(\pi) = e^{\pi\Sigma^{12}}, \ U_2(\pi) = e^{i\pi\Sigma^{23}}, \ U_3(\pi) = e^{i\pi\Sigma^{45}}, \ U_4(\pi) = e^{i\pi\Sigma^{67}}, \ U_5(\pi) = e^{i\pi\Sigma^{89}}.$$

Notice that $U_1(\pi)$ has no i in the exponential. This is because $\{\psi_r^0, \psi_s^0\} = -\delta_{r+s,0}$. The product

$$U_1(\pi)U_2(\pi)...U_5(\pi) = e^{i\pi(-i\Sigma^{01}+\Sigma^{23}+\Sigma^{45}+\Sigma^{67}+\Sigma^{89})} = e^{i\pi F}.$$

This anti-commutes with **all** ψ_r^μ. F is a fermion number operator. e^{iF} will play the role of parity in the harmonic oscillator case. Correction: $e^{i\pi F}V_{\text{gh}}$ will. V_{gh} is the ghost contribution. Since there are no ghost oscillators, all it does is act on the vacuum: $V_{\text{gh}}|0\rangle = -|0\rangle$. Thus restrict Hilbert space to eigenstates of $e^{i\pi F}V_{\text{gh}}$ of eigenvalue $+1$ (invariant states). This gets rid of the tachyon, for $e^{i\pi F}V_{\text{gh}}|0;k\rangle = -|0;k\rangle$ but keeps all massless states $\psi_{-1/2}^\mu|0;k\rangle$.

Consistent truncation

Since $e^{i\pi F}$ is made of Lorentz generators it is guaranteed to be conserved by the OPEs of vertex operators. So even states will produce even states when they interact with other even states.
Thus, we now have a consistent string theory without a tachyon! Or do we? We still need to check modular invariance. The X^μ part of the partition function is modular invariant by itself,

$$Z_X(\tau) = \left(\frac{1}{2\pi\sqrt{\alpha'\tau_2}}|\eta(q)|^{-2}\right)^D, \quad \eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad q = e^{2\pi i\tau}.$$

the fermionic part of the partition function is similarly calculated. The result is a Jacobi-theta function. But, alas, it is **not** modular invariant. This can be seen without doing any calculation as follows.
Before we demanded $\psi^\mu(\sigma + 2\pi) = -\psi^\mu(\sigma)$ (anti-periodic boundary conditions). On a torus, we demand $\psi^\mu(z + 2\pi) = -\psi^\mu(z)$ and also $\psi^\mu(z + 2\pi\tau) = -\psi^\mu(z)$. But then,

$$\psi^\mu(z + 2\pi(\tau + 1)) = -\psi^\mu(z + 2\pi\tau) = +\psi^\mu(z).$$

Therefore the transformation $\tau \to \tau + 1$ changes the boundary conditions to periodic! Therefore $\tau \to \tau + 1$ is not a symmetry of the theory. Our theory is **not** modular invariant. The above argument also shows how to fix the theory. We need to include (somehow) the sector in which ψ^μ obeys periodic boundary conditions. That is the Ramond sector and we study it next.

7.5 The Ramond (R) sector

The R-sector can only exist in two-dimensions, because there is no spin-statistics theorem there. The mode expansion is

$$\psi^\mu(z) = \sum_{n\in\mathbb{Z}} \psi^\mu_n z^{-n-\frac{1}{2}}$$

indices are integers, since $\psi^\mu(z)$ is periodic. The expansion has a factor of $z^{-1/2}$, because the weight of ψ^μ is $h = 1/2$. Therefore this is **not** a Laurent expansion and has a branch cut. We still have

$$\{\psi^\mu_m, \psi^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}$$

as in the NS-sector. We also have the same algebra for L_m, G_r (note it is now $G_m,\ m \in \mathbb{Z}$).

Normal ordering

We only have a problem with L_0. Using $[L_1, L_{-1}] = 2L_0$, we have

$$2\langle 0|L_0|0\rangle = \langle 0|L_{+1}L_{-1}|0\rangle,$$

$$L_{-1}|0\rangle = \left(\frac{1}{2}\sum_n \alpha^\mu_{-1-n}\alpha_{n\mu} + \frac{1}{4}\sum_n (2n+1)\psi^\mu_{-1-n}\psi_{n\mu}\right)|0\rangle$$

For the α's we need $-1-n,\ n<0$, so $-1<n<0$, which is impossible. For the ψ's, we need $-1-n,\ n\le 0$, so $-1\le n\le 0$, so $n=0$, or $n=-1$. Therefore

$$L_{-1}|0\rangle = \frac{1}{4}\left(-\psi^\mu_0\psi_{-1\mu} + \psi^\mu_{-1}\psi_{0\mu}\right)|0\rangle = \frac{1}{2}\psi^\mu_{-1}\psi_{0\mu}|0\rangle.$$

Therefore

$$\begin{aligned}\langle 0|L_1L_{-1}|0\rangle &= \frac{1}{4}\langle 0|\psi_{0\nu}\psi^\nu_1\psi^\mu_{-1}\psi_{0\mu}|0\rangle, \\ &= \frac{1}{4}\langle 0|\psi^\mu_0\psi_{0\mu}|0\rangle \\ &= \frac{1}{8}\langle 0|\{\psi^\mu_0, \psi_{0\mu}\}|0\rangle \\ &= \frac{D}{8}\end{aligned}$$

Therefore $\langle 0|L_0|0\rangle = D/16 = a$ (i.e., $L_0 =: L_0 : -D/16$).

The ghosts

$$b = \sum_{m \in \mathbb{Z}} b_m z^{-m-2}, \; c = \sum_{m \in \mathbb{Z}} c_m z^{-m+1}, \;\; \beta = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \beta_r z^{-r-\frac{3}{2}}, \; \gamma = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \gamma_r z^{-r+\frac{1}{2}}.$$

where $\beta, \; \gamma$ are **not** Laurent expansions. The algebras are

$$\{b_m, c_n\} = \delta_{m+n,0}, \qquad [\gamma_m, \beta_n] = \delta_{m+n,0},$$

which are the same as before, but in addition, the zero modes: $[\gamma_0, \beta_0] = 1$, i.e., $\gamma_0, \; \beta_0$ are creation and annihilation operators respectively. This define $|0\rangle$ by $b_m|0\rangle = 0, \;\; m > 0, \;\; \beta_m|0\rangle = 0, \;\; m \geq 0$ and $c_m|0\rangle == g_m|0\rangle = 0$ for $m > 0$.

Normal ordering

L_0 again has a problem. We can solve as we did before.

$$\begin{aligned} L_{-1}|0\rangle &= \left(\sum_n (n-1) b_{-1n-1} c_n + \frac{1}{2} \sum_n (2n-1) \beta_{-n-1} \gamma_n \right) |0\rangle \\ &= \left(-b_{-1} c_0 - \frac{1}{2} \beta_{-1} \gamma_0 \right) |0\rangle \end{aligned}$$

There is only one possibility since $-1 < n \leq 0$, so

$$\begin{aligned} \langle 0|L_1 L_{-1}|0\rangle &= -\langle 0|b_0 c_1 b_{-1} c_0|0\rangle - \frac{1}{4} \langle 0|\beta_0 \gamma_1 \beta_{-1} \gamma_0|0\rangle \\ &= -1 - \frac{1}{4} = -\frac{5}{4} \end{aligned}$$

and

$$\langle 0|L_0|0\rangle = \frac{1}{2} \langle 0|L_1 L_{-1}|0\rangle = -\frac{5}{8} = a.$$

The spectrum

First observe that the defintion $|0\rangle$ is ambiguous. Indeed $|0\rangle$ is defined by $\psi^\mu_m|0\rangle, \; m > 0$. But then $\psi^\mu_0|0\rangle$ is as good as $|0\rangle$, for $\psi^\nu_m \psi^\nu_0|0\rangle = -\psi^\nu_0 \psi^\mu_m|0\rangle = 0, \; m > 0$. the ground state is then a representation of the algebra of the zero modes, $\{\psi^\mu_0, \psi^\nu_0\} = \eta^{\mu\nu}$ (Clifford - Dirac algebra). $|0\rangle$ therefore is a spinor. Instead of one spin, here we have five, because we are in ten-dimensions. The spin operators are $\Sigma^{01}, \Sigma^{23}, \Sigma^{45}, \Sigma^{67}, \Sigma^{89}$. They commute with each other so they can be simultaneously diagonalized. We can then define a basis of ground states $|s_1, s_2, s_3, s_4, s_5\rangle$ where $s_i = \pm 1/2 (i = 1, 2, 3, 4, 5)$. We will use the notation $\vec{s} = (s_1, s_2, s_3, s_4, s_5)$. There are $2^5 = 32$ such states (c.f. $2^2 = 4$ states in the four-dimensional Dirac spinor). All states built from $|\vec{s}\rangle$ have

integer $+1/2$ spin, because ψ^μ_{-m} has spin one (eigenstate of S^{XXXX+1} with eigenvalue $+1$). to be contrasted with NS-sector where all states have integer spin. Thus, the inclusion of the R-sector is important, because we need all spins to describe Nature.
The Hamiltonian (L_0) has const. $D/16 - 5/8 = 10/16 - 5/8 = 0$, so $H = \alpha' p^2 + N$ (c.f. $H = \alpha' p^2 + N - 1/2$ in the NS-sector)
The lowest level: $N = 0$, so $H = 0$ and $m^2 = -p^2 = 0$. There is no tachyon! The lowest states, $|0;k\rangle$ are massless! Non-trivial constraint: $G_0|\vec{s};k\rangle = 0$. Relevant piece: $G_0 = \sqrt{2\alpha'}p_\mu\psi^\mu_0$, so $k_\mu\psi^\mu_0|\vec{s};k\rangle = 0$ which is the Dirac equation ($\gamma^\mu = \frac{1}{\sqrt{2}}\psi^\mu_0$, then $k_\mu\psi^\mu|\vec{s};k\rangle = 0$). Notice also that the algebra $\{G_0, G_0\} = 2L_0$, i.e., $G_0^2 = L_0$. G_0 is the square root of the Hamiltonian!
This is just like in the Dirac case. It is also a generic feature of a SUSY theory: the Hamiltonian can be written as the square of a SUSY charge.
Notice that this also implies that the ground state has zero eigenvalue, because $G_0|0\rangle = 0$, which makes it very hard to have a finite cosmological constant in a SUSY theory. In terms of the fields, the contribution of the boson always exactly cancels the contribution of the fermions (due to SUSY boson $\leftrightarrow$ fermion) and we get zero vacuum expectation energy (cosmological constant). The R-sector can also be split into two eigenspaces of $e^{i\pi F}$ with eigenvalues ± 1. The ground states belongs to $+1$.

7.6 Superstring Theories

We may now combine the NS and R-sectors to form a consistent superstring theory. We need to have analycity in the OPEs (which is not guaranteed in the R-sector, due to branch cuts in the expansions of the fields). This severely constrains the possibilities (we also do not want a tachyon) to ...

$$\begin{array}{lllll} IIA: & (NS+, NS+) & (R+, NS+) & (NS+, R-) & (R+, R-) \\ IIB: & (NS+, NS+) & (R+, NS+) & (NS+, R+) & (R+, R+) \\ IIA': & (NS+, NS+) & (R-, NS+) & (NS+, R+) & (R-, R+) \\ IIB': & (NS+, NS+) & (R-, NS+) & (NS+, R-) & (R-, R-) \end{array}$$

It can be shown that IIA' is the same as IIA (also, similarly, IIB' is the same as IIB)
Proof: Transform $X^9 \to -X^9$, $\psi^9 \to -\psi^9$, $\tilde{\psi}^9 \to -\tilde{\psi}^9$. Then $e^{i\pi S^{89}} \to e^{-i\pi S^{89}}$ (same eigenvalue), but $S^{89}|0\rangle \to -S^{89}|0\rangle$, so the sign is reversed in the R-sector (S^{89} annihilates the NS vacuum (no zero modes), so no change there). Therefore this transformation maps $R+ \to R-$ and vice versa. *QED*

Open Strings

Only one possibilty: type I: NS+,R+. The projection of eigenspaces of $e^{i\pi F}$ and $e^{i\pi\tilde{F}}$ is known as the Gliozzi-Scherk-Olive (GSO) projection.

The resulting theories turn out to have *space-time* SUSY and obey the spin-statisics theorem (which has to be obeyed for $D > 2$). The fact that space-time SUSY and the spin-statistics theorem emerge is rather unexpected. One would expect that these two should be evident from the start - built in formalism. This fact remains elusive.

Modular Invariance

We have already seen that modular invariance for the NS-NS sector alone cannot possibly work. Now we have a multitude of sectors and a hope that modular transformations will map one onto others and somehow the combination will be invariant. Let us start with the NS-sector. Only the NS+ subsector appears. The partition function for the X^μ's is the same as before and we have already established it is modular invariance, so we will concentrate on the ψ^μ's.
The partition function is as always

$$Z_{NS+} = \text{Tr }(q^H), \quad q = e^{2\pi i\tau}.$$

If $|\psi\rangle$ is in NS+, then $e^{i\pi F}|\psi\rangle = |\psi\rangle$. To find such a $|\psi\rangle$, we can start with an arbitrary state $|\psi'\rangle$ and project onto the eigenspace of $e^{\pi i F}$ of eigenvalue $+1$. The projection operator is

$$P = \frac{1}{2}(1 + e^{i\pi F}), \quad P^2 = P.$$

Also, $e^{i\pi F}P|\psi'\rangle = P|\psi'\rangle$, so eigenvalue $+1$. Thus, to compute the $\text{Tr }_{NS}(PA) = \frac{1}{2}\text{Tr }_{NS}A + \frac{1}{2}\text{Tr }_{NS}(e^{i\pi F}A)$. First trace: for each μ, we have the creation operators $\psi^\mu_{-r},\ r > 0$ where of course $r \in \mathbb{Z} + \frac{1}{2}$. A state can have 0 or 1 ψ^μ_{-r}, since $(\psi^\mu_{-r})^2 = 0$ (fermionic mode). So for fixed r, μ we get a factor $q^0 + q^r = 1 + q^r$ (since $N = 0, r$) the rest of H has already been considered in the X^μ part).
Varying r, we get a product

$$\prod_{r>0}(1 + q^r) = \prod_{m=1}^{\infty}(1 + q^{m-1/2}).$$

Varying μ, we get eight copies of this product (because only the transverse μ's contribute and there are $10 - 2 = 8$ of them). Thus

$$\text{Tr }_{NS}q^H = \left(q^{-1/48}\prod_{m=1}^{\infty}(1 - q^{m-\frac{1}{2}})\right)^8.$$

NB the factor of $q^{-1/48}$ which comes from the new tensor transformation of T (stress-energy "tensor") as we go from z to $\sigma + \tau$ ($z = e^{i(\sigma+\tau)}$) c.f. in the bosonic case we got $q^{-1/24}$, double becuase for a boson $c = 1$ whereas for a fermion $c = 1/2$. We can write this partiion function in terms of the Jacobi ϑ-function. Recall ... $(z = e^{2\pi i\nu},\ \ q = e^{2\pi i\tau})$

$$\begin{aligned}
\vartheta_{00}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2}) \\
\vartheta_{01}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-1/2})(1-z^{-1}q^{m-1/2}) \\
\vartheta_{10}(\nu,\tau) &= 2e^{\pi i\tau/4}\cos\pi\nu\prod_{m=1}(1-q^m)(1+zq^m)(1+z^{-1}q^m) \\
\vartheta_{11}(\nu,\tau) &= -2e^{\pi i\tau/4}\sin\pi\nu\prod_{m=1}(1-q^m)(1-zq^m)(1-z^{-1}q^m)
\end{aligned}$$

For $\nu = 0,\ z = 1$, so

$$\begin{aligned}
\vartheta_{00}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1+q^{m-1/2})(1+q^{m-1/2}) \\
\vartheta_{01}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1-q^{m-1/2})(1-q^{m-1/2}) \\
\vartheta_{10}(\nu,\tau) &= 2q^{1/8}\prod_{m=1}(1-q^m)(1+q^m)(1+q^m) \\
\vartheta_{11}(\nu,\tau) &= -2q^{1/8}\sin\pi 0\prod_{m=1}(1-q^m)(1-q^m)(1-q^m) = 0!
\end{aligned}$$

Also $\eta(\tau) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m)$. Thus,

$$\vartheta_{00}(0,\tau) = q^{-1/24}\eta(\tau)\left(\prod_{m=1}^{\infty}(1+q^{m-1/2})\right)^2.$$

So

$$\text{Tr}_{\ NS}q^H = \left(\frac{\vartheta_{00}(0,\tau)}{\eta(\tau)}\right).$$

Next, let us do $\text{Tr}_{\ NS}e^{\pi iF}q^H$. Let us fix μ and r again. If the state has zero ψ^μ_{-r}'s then $F = 0$ and $N = 0$. So we get $q^0 = 1$. If the state has one ψ^μ_{-r}, then $e^{i\pi F} = -1$ (recall $e^{i\pi F}\psi^\mu_{-r}e^{-\pi iF} = -\psi^\mu_{-r}$) and $N = r$, so we get $-q^r$.
So this case differs from the previous one by a mere sign change which implies that $\vartheta_{00} \to \vartheta_{01}$. Moreover, the ground state $|0\rangle$ has eigenvalue -1 $(e^{i\pi F}|0\rangle_{NS} = -|0\rangle_{NS})$, so we get an overall "-" sign. Thus

$$\text{Tr}_{\ NS}e^{i\pi F}q^H = -\left(\frac{\vartheta_{01}(0,\tau)}{\eta(\tau)}\right)^4.$$

Similarly, in the Ramond sector,

$$Z_{R+} = \text{Tr}_{\ R+}Pq^H = \frac{1}{2}\left(\text{Tr}_{\ R}q^H + \text{Tr}_{\ R}e^{\pi iF}q^H\right).$$

The creation modes are ψ^μ_{-m} , $m > 0$ and we need to take special care of the zero modes ψ^μ_0.
Fix μ and $m > 0$. Then, for zero ψ^μ_{-m}'s, we obtain $q^0 = 1$ and for one $\psi^m_{-m}u$, we obtain q^m, so overall, $1 + q^m$. The ground state has energy $H = 1/16$ (normal ordering constant we obtained earlier, so $1 + q^m \to q^{1/16}(1 + q^m)$. Varying $\mu,\ m$ we obtain the product

$$\text{Tr }_R q^H = \left(q^{\frac{1}{16}} q^{-\frac{1}{48}} \prod_{m=1}^{\infty} (1 + q^m) \right)^8 \times \text{“0”}$$

where “0” is the contribution of the zero-modes. Recall the ground state $|\vec{s}\rangle$. Consider the Σ_{23} spin, for example. There are two states $|\uparrow\rangle,\ |\downarrow\rangle$, with spin $\pm\frac{1}{2}$ respectively. Each contributes 1, so overall $1 + 1 = 2$. we have four independent such states (since we have eight transverse dimensions Σ^{01} does not produce independent physical states). Therefore the overall factor “0” $= 2^4$ c.f. with

$$\vartheta_{01}(0,\tau) = 2q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + q^m)^2 = 2q^{1/8} q^{-1/24} \eta(\tau) \left(\prod_{m=1}^{\infty} (1 + q^m) \right)^2$$

$$\text{Tr }_R q^H = \left(2q^{1/8} q^{-1/24} \left(\prod_{m=1}^{\infty} (1 + q^m) \right)^2 \right)^4 = - \left(\frac{\vartheta_{10}(0,\tau)}{\eta(\tau)} \right)^4$$

where the minus sign comes from space-time spin-statistics (ghosts). Finally, $\text{Tr }_R e^{i\pi F} q^H$ gives a similar product, but with two changes

- $1 + q^m \to 1 - q^m$ (- form $e^{i\pi F}$, as in NS-sector
- $|\uparrow\rangle$ and $|\downarrow\rangle$ have opposite eigenvalues, contributing $1 - 1 = 0$!

Therefore

$$\text{Tr }_R = q^H = - \left(\frac{\vartheta_{10}(0,\tau)}{\eta(\tau)} \right)^4 = 0.$$

Putting everything together, the partition function for ψ^μ in NS+ and R+ sector is

$$\begin{aligned} Z_\psi(\tau) &= \text{Tr }_{NS+} q^H + \text{Tr }_{R+} q^H \\ &= \frac{1}{2} \text{Tr }_{NS} q^H + \text{Tr }_{NS} \frac{1}{2} e^{\pi i F} q^H + \frac{1}{2} \text{Tr }_R q^H + \frac{1}{2} \text{Tr }_R e^{\pi i F} q^H \\ &= \frac{1}{2(\eta(\tau)^4} \left(\vartheta_{00}(0,\tau)^4 - \vartheta_{01}(0,\tau)^4 - \vartheta_{10}(0,\tau)^4 + \vartheta_{11}(0,\tau)^4 \right). \end{aligned}$$

This is complicated combination of Jacobi-theta functions, yet not only is it modular invariant, but it vanishes identically! This should not be too surprising, since we have space-time SUSY and so the cosmological constant should vanish. This fact was known to Jacobi himself, for he proved the "abstruse identity"

$$\vartheta_{00}(0,\tau)^4 - \vartheta_{01}(0,\tau)^4 - \vartheta_{10}(0,\tau)^4 = 0,$$

and we have already seen $\vartheta_{11}(0,\tau)^4 = 0$.
Of course the total Z is a product of Z_ψ and $Z_\rightarrow = Z_\psi^*$ in the case of IIB and

$$(Z_\psi')^* = \frac{1}{2\eta(\tau)^4}\left(\vartheta_{00}(0,\tau)^4 - \vartheta_{01}(0,\tau)^4 - \vartheta_{10}(0,\tau)^4 - \vartheta_{11}(0,\tau)^4\right)$$

which of course $Z_\psi' = Z_\psi$.

Modular invariance of type-I

Type-I is an open string theory. Instead of a torus, we have a cylinder.
The cylinder can easily be deduced from the torus. Recall for the torus

$$Z = \int_{F_0} \frac{d\tau d\bar\tau}{4\tau_2} Z(\tau), \qquad Z(\tau) = \mathrm{Tr}\; q^H,\;\; q = e^{2\pi i \tau}$$

where F_0 is the fundamental region and $\frac{d\tau d\bar\tau}{4\tau_2}$ is a modular invariant measure on the torus $(\tau_2(2\pi)^2$ is the volume of the torus = volume of the group of translations). The cylinder defines a more honest partition function, because $\tau \to t \in \mathbb{R}$ and $Z(t) = \mathrm{Tr}\; q^H,\;\; q = e^{-2\pi t}$, i.e., $\tau = i\tau_2,\; \tau_2 = t$, and

$$Z = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}\; e^{-2\pi t L_0}.$$

Notice that thee is no fundamental region, so we have potential divergences from both limits $t \to \infty$ and $t \to 0$. $t \to \infty$ is usually associated with the IR region (long-distance, low energy). $t \to 0$ is associated with UV divergences (short-distances - high energies). In closed strings, there is no $t \to 0$ limit, for it is cut by the restriction to the fundamental region F_0. In the open string case, it is there. But does open string theory have UV divergences? That would make it as bad as field (particle) theory. To answer this, concentrate on $X^\mu,\; \mu = 0, 1, .., D-1$. The partition function (easily deduced from the torus) is

$$Z(t) = \mathrm{Tr}\; e^{-2\pi t L_0} = iV\left(\sqrt{8\pi^2\alpha' t}\right)^{-D} (\eta(it))^{-(D-2)}$$

c.f. on torus:

$$Z(t) = \mathrm{Tr}\; e^{-2\pi i \tau L_0} e^{-2\pi i \bar\tau \tilde L_0} = iV\left(\sqrt{4\pi^2\alpha'\tau_2}\right)^{-D} |\eta(it)|^{-2(D-2)}$$

where

$$\eta(it) = e^{-\pi t/12}\prod_{m=1}^\infty (1 - e^{-2\pi m t}).$$

Let $D = 26$. In the $t \to \infty$ limit, we may expand

$$\begin{aligned}(\eta(it))^{-24} &= e^{2\pi t}\prod_{m=1}^{\infty}(1-e^{-2\pi m t})^{-24} = e^{2\pi t}(1+24e^{-2\pi t}+...)\\ &= e^{2\pi t}+24+...\end{aligned}$$

Each term in the expansion comes from a certain mass level. The first term is from the tachyon, and diverges, because $m^2 < 0$. The second term is from the massless modes (24 transverse photons). Again, it diverges, but only logarithmically. This is expected and is similar to field theory. These divergences cancel in physical quantities.
Now look at $t \to 0$. This appears to be a high energy effect, but it is not! The cylinder becomes very thin and it looks like a closed string is being created, propogating and disappearing again (NB: t does **not** represent a *physical* distance). So $t \to 0$ is *still* an IR effect (long-distance). To show this, use the modular property, $\eta(-1/\tau) = \sqrt{i\tau}\eta(\tau)$. For $\tau = it$, we get $\eta(i/t) = \sqrt{t}\eta(it)$, so

$$\eta(it) = \frac{1}{\sqrt{t}}\eta\left(\frac{i}{t}\right).$$

Change variables to $s = \frac{\pi}{t}$. Then, apart from constants

$$Z \sim \int_0^\infty \frac{dt}{t}t^{-13}\eta(it)^{-24} = \int_0^\infty \frac{dt}{t^2}\eta\left(\frac{i}{t}\right)^{-24} \sim \int_0^\infty ds\eta\left(\frac{is}{\pi}\right)^{-24}.$$

$t \to 0$ is obtained by expanding in large s,

$$\eta\left(\frac{is}{\pi}\right)^{-24} = e^{2s}+24+...$$

(same expansion as before). The first term is from the tachyon (pathological). The second term is from the massless modes. The propagator for them is $1/k^2$ and since $k^2 = -m^2 = 0$, we have $1/0 = \infty$. The pole is due to the propagator for a long time (on-shell).
Let us return to type-I. In this case $d = 10$, so for the X^μ's, we have

$$Z_X(t) = iV(8\pi^2\alpha' t)^{-5}\eta(it)^{-8}.$$

Moreover, there is a subtlety: define the world-sheet parity Ω by

$$\Omega: \;\; X^\mu(\sigma) \to X^\mu(\pi-\sigma).$$

In terms of modes, $\Omega\alpha_n^m u\Omega^{-1} = (-1)^n\alpha_n^\mu$ (recall $X^\mu(\sigma) \sim \sum\alpha_n^\mu e^{-in\sigma}$). Obviously, $\Omega^2 = 1$, sp Ω has two eigenvalues, ± 1. We need to restrict to the $+1$ eigenspace for consistency of the theory. This is easily implemented: we need to keep the states with and **even** number of α_{-n}^μ modes. [NB: In bosonic theory, this would give garbage, for it would exclude the photon! Here, the photon is $\psi_{-1/2}^\mu|0;k\rangle$, so it has 0 (even!) α_{-n}^μ's.

The various partition functions are not affected by the presence of Ω, but we get an extra factor of $1/2$ from the projection $\frac{1}{2}(1+\Omega)$. Thus

$$Z = \int_0^\infty \frac{dt}{2t}\frac{1}{2}\frac{1}{2}Z_X(t)Z_\psi(t)$$

where

$$Z_\psi(t) = \vartheta_{00}(0,it)^4 - \vartheta_{01}(0,it)^4 - \vartheta_{10}(0,it)^4 - \vartheta_{11}(0,it)^4$$

and the two factors of $1/2$ come from Ω and the GSO projections respectively. To study the divergences (even though $Z_\psi = 0$! - we still need to study the, otherwise $Z_\psi = 0$ is a $\infty - \infty = 0$ statement; also these divergences appear (and did not cancel) in other amplitudes) Define $s = \pi/t$. Then

$$Z_X(t) = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds\, \eta\left(\frac{is}{\pi}\right)^{-8} Z_\psi\left(\frac{\pi}{s}\right).$$

Modular properties

$$\eta(it) = \frac{1}{\sqrt{t}}\eta(i/t), \qquad \vartheta_{00}(0,it) = \frac{1}{\sqrt{t}}\vartheta_{00}(0,i/t).$$

Separate NS and R. Then in the NS-sector

$$Z_{NS}(t) = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds\, \eta\left(\frac{is}{\pi}\right)^{-12}\left(\vartheta_{00}(0,is/\pi)^4 - \vartheta_{10}(0,is/\pi)^4\right).$$

To leading order, $\eta\left(\frac{is}{\pi}\right)^{-12} = q^{-1/2} = e^s$ and

$$\vartheta_{10}(0,is/\pi)^4 = 2^4 q^{1/2} = 2^4 e^s, \qquad \vartheta_{00}(0,is/\pi)^4 = 1 + ...$$

So

$$Z_{NS} = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16 + o(e^{-2s})).$$

Notice that the tachyon has disappeared, but of course, we still have the divergence from the sixteen massless modes, as expected. What can we do? Well, the cylinder is not the only possibility. We also have the Möbius strip and the Klein bottle.

The Möbius Strip

Same as the cylinder, but we twist before we identify the ends. In other words, $X^\mu(\omega, 2\pi t) = X^\mu(\pi-\sigma, 0) = \Omega X^\mu(\sigma,0)\Omega^{-1}$. The partition function is very similar to the cylinder. The only difference is the insertion of the parity operator, Ω. Thus $Z_{\text{Mobius}} = \text{Tr}\,(q^{L_0}\Omega)$. The action of Ω is simple. If a state has an even (odd) number of α^μ_{-n}'s, $\Omega = +1(-1)$. Thus, $(1-q^m)^{-1}$ is replaced by $(1-(-)^m q^m)^{-1}$, $q = e^{=2\pi t}$ and so

$$\eta(it) = e^{-\pi t/12}\prod_{m=1}^\infty(1-e^{-2\pi m t}) \Rightarrow e^{-\pi t/12}\prod_{m=1}^\infty(1-(-)^m e^{-2\pi m t})$$

which can be written in terms of

$$\vartheta_{00}(0,\tau) = \prod_{m=1}^{\infty}(1-q^m)(1-q^{m-1/2})^2.$$

As follows: let $\tau = 2it$,

$$\begin{aligned}
\vartheta_{00}(0,2it) &= \prod_{m=1}^{\infty}(1-q^m)(\prod(1+e^{-2\pi t(2m-1)}))^2 \\
&= \prod_{m=1}^{\infty}(1-q^m)^{-1}\left[\prod(1+e^{-2\pi t(2m)})(1+e^{-2\pi t(2m-1)})\right]^2 \\
&= e^{-\pi t/16}\frac{1}{\eta(2it)}\left[\prod(1-(-)^m e^{-2\pi t m})\right]^2
\end{aligned}$$

so

$$e^{-\pi t/12}\prod(1-(-)^m e^{-2\pi t m}) = \sqrt{\vartheta_{00}(0,2it)\eta(2it)}$$

replaces $\eta(it)$. zero modes are still the same , so... Recall the cylinder

$$Z_X = iV\left(\frac{1}{\sqrt{8\pi^2\alpha' t}}\right)^D \eta(it)^{-(D-2)}$$

The partition function for the Möbius strip is

$$Z_X = iV\left(\frac{1}{\sqrt{8\pi^2\alpha' t}}\right)^D (\vartheta_{00}(0,2it)\eta(it))^{-(D-2)/2}.$$

Next, do the ψ's. Easier to work in the R-sector (only one contribution)

$$Z_{\psi,R} = \mathrm{Tr}\ \Omega q^{N_\psi} = -2^4\left[q^{1/16}q^{-1/48}\prod_{m=1}^{\infty}(1+(-)^m q^m)\right]^8$$

which can be written in terms of Jacobi-theta functions as follows:

$$\begin{aligned}
\vartheta_{01}(0,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1-q^{m-1/2}) \\
\vartheta_{10}(0,\tau) &= 2q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+q^m)^2
\end{aligned}$$

so

$$\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau) = 2q^{1/8}\prod_{m=1}^{\infty}(1-q^m)\prod_m(1+(-)(\sqrt{q})^m)^2,$$

so let $q = e^{-4\pi t}$.

$$\frac{\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)}{\eta(0,2it)^2} = 2e^{\pi t/3}e^{-\pi t/2}\prod_m(1+(-)(\sqrt{q})^m)^2$$

so

$$Z_{\psi,R} = -\left(\frac{\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)}{\eta(0,2it)^2}\right)^4.$$

At $D = 10$

$$Z_R = \int_0 \int \frac{dt}{2t}\frac{1}{2}\frac{1}{2}Z_X Z_{\psi,R} = iV\int_0 \int \frac{dt}{8t}(8\pi^2\alpha' t)^{-5}\left(\frac{\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)}{\eta(0,2it)^2}\right)^4.$$

To study the $t \to 0$ limit, switch the variable to $s = \pi/t$. Then

$$Z_R = iV\frac{8}{(8\pi^2\alpha')^5}\int_0^\infty ds\left(\frac{\vartheta_{01}(0,2is/\pi)\vartheta_{10}(0,2is/\pi)}{\eta^3(2is/\pi)\vartheta_{00}(2is/\pi)}\right)^4$$

for small s, we have

$$\begin{aligned}
\vartheta_{01}(0,2is/\pi) &\simeq 1 + ..., \\
\vartheta_{01}(0,2is/\pi) &\simeq 2q^{1/8} + ... \\
\vartheta_{00}(0,2is/\pi) &\simeq 1 + ..., \\
\eta(2is/\pi) &\simeq q^{1/24} + ...
\end{aligned}$$

so

$$\frac{\vartheta_{01}\vartheta_{10}}{\eta^3\vartheta_{00}} = \frac{2q^{1/8}}{q^{1/8}} = 2 + ... \Rightarrow \left(\frac{\vartheta_{01}\vartheta_{10}}{\eta^3\vartheta_{00}}\right)^4 = 2^4 + ... = 16 + ...$$

no tachyon, and sixteen massless modes contributing, as expected. c.f. for the cylinder,

$$Z_R = -Z_{NS} = i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16 + ...).$$

of opposite sign, but they do not cancel!

The Klein Bottle

Even though a bottle looks more appropriate for closed strings, and amplitudes are defined in terms of closed string modes, the Klein bottle contributes to open strings.
DEFINITION: Consider a torus with $\tau = it$. we identify the sides $\sigma = 0$ and $\sigma = 2\pi$ and obtain a cylinder, but just like with the Möbius strip, we identify the sides $\tau = 0$ and $\tau = 2\pi\tau$ by twisting them first

$$X^\mu(\sigma, 0) = X^\mu(-\sigma, 2\pi t) = \Omega X^\mu(\sigma, 2\pi t)\Omega^{-1}$$

The partition function is given by

$$Z_X = \text{Tr } \Omega e^{-2\pi t L_0} e^{-2\pi t \tilde{L}_0}$$

In this case, $\Omega\alpha_n^\mu\Omega^{-1} = -\tilde{\alpha}_n^\mu$ (unlike for open strings, where $\alpha_n^\mu \to -\alpha_n^\mu$) Therefore, the diagonal elements if Ω have exactly the same α_n^μ's as $\tilde{\alpha}_n^\mu$'s.
So the trace is effectively over the α_n^μ's only, which explains why this is an open string amplitude.
For the diagonal elements of Ω we have $\Omega = +1$ (even total # of $\alpha_n^\mu, \tilde{\alpha}_n^\mu$.) and Ł$_0 = \tilde{L}_0$, so

$$Z_X = \text{ Tr } e^{-4\pi t L_o}\Omega\big|_{\Omega=1}$$

which is the same as open string partition function, but with q^2 instead of q (or $2t$ instead of t)

$$Z_X = iV(4\pi\alpha' t)^{-D/2}(\eta(2it))^{-(D-2)}$$

Note the first factor has a 4 rather than an 8 due to the closed string.
The partition function for the ψ^μ's is obtained similarly. The result is the same as the open string (cylinder) again, but with $t \to 2t$.

$$Z_\psi^{NS} = \frac{1}{(\eta(2it))^4}\left[\vartheta_{00}^4(2it) - \vartheta_{10}^4(2it)\right]$$

and $Z_\psi^{NS} = -Z_\psi^R$. Overall

$$Z_{NS} = \int_0^\infty \frac{dt}{2t}\frac{1}{2}\frac{1}{2}Z_X Z_\psi^{NS} = iV\int_0^\infty \frac{dt}{8t}(4\pi^2\alpha' t)^{-}s(\eta(2it))^{-12}\left[\vartheta_{00}^4(2it) - \vartheta_{10}^4(2it)\right]$$

and $Z_R = -Z_{NS}$. The study of the $t \to -$ limit can be copied from the cylinder with an extra 2^10 factor

$$Z_{NS} = i\frac{2^10V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16 + ...)$$

Again there is no tachyon, but alas, $Z_{cylinder} + Z_{mobius} + Z_{klein}$ still has a non-vanishing divergence. What do we do? We need to introduce **Chan-Paton** factors!
Chan-Paton factors were first introduced in QCD, where the string was made of glue. They attached quarks at the ends of the string which carried indices labeling *color*.
In the present setting, we will introduce them because we can. They do not spoil Lorentz invariance, because they live at the ends of the string. They are useful because they give us extra degrees of freedom, which are needed to describe gauge interactions.
e.g. E&M: Kaluza-Klein added an extra index throughout the string (didn't know about strings, but that is what they effectively did.) $(X^0, ..., X^3, X^4)$: X^4 was the extra-dimension. This spoiled Lorentz invariance, but that was ok, because we only care about Lorentz invariance in four dimensions. The extra dimension gave us a gauge group (U(1)) corresponding to a photon. More dimensions give us more complicated gauge groups and extra degrees of freedom.

With Chan-Paton factors, the gauge group does not come from extra dimensions, but from extra degrees of freedom at the ends of the string (open of course). Yet another innovation of string theory!
So all states now carry two more indices $|0\rangle \to |0, ij\rangle$, so, e.g. we now have n^2 tachyons or photons, if $i, j = 1, 2, ..., n$. Thus, the photon can be the weak boson multiplet $(W^\pm, Z^0)$, or the gluon.
How does this effect the partition function? For the cylinder, all n^2 states contribute equally, so Z is multiplied by n^2. For the Möbius strip ,because of the twist, i needs to be identified with j, and there are n possibilities, $Z_{\text{Mobius}} \to nZ_{\text{Mobius}}$.
For the Klein bottle, we have no indices, because we have closed strings, so $Z_{\text{Klein}} \to Z_{\text{Klein}}$.
Overall, the partition function is now

$$Z = n^2 Z_{\text{cylinder}} + nZ_{\text{Mobius}} + Z_{\text{Klein}}.$$

Recall for the R-sector

$$\begin{aligned} Z_{\text{cylinder}} &= -i\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...) \\ Z_{\text{Mobius}} &= i\frac{2^6 V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...) \\ Z_{\text{Klein}} &= -i\frac{2^{10} V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...) \end{aligned}$$

$$Z = -i(n-2^5)^2\frac{V}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds(16+...)$$

We obtain a **finite** answer if and only if $n = 2^5 = 32$. This implies that out of all possible gauge groups, type I string theory makes a unique choice: SO(32). This was a crucial discovery that led to the explosion of interest in string theory.

UNIT 8

Heterotic Strings

8.1 Introduction

The Heterotic string was introduced in 1985 by the "string quartet" (Gross Harvey, Martinec and Rohm).
Basic idea: left and right movers need <u>not</u> be in the same theory. not even in the same dimension!
Thus, take the holomorphic part as bosonic $(\partial X^\mu(z), \mu = 0, 1, ..., 25)$ and the anti-holomorphic part as a superstring $(\bar{\partial} X^\mu(\bar{z}), \tilde{\psi}^\mu(\bar{z}), \mu = 0, 1, ..., 9)$.
Then the holomorphic piece has $c = 26$, so we need to throw in the bc ghosts with $c = -26$ in order to have a vanishing central charge. The anti-holomorphic piece has a central charge $c = 15$, so we need couple it to the superconformal ghosts (b,c and γ, β) with central charge, $\tilde{c} = -15$.
It is convenient to split ∂X^μ into $\partial X^\mu, \mu = 0, 1, ..., 9$, which will then combine with the anti-holomorphic piece to $X^\mu(z, \bar{z})$ and the rest $\partial X^\mu, \mu = 10, 11, ..., 25$ have $c = 16$ and have no anti-holomorphic partners. We may replace them with $\psi^A(z), A = 1, 2, ..., 32$ which have the same central charge (c=32/2=16).
The two theories are the same even though it may not be obvious.
Now we can write the action

$$S = \int d^2z \left(\frac{1}{2\pi\alpha'} \partial X^\mu \bar{\partial} X_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu + \partial^A \bar{\partial} \psi_A \right), \; \mu = 0, 1, ..., 9, \; A = 1, 2, ..., 32.$$

so μ is a space-time index and A is an internal index which represents the gauge degrees of freedom. Think of ψ^A as a 32 dimensional vector rotated in an abstract space. The gauge group is then the group of rotations, SO(32), which is large enough (too large!) to accommodate all interactions we see in Nature.
Notice that SO(32) is also the gauge group in type-I theory, yet they are different, for in type-I, SO(32) is at the ends of the strings, whereas in the heterotic theory, SO(32) is along the entire string.

Modern wisdom holds that type-I and heterotic only look different. Deep inside they are different manifestations of the same theory. The operator product expansions are as the usual

$$\begin{aligned} X^\mu(z,\bar z)X^\nu(0,0) &\sim -\eta^{\mu\nu}\frac{\alpha'}{2}\ln|z|^2 \\ \tilde\psi^\mu(\bar z)\bar\partial^\nu(0) &\sim \eta^{\mu\nu}\frac{1}{z} \\ \psi^A(z)\psi^B(0) &\sim \delta^{AB}\frac{1}{z} \quad \leftarrow \text{Euclidean signature!} \end{aligned}$$

Energy momentum tensor:

$$T(z) = -\frac{1}{\alpha}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^A\partial\psi_A \quad , \quad \tilde T(z) = -\frac{1}{\alpha}\bar\partial X^\mu\bar\partial X_\mu - \frac{1}{2}\tilde\psi^A\bar\partial\tilde\psi_A .$$

SUSY currents:

$$\tilde T_F = i\sqrt{\frac{2}{\alpha'}}\tilde\psi^\mu\bar\partial X_\mu, \quad , \quad T_F = 0$$

so this theory has $N = 0,\ \tilde N = 1$ SUSY (world-sheet)
To build the Hilbert space, we need to specify the boundary conditions on ψ^A and $\tilde\psi^\mu$. $\tilde\psi^\mu$ is as before, leading to NS and R sectors - we may apply GSO projection to split the sectors in NS$\pm$ and R$\pm$.
ψ^A is tricky. It is not restricted by Lorentz invariance, because A is an internal index. We may only require that $T(z)$ be periodic, so $\psi^A(\sigma+2\pi,\tau) = O^{AB}\psi^A(\sigma,\tau)$ where O is a 32×32 orthogonal matrix. This leaves the quadratic form $\psi^A\partial\psi_A$ and $T(z)$ invariant. A host of possibilities, but only two work!
Possibility 1

$$\psi^A(\sigma+2\pi,\tau) = \pm\psi^A(\sigma,\tau)$$

same sign from all components. This is easy and is the same as before. Define the fermion number operator $F = \Sigma^{12} + \Sigma^{34} + ... + \Sigma^{31\ 32}$ (we now have 16 spins). The GSO projection is onto eigenspaces of $e^{i\pi F}$.
We will select $e^{i\pi F} = +1$, thus restricting to NS+, R+ for ψ^A. Partition Function
Recall for ψ^μ:

$$Z_\psi = \frac{1}{2}\left[\left(\frac{\vartheta_{00}(0,\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_{10}(0,\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_{01}(0,\tau)}{\eta(\tau)}\right)^4 \pm \left(\frac{\vartheta_{11}(0,\tau)}{\eta(\tau)}\right)^4\right] = 0$$

which vanished by the abstruse identity.
In our case, instead of 4=8/2, we have 16=32/2 (no time-like coordinate therefore all components contribute) Answer:

$$Z_\psi = \frac{1}{2}\left[\left(\frac{\vartheta_{00}(0,\tau)}{\eta(\tau)}\right)^{16} + \left(\frac{\vartheta_{10}(0,\tau)}{\eta(\tau)}\right)^{16} + \left(\frac{\vartheta_{01}(0,\tau)}{\eta(\tau)}\right)^{16} \pm \left(\frac{\vartheta_{11}(0,\tau)}{\eta(\tau)}\right)^{16}\right]$$

Where the first "+" is due to the ghosts (β,γ) from which are absent in this case. The second "+" sign is due to space-time statistics, ψ^A is a scalar.

This partition function is multiplied by $Z_{\tilde{\psi}}$, which vanishes, but it is still useful to demonstrate the modular invariance of $Z_\psi Z^*_{\tilde{\psi}}$.
Under $\tau \to -\frac{1}{\tau}$

$$\begin{aligned} \vartheta_{01}(0,-1/\tau) &= \sqrt{-i\tau}\vartheta_{00}(0,\tau) \\ \vartheta_{01}(0,-1/\tau) &= \sqrt{-i\tau}\vartheta_{10}(0,\tau) \\ \vartheta_{10}(0,-1/\tau) &= \sqrt{-i\tau}\vartheta_{01}(0,\tau) \\ \eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau) \end{aligned}$$

It is obvious that both Z_ψ and $Z_{\tilde{\psi}}$ are invariant since $\vartheta_{10} \leftrightarrow \vartheta_{01}$.
Under $\tau \to \tau + 1$

$$\begin{aligned} \vartheta_{00}(0,\tau+1) &= \vartheta_{01}(0,\tau) \\ \vartheta_{01}(0,\tau+1) &= \vartheta_{00}(0,\tau) \\ \vartheta_{10}(0,\tau+1) &= e^{i\pi/4}\vartheta_{10}(0,\tau) \\ \eta(\tau+1) &= e^{i\pi/12}\eta(\tau) \end{aligned}$$

$Z_\psi \to e^{-16\pi i/12}Z_\psi = e^{2\pi i/3}Z_\psi$ and $Z_{\tilde{\psi}} \to -e^{-4\pi i/12}Z_{\tilde{\psi}} = e^{2\pi i/3}Z_{\tilde{\psi}}$. Therefore the product $Z_\psi Z^*_{\tilde{\psi}}$ is invariant under modular transformations.

8.2 The spectrum

Constraints: $L_0 = \tilde{L}_0 = 0$. Do $\tilde{L}_0$ first (anti-holomorphic part). This is the same as before.

$$\tilde{NS}: \quad \tilde{L}_0 = \frac{\alpha'}{4}p^2 + \tilde{N} - \frac{1}{2} = 0.$$

Lowest state: $|0,k\rangle$ has no $\tilde{N} = 0$, so $m^2 = -k^2 = -2/\alpha'$, a tachyon. This has $e^{i\pi\tilde{F}}|0,k\rangle = -|0,k\rangle$, where the minus sign is due to the (β,γ) ghosts. Therefore it is not a $\tilde{NS}+$.
Next level: $\tilde{\psi}^\mu_{-\frac{1}{2}}|0,k\rangle$ has $\tilde{N} = \frac{1}{2}$, so $m^2 = -k^2 = 0$. It is a vector transforming under SO(8). It has $e^{i\pi\tilde{F}}\tilde{\psi}^\mu_{-\frac{1}{2}}|0,k\rangle = \tilde{\psi}^\mu_{-\frac{1}{2}}|0,k\rangle \in \tilde{NS}+$.
R-Sector $\tilde{L}_0 = \frac{\alpha'}{4}p^2 + N + a$
a can be deduced as follows: recall in bosonic theory $a = -1$. This is because $a = -(D-2)/24$. Here $D = 10$, so we get $a = -1/3$. In the NS sector, we get $a = -1/2$, because the fermions contributed $-1/6(1/2 = 1/3 + 1/6)$-each fermion contributes $-1/48$. Now we have 32 fermions, so they contribute $-32/48 = -2/3$. Overall $a = -1/3 - 2/3 = -1$ in NS.
In the R sector, we get $a = 0$, so fermions contribute $1/24$ each. Overall, $a = -1/3 + 32/24 = +1$ in the R sector which implies all modes are massive in the R sector.
In NS, the lowest state has $m^2 = -k^2 = -\frac{4}{a'}$, which represents a tachyon! It has $e^{i\pi F}|0,k\rangle = |0,k\rangle$, so we keep it.

The next level has $N = 1/2: \ \ \psi^A_{-1/2}|0,k\rangle$, so $m^2 = k^2 = -\frac{2}{\alpha'}$, which is another tachyon, but $e^{i\pi F}\psi^A_{-1/2}|0,k\rangle = -\psi^A_{-1/2}|0,k\rangle$, so we must reject it.
The next level has $N = 1: \ \ m^2 = k^2 = 0$, massless! There are two possibilities:

$$A_\mu(k)\alpha^\mu_{-1}|0;k\rangle \ , \quad B_{AB}\psi^A_{1/2}\psi^B_{-1/2}|0;k\rangle$$

Both possibilities have the correct GSO projection i.e., $e^{i\pi F} = +1$, so we keep them.
$A_\mu(k)$ represents a photon with 8 transverse polarizations. B_AB is an antisymmetric 32×32 matrix with $\frac{32\times 31}{2} = 496$ components.
Summary

m^2	$NS+$	$R+$	$\tilde{NS}+$	$\tilde{R}+$
$-4/\alpha'$	$\|0;k\rangle$	$-$	$-$	$-$
0	A_μ, B_{AB}	$-$	$\tilde{\psi}^\mu_{-1/2}\|0;k\rangle$	$\|\vec{s};k\rangle$

Closed strings states must have $L_0 = \tilde{L}_0$, so the tachyon in NS+ is rejected, because there is no tachyon in $\tilde{NS}+$ or $\tilde{R}+$. At the massless level we have

$$A_{\mu\nu}\alpha^\mu_{-1}\tilde{\psi}^\nu_{-1/2}|0;k\rangle, \ \ A_{\mu,\vec{s}}\alpha^\mu_{-1}|\vec{s};k\rangle$$

where $A_{\mu\nu}$ may be decomposed via a scalar, antisymmetric and traceless symmetric tensors as $8\times 8 = 64 = 1+28+35$. $A_{\mu\vec{s}}$ is the supersymmetric partner to $A_{\mu\nu}$ and may be decomposed into $8+56$ irreducible representations of SO(8). We also have

$$B^{(\vec{s})}_{AB}\psi^A_{-1/2}\psi^B_{-1/2}|\vec{s};k\rangle, \ \ B^\mu_{AB}\psi^A_{-1/2}\psi^B_{-1/2}\tilde{\psi}_{\mu\ -1/2}|0;k\rangle$$

so B^μ_{AB} represents a gauge boson with 496 components. c.f. gluon has A^μ_i, where $i = 1,2,...,8$. So SO(32) is a gauge symmetry in space-time, just like SU(3) is a gauge symmetry for QCD.
Comparing with SO(32) type-I theory, we see big differences. In type-I theory SO(32) resides at the ends of the strings and the spectrum does not match that of the heterotic string (except at the massless level). Yet, these two theories are one and the same (to be proved)!
Possibility #2
Divide $\psi^A(z)$ into two groups, $\psi^A, \ A = 1,2,...,16, \ \psi^B, \ B = 17,18,...,32$. This is possible because

$$\begin{array}{ll} \psi^A(\sigma+2\pi) = -\psi^A(\sigma) \ \ (NS_1) & \psi^A(\sigma+2\pi) = +\psi^A(\sigma) \ \ (R_1) \\ \psi^B(\sigma+2\pi) = -\psi^B(\sigma) \ \ (NS_2) & \psi^B(\sigma+2\pi) = +\psi^B(\sigma) \ \ (R_2) \end{array}$$

Total of four possibilities: $NS_1+NS_2, \ NS_1+R_2, \ R_1+NS_2, \ R_1+R_2$. There are two GSO projections because we have two fermion number operators. We will restrict to $e^{i\pi F_1} = e^{i\pi F_2} = +1$.
Partition Function

$$Z_\psi = Z_{\psi^A}Z_{\psi^B} = Z^2_{\psi^A},$$

because $Z_{\psi^A} = Z_{\psi^B}$. Z_{ψ^A} is the same as Z_ψ we derived in possibility #1, except $16 \to 8$ (we have half as many ψ^As now). Therefore

$$Z_{\psi^A} = \frac{1}{2}\left[\left(\frac{\vartheta_{00}}{\eta}\right)^8 + \left(\frac{\vartheta_{10}}{\eta}\right)^8 + \left(\frac{\vartheta_{01}}{\eta}\right)^8 \mp \left(\frac{\vartheta_{11}}{\eta}\right)^8\right].$$

Modular invariance: $\tau \to -1/\tau$ leave the partition function invariant (trivial). $\tau \to \tau + 1$ takes $\vartheta_{00} \leftrightarrow \vartheta_{01},\ \vartheta_{10} \to e^{i\pi/4}\vartheta_{10}$, so $\vartheta_{10}^8 \to \vartheta_{10}^8$. So only $\eta \to e^{i\pi/12}\eta$, i.e., $\eta^{-8} \to e^{-2\pi i/3}\eta^{-8}$, and the partition transforms as $Z_{\psi^A} \to e^{-2\pi i/3} Z_{\psi^A} \Rightarrow Z_{\psi^A}^2 \to e^{-4\pi i/3} Z_{\psi^A}^2 = e^{2\pi i/3} Z_{\psi^A}^2$. The additional factor cancels when we multiple by $Z_{\tilde{\psi}}^* \to e^{-2\pi i/3} Z_{\tilde{\psi}}^*$.
Gauge Group: Obviously, this theory has $SO(16) \times SO(16)$ symmetry. However, this is only a subgroup os the full gauge group, which is $E_8 \times E_8$ (exceptional group).
To summarize: there exist only two possibilities for heterotic strings, with gauge groups SO(32) and $E_8 \times E_8$, respectively.

UNIT 9

Low Energy Physics

9.1 Type IIA Superstring

Sectors: (Ns+,NS+), (R+,NS+), (NS+,R-), (R+,R-).
(NS+, NS+): massless states: $A_{\mu\nu}\psi^{\mu}_{-1/2}\psi^{\nu}_{-1/2}|0;k\rangle$ $A_{\mu\nu}$ is decomposed into a scalar, antisymmetric field and traceless symmetric field: $8\times 8 = 1+28+35$ The scalar field represents the dilaton. We do not see it in Nature and it is believed to have settled into it's ground state value and not affect dynamics any further. We will set it to zero to avoid complications in an already complicated discussion (it should be set to a constant, but we can always tweak couplings, etc., so setting it to zero will be fine). Let $B_{\mu\nu}$ be the antisymmetric tensor and $g_{\mu\nu}$ the traceless symmetric tensor (graviton). The dynamics of $g_{\mu\nu}$ is described by the Einstein action

$$S_g = \frac{1}{4\pi G_{10}}\int d^{10}x\sqrt{-g}\;R,$$

where G_{10} is the ten-dimensional Newton's constant. This can be derived from string theory tree-level amplitudes and loop amplitudes introduce corrections. $B_{\mu\nu}$ has field strength

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \partial_\nu B_{\mu\rho} + \partial_\rho B_{\mu\nu}.$$

$H_{\mu\nu\rho}$ is totally antisymmetric in it's indices. The action is given by

$$S_B = -\frac{1}{8\pi G}\int d^{10}x\sqrt{-g}H^{\mu\nu\rho}H_{\mu\nu\rho},$$

where indices are raised and lowered by $g_{\mu\nu}$.
(R+,R-): massless states: $|\vec{s};k\rangle\otimes|\vec{s'};k\rangle,\;\; 8\times 8$ of them.
Recall $\psi^{\mu}_{0}|\vec{s};k\rangle$ is also a ground state (annihilated by all $\psi^{\mu}_{r},\; r>0$).
States decompose into $C_\mu\psi^{\mu}_{0}|0\rangle$ and $C_{\mu\nu\rho}\psi^{[\mu,}_{0}\psi^{[\nu,}_{0}\psi^{\rho]]}_{0}|0\rangle$ where we antisymmetrize over all indices. There are 8 C_μ (transverse μ) and 56 $C_{\mu\nu\rho}$ $(8{+}56=64)$

so they span the ground states. The action is given by

$$S_R = -\frac{1}{8\pi G_{10}} \int d^{10}x\sqrt{-g}\left(F_{\mu\nu}F^{\mu\nu} + \tilde{F}_{\mu\nu\rho\sigma}\tilde{F}^{\mu\nu\rho\sigma}\right)$$

where $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ is the field strength of C_μ.

$$\tilde{F}_{\mu\nu\rho\sigma} = F_{\mu\nu\rho\sigma} - \frac{1}{4}\left(C_\mu H_{\nu\rho\sigma} + C_\nu H_{\rho\sigma\mu} + C_\rho H_{\sigma\mu\nu} + C_\sigma H_{\mu\nu\rho}\right)$$

where $F_{\mu\nu\rho\sigma} = \partial_\mu C_{\nu\rho\sigma} + ...$ (add terms such that $F_{\mu\nu\rho\sigma}$ is completely antisymmetric) and is the field strength of $C_{\mu\nu\rho}$.
There is one more contribution to the action that does not involve the metric (topological). This is a Chern-Simons term given by

$$S_{CS} = -\frac{1}{8\pi G_{10}} \int d^{10}x\epsilon^{\mu_1\mu_2\ldots\mu_{10}} B_{\mu_1\mu_2} F_{\mu_3\mu_4\mu_5\mu_6} F_{\mu_7\mu_8\mu_9\mu_{10}}$$

The total action is the sum of all the actions and is given by

$$S = S_g + S_B + S_R + S_{CS}$$

There is a fermionic counterpart which we will not discuss.

9.2 Supergravity

Let us compare with supergravity (SUGRA). Unfortunately, SUGRA lives in 11 dimensions, yet it looks so much like the type-II superstring, that is hard to ignore. It turns out that (modern wisdom holds) we really live in eleven dimensions and ten dimensional strings are really an eleven dimensional theory! What theory? Nobody knows ... M-Theory.
We have seen this problem with dimensions before. We compactified one dimension a la Kaluza-Klein, then let $R \to 0$ and the extra dimension would not go away. Same here. We will compactify the eleventh dimension to get 10D superstrings, but the eleventh dimension will remain lurking in the background.
Action:

$$\begin{aligned} S_{11}^{SUGRA} &= \frac{1}{4\pi G_{11}} \int d^{11}x\sqrt{-G}\left(R^{(11)} - \frac{1}{2}F_{MNQR}F^{MNQR}\right) \\ &\quad -\frac{1}{24\pi G_{11}} \int d^{11}x\ \epsilon^{M_1M_2\ldots M_{11}} A_{M_1M_2M_3} F_{M_4M_5M_6M_7} F_{M_8M_9M_{10}M_{11}} \end{aligned}$$

where F_{MNQR} is the field strength of A_{MNQ} $(F_{MNQR} = \partial_M A_{NQR} + ...)$, where F is completely antisymmetric. The last term is a Chern-Simons term and is gauge invariant

$$\delta A_{MNQ} = \partial_M \lambda_{NQ} + ...$$

even though it doesn't look like it.
We need to reduce the dimension from eleven to ten to compare with superstrings. We will do that a la Kaluza-Klein. Assume nothing depends on the eleventh coordinate and call it "u".
The metric: $ds^2 = G_{MN}(x^\mu)dx^\mu dx^\nu, \;\; M,N = 0,1,...,10, \; \mu = 0,1,..,9$ We may decompose the metric as such

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu + 2G_{\mu u}dx^\mu du + G_{uu}du^2$$

Let $G_u u = 1$ for simplicity (fixes the size of the extra dimension, which can be rescaled later).
Introduce vector $A_m u = G_{\mu u}$ and metric $g_{\mu\nu} = G_{\mu\nu} - A_\mu A_\nu$. Then we may write $ds^2 = g_{\mu\nu}dx^\mu dx^\nu + (du + A_\mu dx^\mu)^2$. The potential A_{MNQ} may also be grouped into $A_{\mu\nu\rho}$ and $A_{\mu\nu} = A_{\mu\nu u}$ (no other components exist, because A_{MNQ} is antisymmetric, so we can not have two u indices). So now the field content becomes (from the 10D perspective) $g_{\mu\nu}, \; A_\mu, \; A_{\mu\nu}, \; A_{\mu\nu\rho}$, very similar to the type-II superstring.
Futhermore,

$$R^{(11)} = R^{(10)} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$F_{MNQR}F^{MNQR} = F_{\mu\nu\rho}F^{\mu\nu\rho} + \tilde{F}_{\mu\nu\rho\sigma}\tilde{F}^{\mu\nu\rho\sigma}$$

where $F_{\mu\nu\rho} = \partial_\mu A_{\mu\rho} + ...$, $\tilde{F}_{\mu\nu\rho\sigma} = F_{\mu\nu\rho\sigma} - A_\mu F_{\nu\rho\sigma} + ...$ and $F_{\mu\nu\rho\sigma} = \partial_\mu A_{\nu\rho\sigma} + ...$ and

$$\begin{aligned} \frac{1}{6}\epsilon^{M_1M_2...M_{11}}A_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}} &= \epsilon^{\mu_1\mu_2...\mu_{10}}A_{\mu_1\mu_2}F_{\mu_3\mu_4\mu_5\mu_6}F_{\mu_7\mu_8\mu_9\mu_{10}} \\ &= \epsilon^{\mu_1\mu_2...\mu_{10}}A_{\mu_1\mu_2\mu_3}F_{\mu_4\mu_5\mu_6}F_{\mu_7\mu_8\mu_9\mu_{10}} + \text{total derivative} \end{aligned}$$

The last equality can easily be verified by integrating by parts.
Also $\sqrt{-G} = \sqrt{-g}$.
If the eleventh dimension has length $2\pi R$, then the action becomes

$$\begin{aligned} S_{11}^{SUGRA} &= \frac{1}{4\pi G_{10}}\int d^{10}x\sqrt{-g}\left(R^{(10)} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{\mu\nu\rho}F^{\mu\nu\rho} - \frac{1}{2}\tilde{F}_{\mu\nu\rho\sigma}\tilde{F}^{\mu\nu\rho\sigma}\right. \\ &\left. -\frac{1}{2}\epsilon^{\mu_1\mu_2...\mu_{10}}A_{\mu_1\mu_2}F_{\mu_3\mu_4\mu_5\mu_6}F_{\mu_7\mu_8\mu_9\mu_{10}}\right) \end{aligned}$$

where $G_{10} = 2\pi R G_{11}$ is the 10D Newton's constant and we have rescaled $A_{\mu\nu} \to \frac{1}{\sqrt{2\pi R}}A_{\mu\nu}, \;\; A_{\mu\nu\rho} \to \frac{1}{\sqrt{2\pi R}}A_{\mu\nu\rho}$.
This action is identical to the one obtained from type-IIA superstring if we identify

$$g_{\mu\nu} \to g_{\mu\nu}, \;\; A_\mu \to C_\mu, \;\; A_{\mu\nu} \to B_{\mu\nu}, \;\; A_{\mu\nu\rho} \to C_{\mu\nu\rho}.$$

UNIT 10

D-Branes

10.1 T-duality (again)

Consider type-II theories. We have

$$\begin{array}{lllll} IIA: & (NS+, NS+) & (R+, NS+) & (NS+, R-) & (R+, R-) \\ IIB: & (NS+, NS+) & (R+, NS+) & (NS+, R+) & (R+, R+) \end{array}$$

Compactify the 10th dimension on a circle of radius R in IIA, say. As we showed in the bosonic theory (argument is the same) the theory at R is identical to the theory at $R' = \frac{\alpha'}{R}$ (T-duality).
To show this, we started with the coordinate $X^9 = U = U_L(z) + U_R(\bar{z})$ and introduced the coordinate $z = U_L(z) - U_R(\bar{z})$. The resulting theory is at $R' = \frac{\alpha'}{R}$. In other words, the parity transformation on the right-moving part (only!).

$$X_R^9(\bar{z}) \to -X_R^9(\bar{z})$$

relates the theory at R with the theory at $R' = \frac{\alpha'}{R}$. Because of the superconformal invariance, this parity transformation is also applied tp the superpartner, $\tilde{\psi}^9(\bar{z})$

$$\tilde{\psi}^9(\bar{z}) \to -\tilde{\psi}^9(\bar{z}).$$

This, in particular, reverses the chirality of the states in the antiholomorphic part, so R- $\leftrightarrow$ R+. Therefore IIA $\leftrightarrow$ IIB, because that is the only difference between the two theories. Therefore IIA at R is equivalent to IIV at $R' = \frac{\alpha'}{R}$.
In particular, the IIA R-R fields, $C_\mu,\ C_{\mu\nu\lambda}$ are mapped onto the IIB R-R fields, $C,\ C_{\mu\nu},\ C_{\mu\nu\rho\sigma}$ as follows:

$$C_9 \to C,\ \ C_\mu \to C_{\mu 9},\ \ C_{\mu\nu 9} \to C_{\mu\nu},\ \ C_{\mu\nu\lambda} \to C_{\mu\nu\lambda 9}.$$

Of course, e.g., $C_{\mu\nu\rho\sigma}$ is obtained from $C_{\mu\nu\rho\sigma 9}$ in IIA, but $C_{\mu\nu\rho\sigma 9}$ is <u>not</u> an independent field (can be expressed in terms of $C_\mu,\ C_{\mu\nu\lambda}$) $8 + 56 = 64$.

Type-I Strings If we compactify the 10th dimension, $X^9(z,\bar{z})$, then the theory in the $R \to 0$ limit is mapped onto a T-dual theory at $R = \frac{\alpha'}{R} \to \infty$ which contains a D-brane.
Recall the argument, in the R' theory, the 10th dimension is

$$Z(z,\bar{z}) = X^9_L(z) - X^9_R(\bar{z}), \;\; \partial_\sigma Z = \partial_\tau X^9$$

so

$$\begin{aligned} Z(\sigma=\pi) - Z(\sigma=0) &= \int_0^\pi d\sigma \partial_\sigma Z = \int_0^\pi d\sigma \partial_\tau X^9 = \int_0^\pi \partial_\tau (2\alpha' p \tau) \\ &= 2\alpha' p \pi = 2\alpha' \frac{n}{R}\pi = 2\pi n R' = 0. \end{aligned}$$

Translation invariance is broken in the T-dual theory. Massless modes (same as in uncompactified theory)

$$NS: \;\; A_\mu \psi^\mu_{-1/2}|k\rangle, \;\; A\psi^9_{-1/2}|k\rangle, \quad R: \;\; |\vec{s};k\rangle$$

where A_μ represents a photon tangent to the brane. The second state shifts the position of the brane making it a dynamical object. (A is a function of k $\to$ its F.T. is a function of $X^\mu, \; \mu = 0,1,...,8$).
Even though the translation invariance is broken, the original theory has 32 supersymmetries! Of these, only half are broken. Thus the brane is a supersymmetric object with 16 supersymmetries! This large amount of symmetry implies the existence of conserved charges. What are they?
Our brane has 8+1 dimensions, so its volume element couples to the R-R potential, $C_{\mu_1\mu_2,...,\mu_9}$ $(dV \sim \epsilon_{\mu_1\mu_2...\mu_9} dx^{\mu_1}...dx^{\mu_9})$.
Recall familiar examples:

- A point charge q moving along a trajectory $x^\mu(\tau)$ has the action $q\int d\tau v^\mu A_\mu = \int d\tau j^\mu A_\mu = q\int dx^\mu A_\mu$. The charge q is conserved.
- The magnetic flux: $\Phi = \int \vec{B}\cdot d\vec{s}$, $\vec{B} = \nabla \times \vec{A}$ Define a field strength: $F_{ij} = \partial_i A_j - \partial_j A_i$. Then $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$, so $\Phi = \int F_{jk} dS^{jk}$ where $dS^{jk} = \frac{1}{2}\epsilon^{ijk} dS_i$ is the surface element. This is the magnetic charge, i.e., 0. Similarly, for the electric charge field, $\Phi_E = \int F_{0i} d\Sigma^{0i} \propto q$.

For the R-R charge on the D8 brane, we have

$$Q \propto \int dx^{\mu_1}...dx^{\mu_9} C_{\mu_1...\mu_9}$$

If we dualize two more dimension, the brane becomes a 6+1 dimensional object, (D6-brane). Two more gives D4, two more gives D2 and two more gives a D0 brane which represents a point particle. The charges are $\int dx^\mu C_\mu, \;\int dx^{\mu_1} dx^{\mu_2} ... C_{\mu_1\mu_2...}$ which are the R-R fields in type IIA theory! On the other hand, the D(2p+1)-branes couple to $C_{\mu_1\mu_2}, \; C_{\mu_1\mu_2\mu_3\mu_4}$, etc., which are the potentials in the type-IIB theory!

Not all these potentials are independent. Consider, e.g., D0-brane coupled to C_μ. The D0-brane is a point particle (with strings attached- hairy) with charge q which is the source of C_μ and field strength $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. q is an electric charge. The flux $\int F_{\mu\nu} d\Sigma^{\mu\nu} \propto q$ (Gauss' Law).
In four dimensional electromagnetism we may define the dual of $F_{\mu\nu}$ as $\tilde{F}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ which interchanges $\vec{E} \leftrightarrow \vec{B}$. Then the electric charges become magnetic charges. One may define a vector potential $\tilde{A}_\mu$ corresponding to $\tilde{F}_{\mu\nu}$ and describe electromagnetics in terms of $\tilde{A}_\mu$ instead of A_μ. $\tilde{A}_\mu$ can not be defined globally, since the magnetic flux around a charge is no longer zero, but it can be defined in patches, or almost everywhere apart from the string (Dirac string). If we include both electric and magnetic charges, then **no action** can be defined, yet the theory still makes sense. The existence of a monopole leads to quantization of the electric charge (Dirac).
Proof: Consider a point particle moving from $\vec{x_1} \to \vec{x_2}$. Its wavefunction changes $\psi(\vec{x}_1) \to \psi(\vec{x}_2)$. If I want to compare $\psi(\vec{x}_1)$ and $\psi(\vec{x}_2)$, then I will define the quantity $\psi(\vec{x}_2) * \psi(\vec{x}_1)$. In the limit $\vec{x}_2 \to \vec{x}_1$ (closed path) we obtain $|\psi(\vec{x}_1)|^2$. Gauge invariance: $\psi(\vec{x}) \to e^{iq\lambda(x)}\psi(\vec{x})$, so $\psi^*(\vec{x}_2)\psi(\vec{x}_1) \to e^{iq(\lambda(x_2)-\lambda(x_1))}\psi^*(\vec{x}_2)\psi(\vec{x}_1)$ This is not a gauge-invariant oblect. To make it gauge-invariant, multiply by $e^{iq\int \vec{A}\cdot d\vec{\ell}}$, $\vec{A} \to \vec{A} - \nabla\lambda$, so $\delta e^{iq\int \vec{A}\cdot d\vec{\ell}} = e^{-iq(\lambda(\vec{x}_1)-\lambda(\vec{x}_2))}$, so $\psi^*(\vec{x}_2)e^{iq\int \vec{A}\cdot d\vec{\ell}}\psi(\vec{x}_1)$ is gauge-invariant (physical)!
Go around a loop: we have $e^{iq\oint \vec{A}\cdot d\vec{\ell}}|\psi(\vec{x}_1)|^2$. By Stoke's theorem, $\oint_{\mathcal{C}} \vec{A}\cdot d\vec{\ell} = \int_{\mathcal{S}} \vec{B}\cdot d\vec{s}$ (flux through S).
If the path shrinks to zero, then $\oint_{\mathcal{C}} \vec{A}\cdot d\vec{\ell} = \int_{\mathcal{S}} \vec{B}\cdot d\vec{s} = 0$.
In the presence of a magnetic monopole, $\int_{\mathcal{S}} \vec{B}\cdot d\vec{s} = m$, the magnetic charge, so $e^{iq\int_{\mathcal{S}} \vec{B}\cdot d\vec{s}} = e^{iqm}$. We must have $e^{iqm} = 1$, therefore $qm = 2\pi n$, i.e., q is quantized even if only one magnetic monopole exists in the entire Universe.
Returning to D-branes, the C_μ potential on the D0-brane has field strength $F_{\mu\nu}$ whose dual is $\epsilon^{\mu_1\mu_2\cdots\mu_{10}}F_{\mu_9\mu_{10}}$ (8 indices). It corresponds tp a potential with seven indices, $C_{\mu_1\mu_2\ldots\mu_7}$ which resides on a D6-brane.
Thus the D0 electric charge is a source for the same field for which the D6-branes magnetic charge is a source. More generally, the electric Dp-brane charge and the magnetic D(6-p)-brane charge are sources for the same field.
Action for D0-branes electromagnetism:

$$S = -\frac{1}{2}\int d^{10}x\,\sqrt{-g}F_{\mu\nu}F^{\mu\nu} + q\int dx^\mu A_\mu$$

The potential between two points (D0-branes) is a Coulomb potential (in 10D)

$$V(y) \propto \frac{q^2}{y^7}$$

In momentum space, this is obtained from the propagator $-\frac{i}{k^2}$ where k^μ is the momentum of the exchanged boson (photon). Then

$$V(y) = -i\int d^{10}k e^{i\vec{k}\cdot\vec{y}}\frac{q^2}{k^2} = -i\frac{15V}{32\pi^4}\frac{q^2}{y^7}$$

where $V = \int d\omega$.
With D-branes, the potential comes from the exchange of closed strings. This may also be viewed as an open string with ends at $y = 0$ and $y = y$ moving around a loop. We already know how to calculate it.
The answer is

$$Z = \int_0^\infty \frac{dt}{2t} Z(t), \;\; Z(t) = \text{Tr}\, e^{-2\pi t L_0}.$$

Recall our result earlier

$$Z_{NS} = i \frac{V}{8\pi(8\pi^2\alpha')^5} \int_0^\infty ds(16 + o(e^{-2s})), \;\; s = \frac{\pi}{t}.$$

Now we have 9 dimensions (16 compactified), so $s = \frac{9}{2}$. Also there is no integral over spatial momenta, only the energy D0-branes have world-lines, so the contribution from 0-modes $\left(8\pi^2\alpha' t\right)^{-D/2} \to \left(8\pi^2\alpha' t\right)^{-1/2}$, therefore, there is an additional factor $\left(8\pi^2\alpha' t\right)^{-(1-D)/2} \to \left(8\pi^2\alpha' t\right)^{9/2}$.
An extra factor of $4 = 2 \times 2$ (2 from **XXXXX** and no need to average over orientations). Finally, since $Z(\sigma = \pi) - Z(\sigma = 0) = y$, the expansion contains an extra term $Z = y\frac{\sigma}{\pi} + ...$ which gives an extra contribution to $L_0 = \frac{y^2}{4\pi^2\alpha'} + ...$
Therefore the extra factor is given by $e^{-2\pi t \frac{y^2}{4\pi^2\alpha'}} = e^{-ty^2/2\pi\alpha'}$. The partition function becomes

$$\begin{aligned} Z &\to \frac{iV(4 \times 16)}{8\pi(8\pi^2\alpha')^5} \int_0^\infty \frac{\pi dt}{t^2} (8\pi^2\alpha' t)^{9/2} e^{-ty^2/2\pi\alpha'} \\ &= iV(2\pi)(4\pi^2\alpha')^3 \frac{15}{32\pi^4} \frac{1}{y^7} \end{aligned}$$

This is compared to the potential $V(y) = -i\frac{15V}{32\pi^4}\frac{q^2}{y^7}$. In fact it is the R-sector $Z_R = V(y)$, but $Z_R = -Z_{NS}$, so $q^2 = 2\pi(4\pi^2\alpha')^3$.
This generalizes to Dp-branes: $(8\pi^2\alpha' t)^{9/2} \to (8\pi^2\alpha' t)^{(9-p)/2}$. The potential generalizes to $V_p(y) \sim \frac{1}{y^{7-p}}$ and the charges become $q_p^2 = 2\pi(4\pi^2\alpha')^{3-p}$.
For the D6-brane, $q_6^2 = \frac{2\pi}{(4\pi^2\alpha')^3}$, so $q_6 q_0 = 2\pi$, the Dirac quantization condition with $n = 1$! In general, $q_p q_{6-p} = 2\pi$, confirming that the Dp-brane and D(6-p)-brane act as electric and magnetic sources for the same field.

10.2 D-branes at angles

So far we have considered similar D-branes separated by a distance y. These are parallel D-branes. More generally, we can have a Dp-brane and a Dp'-brane along different subspaces and they may even intersect e.g., a D8-brane obtained by dualizing X^9 and a D8-brane from X^8 dualization. These two branes have the space $(X^1, X^2, ..., X^7)$ common. One may be obtained by rotating the other by 90° in the (X^8, X^9) plane. An open string may stretch between these two branes. Then its X^9 coordinate will obey Dirichlet boundary conditions at one end and Neumann boundary conditions at the other. The X^8 coordinate is reversed: Neumann boundary conditions at one end and Dirichlet at the other. Thus the modes expansions will be different. Recall for Neumann boundary conditions on both ends (set $\tau = 0$ for simplicity)

$$X^\mu_{NN}(\sigma) = x^\mu + i\sqrt{2\alpha'}\sum \frac{1}{n}\alpha^\mu_n \cos(n\sigma).$$

Check $\partial_\sigma X_{NN} 6\mu = 0$ at $\sigma = 0, \pi$.
For Dirichlet boundary conditions on both ends,

$$X^\mu_{DD}(\sigma) = \frac{y\sigma}{\pi} - i\sqrt{2\alpha'}\sum \frac{1}{n}\alpha^\mu_n \sin(n\sigma)$$

where y is the separation of the two (parallel in the μ-direction) branes. Check $X^\mu_{DD}(0) = 0,\ X^\mu_{DD}(\pi) = y$. X^μ_{NN} is split into holomorphic and antiholomorphic pieces as such

$$\begin{aligned} X^\mu_L &= \frac{1}{2}x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum \frac{1}{n}\alpha^\mu_n e^{in\sigma} \\ X^\mu_R &= \frac{1}{2}x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum \frac{1}{n}\alpha^\mu_n e^{-in\sigma} \end{aligned}$$

X_{DD} is split as $X^\mu_{DD} = X^\mu_L - X^\mu_R$ (dual!) For DN-b.c., i.e., $X^\mu_{DN}(\sigma = 0) = 0$, $\partial_\sigma X^\mu_{DN}(\sigma = \pi) = 0$, we obtain

$$X^\mu_{DN}(\sigma) = -\sqrt{2\alpha'}\sum_{r\in\mathbb{Z}+1/2} \frac{\alpha^\mu_r}{r}\sin(r\sigma).$$

For ND-boundary conditions, we have $X^\mu_{ND}(\sigma) = i\sqrt{2\alpha'}\sum_{r \int \mathbb{Z}+1/2} \frac{\alpha^\mu_r}{r}\cos(r\sigma)$.
The superpartners ψ^μ and $\tilde{\psi}^\mu$ are similar.
Generalize to general angles. Suppose that there is an angle ϕ between the branes and consider strings stretched between the two. Define $Z = X^8 + iX^9$ (the brane at $X^9 = 0$ is not rotated-no loss of generality). At $\sigma = 0,\ X^9 = 0$ and $\partial_\sigma X^8 = 0$, so $\text{Im}(Z) = 0$, $\text{Re}(Z) = 0$. At $\sigma = \pi$, the brane is rotated by ϕ, so $Z \to e^{i\phi}Z$, so $\text{Im}(Ze^{-i\phi} = \partial_\sigma\ \text{Re}(Ze^{-i\phi}) = 0$.
We may expand in terms of the modes

$$Z = \sqrt{\frac{2}{\alpha'}}\sum_{r\in\mathbb{Z}+\frac{\phi}{\pi}} \frac{\alpha_r}{r}e^{ir\sigma} + \sqrt{\frac{2}{\alpha'}}\sum_{r\in\mathbb{Z}-\frac{\phi}{\pi}} \frac{\alpha^+_r}{r}e^{-ir\sigma}$$

α_r and α^+_r are independent, because they involve α^8_r and α^9_r ($\alpha_r = \alpha^8_r + \alpha^9_r$).
At $\sigma = 0$: $Z \sim O(\alpha_r + \alpha^+_r)$, so, $\text{Im}(Z) = 0$, and $\partial_\sigma Z \sim i(\alpha_r - \alpha^+_r)$, so, $\text{Re}(\partial_\sigma Z) = 0$.
At $\sigma = \pi$: $Z \sim (\alpha_r + \alpha^+_r)e^{i\phi}$, so, $\text{Im}(Ze^{-i\phi}) = 0$, and $\partial_\sigma Z \sim i(\alpha_r - \alpha^+_r)e^{i\phi}$, so, $\text{Re}(\partial_\sigma Ze^{-i\phi}) = 0$.

10.3 Partition Function

The partition function has contributions from both the α_r's and α_r^+'s. It is easy to see that for $q = e^{-2\pi t}$

$$\begin{aligned} Z &= q^a \prod_{r\in Z+\frac{\phi}{\pi}} (1-q^r)^{-1} \prod_{r\in Z-\frac{\phi}{\pi}} (1-q^r)^{-1}, \\ &= q^a \prod_{m=0}^{\infty} (1-q^{m+\frac{\phi}{\pi}})^{-1} \prod_{m=1}^{\infty} (1-q^{m-\frac{\phi}{\pi}})^{-1}, \end{aligned}$$

where a is the Casimir energy (normal ordering constant in $L_0 =: L_0 : -a$). Recall $a = -1/24$ for a boson, because $a = \frac{1}{2}\sum_{n=1}^{\infty} n = \frac{1}{2}\zeta(1) = -1/24$. Here the sum becomes

$$\frac{1}{2}\sum_{r\in Z-\frac{\phi}{\pi}} r = \frac{1}{2}\sum_{m=1}^{\infty}\left(m - \frac{\phi}{\pi}\right) = \frac{1}{2}\left[\frac{1}{24} - \frac{1}{8}\left(2\frac{\phi}{\pi} - 1\right)^2\right].$$

To prove this, look at the twisted sum problem (Polchinski 2.9.19) done last semester.
Also,

$$\frac{1}{2}\sum_{\substack{r\in Z+\frac{\phi}{\pi} \\ r>0}} r = \frac{1}{2}\sum_{m=0}^{\infty}\left(m + \frac{\phi}{\pi}\right) = \frac{1}{2}\left[\frac{1}{24} - \frac{1}{8}\left(2\left(1-\frac{\phi}{\pi}\right) - 1\right)^2\right] = \frac{1}{2}\left[\frac{1}{24} - \frac{1}{8}\left(1 - 2\frac{\phi}{\pi}\right)^2\right],$$

which is the same as before. So, $a = \frac{1}{24} - \frac{1}{8}\left(1 - 2\frac{\phi}{\pi}\right)^2$. Therefore,

$$Z = q^a (1-z)^{-1}\left[\prod_{m=1}^{\infty}(1-zq^m)(1-z^{-1}q^m)\right]^{-1}, \quad z = q^{\phi/\pi} = e^{-2\phi t} = e^{2\pi i \nu}$$

This can be expressed in terms of

$$\vartheta_{11}(\nu, it) = -2q^{1/8}\sin\pi\nu \prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m),$$

and

$$\eta(it) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m).$$

Indeed,

$$\begin{aligned} \frac{\eta(it)}{\vartheta_{11}(\nu, it)} &= -\frac{1}{2}q^{-1/24-1/8-a}\frac{1}{\sin\pi\nu}\left[\prod(1-zq^m)(1-z^{-1}q^m)\right]^{-1} \\ &= -\frac{1}{2}q^{1/24-1/8-a}\frac{1-z}{\sin\pi\nu}Z \\ &= iq^{\phi^2/2\pi^2}Z \end{aligned}$$

Therefore

$$Z = -1\frac{e^{\phi^2 t/\pi\eta(it)}}{\vartheta_{11}(\nu, it)}.$$

Similarly for the fermion, we obtain

$$Z = \frac{\vartheta_{ab}(\nu, it)}{e^{\phi^2 t/\pi}\eta(it)},$$

for $a, b = 0, 1$ (NS-NS, NS-R, etc.)
Notice that the bosonic Z diverges as $\nu \to 0$, i.e., $\phi \to 0$. In this limit the two branes become parallel to each other, and the string is free to move along them, i.e., it has an additional (continuous) momentum, whose trace gives $\mathrm{Tr}\, q^{L_0} \sim \frac{V}{\sqrt{8\pi\alpha' t}}$, where V is the volume of the dimension along the brane. Therefore,

$$\begin{aligned} Z &= q^a \frac{1}{\sqrt{8\pi^2\alpha' t}} \prod_{m=1}^{\infty}(1-q^m)^{-2}, \quad a = 1/24 - 1/8 = -1/12 \\ &= \frac{V}{\sqrt{8\pi^2\alpha' t}}(\eta(it))^{-2}. \end{aligned}$$

The fermionic partition functions Z_{ab} do not change. Suppose as $\phi \to 0$, both branes are in the X^8 direction. Now take the dual of X^8. Since we have Neumann boundary conditions in X^8 (Dirichlet in X^9), in the dual, we will have Dirichlet in X^8. So in the dual picture, the two branes will become distinct points separated by a distinct y.
If originally we had Dp-branes, we end up with D(p-1) branes in the dual space. Open strings are stretched between the two branes. Thus, instead of a continuous momentum, we now have a contribution $\frac{y^2}{4\pi^2\alpha'}$ in L_0, therefore $e^{-ty^2/2\pi\alpha'}, \quad , y^2 = y_8^2 + y_9^2$, in general. The partition function is

$$Z = q^a e^{-ty^2/2\pi\alpha'} \prod_{m=1}^{\infty}(1-q^m)^{-2} = e^{-ty^2/2\pi\alpha'}(\eta(it))^{-2}$$

Example: Consider two D4-branes at an angle ϕ_1 in the 23-plane, ϕ_2 in the 45-plane, ϕ_3 in the 67-plane, ϕ_4 in the 89-plane and separated by a ditance y in the 1-directioin. In each plane, we obtain a partition function for the fermions:

$$Z_{ab}(\phi_i, it) = \frac{\vartheta_{ab}(\nu_i, it)}{e^{\phi_6^2 t/\pi}\eta(it)}, \quad , \nu_= i\phi_i t/pi, \;\; i = 1, 2, 3, 4.$$

Putting them together, the fermionic partition function is

$$Z_f = \frac{1}{2}\left[\prod_{i=1}^{4}\frac{\vartheta_{00}(\nu_i, it)}{e^{\phi_i^2 t/\pi}\eta(it)} - \prod_{i=1}^{4}\frac{\vartheta_{10}(\nu_i, it)}{e^{\phi_i^2 t/\pi}\eta(it)} - \prod_{i=1}^{4}\frac{\vartheta_{01}(\nu_i, it)}{e^{\phi_i^2 t/\pi}\eta(it)} - \prod_{i=1}^{4}\frac{\vartheta_{11}(\nu_i, it)}{e^{\phi_i^2 t/\pi}\eta(it)}\right].$$

Generalizing our earlier result, when $\phi_i = 0 = \nu_i$,

$$Z_\psi = \frac{1}{2\eta^4(it)} \left(\vartheta_{00}^4(0,i\tau) - \vartheta_{10}^4(0,i\tau) - \vartheta_{01}^4(0,i\tau) - \vartheta_{11}^4(0,i\tau) \right).$$

Earlier we used the abtruse identity to show $Z_\psi = 0$. Now, we shall use the generalization of the abstruse identity:

$$\prod_{m=1}^{\infty} \vartheta_{00}^4(0,i\tau) - \prod_{m=1}^{\infty} \vartheta_{10}^4(0,i\tau) - \prod_{m=1}^{\infty} \vartheta_{01}^4(0,i\tau) - \prod_{m=1}^{\infty} \vartheta_{11}^4(0,i\tau) = 2\prod_{m=1}^{\infty} \vartheta_{11}(\nu_i', it)$$

$$\nu_i' = i\phi_i' t/\pi, \quad \phi_1' = \tfrac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4), \quad \phi_2' = \tfrac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4)$$
$$\phi_3' = \tfrac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4), \quad \phi_4' = \tfrac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4)$$

Notice $\sum_{i=1}^4 \phi_i'^2 = \sum_{i=1}^4 \phi_i^2$, so $\prod_{i=1}^4 e^{\phi_i^2 t/\pi} = \prod_{i=1}^4 e^{\phi_i'^2 t\pi}$ and

$$Z_f = \frac{\prod_{i=1}^4 \vartheta_{11}(\nu_i', it) e^{-\phi_i'^2 t/\pi}}{\eta^4(it)}.$$

Bosons: Recall in the 89-plane

$$Z_{boson} = -i\frac{e^{\phi^2 t/\pi}\eta(it)}{\vartheta_{11}(\nu, it)}$$

so in the $234...9$ direction

$$Z_{boson} = \eta^4(it)\prod_{i=1}^4 \frac{e^{\phi_i^2 t/\pi}}{\vartheta_{11}(\nu_i, it)}.$$

In the 0(time)-direction, we have a continuous distribution, so $Z \sim \frac{1}{\sqrt{8\pi^2\alpha' t}}$.
In the 1-direction, we have branes separated by a distance y, so $L_0 = \frac{y^2}{4\pi^2\alpha'} + ...$, so $Z_1 \sim e^{-ty^2/2\pi\alpha'}$.
Multiplying everything, the partition function becomes (potential)

$$V = -\int_0^\infty \frac{dt}{t}\frac{1}{\sqrt{8\pi^2\alpha' t}} e^{-ty^2/2\pi\alpha'} \prod_{i=1}^4 \frac{\vartheta_{11}(\nu_i', it)}{\vartheta_{11}(\nu_i, it)}$$

This is a complicated function of y. We will calculate it for large distances. If y is large, the dominant contribution to the integral comes from small t (due to the $e^{ty^2/2\pi\alpha'}$ factor). If we set $t = 0$ in the ϑ-function, we obtain a constant and the integral diverges. We will calculate the force, which is a physical quantity and define the potential on the integral, $V = -\int F dy$.

$$F = -\frac{dV}{dy} = -y\int_0^\infty \frac{dt}{\pi\alpha'}\frac{e^{-ty^2/2\pi\alpha'}}{\sqrt{8\pi^2\alpha' t}} \prod_{i=1}^4 \frac{\vartheta_{11}(\nu_i', it)}{\vartheta_{11}(\nu_i, it)}$$

From

$$\vartheta_{11}(\nu, it) = -2q^{1/8}\sin\pi\nu\prod_{m=1}^{4}(1-q^m)(1-zq^m)(1-z^{-1}q^m),$$

and

$$\vartheta_{11}(-i\nu/t, i/t) = -i\sqrt{t}e^{\pi\nu^2/t}\vartheta_{11}(\nu, it),$$

we obtain

$$\prod_{i=1}^{4}\frac{\vartheta_{11}(\nu_i', it)}{\vartheta_{11}(\nu_i, it)} = \prod_{i=1}^{4}\frac{\vartheta_{11}(-i\nu_i'/t, i/t)}{\vartheta_{11}(-i\nu_i/t, i/t)}.$$

As $t \to 0$, $q = e^{-2\pi/t} \to 0$, so $\Pi \to \prod_{i=1}^{4}\frac{\sin i\pi\nu_i'/t}{\sin i\pi\nu_i/t}$

$$\nu_i = i\phi t/\pi \to i\pi\nu_i/t = -\phi_i, \quad \Pi = \prod_{i=1}^{4}\frac{\sin\phi_i'}{\sin\phi_i}.$$

So

$$F \sim Cy\int_0^\infty \frac{dt}{\sqrt{t}}e^{-ty^2/2\pi\alpha'}, \;\; y\to\infty \quad const.: \frac{1}{\pi\alpha'\sqrt{8\pi^2\alpha'}}\prod_{i=1}^{4}\frac{\sin\phi_i'}{\sin\phi_i}.$$

and the potential is $V \sim Cy$.

10.4 Scattering

How do you make a D-brane move? Simple. Motion in e.g., the 1-direction is motion in Minkowski space (X^0, X^1) just like a rotation in Euclidean space (X^8, X^9) we studied above.

$$\begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \to \begin{pmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{pmatrix}\begin{pmatrix} X^0 \\ X^1 \end{pmatrix}, \qquad \begin{pmatrix} X^8 \\ X^9 \end{pmatrix} \to \begin{pmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix}\begin{pmatrix} X^8 \\ X^9 \end{pmatrix}.$$

where$X^1 = vX^0$ and the speed (v) is defined by the rapidity (ζ) as $v = \tanh\zeta$. The rapidity is related to the velocity via

$$\cosh\zeta = \frac{1}{\sqrt{1-v^2}}, \qquad \sinh\zeta = \frac{v}{\sqrt{1-v^2}}.$$

Consider two parallel Dp-branes moving with relative velocity v in the X^1-direction and separated by a distance y in the z-direction (branes are perpendicular to bot X^1 and X^2). In the 01-plane (Minkowski), we may copy our earlier result with the substitution $\phi = -i\zeta$: The bosonic part of the partition function is

$$Z_{bosonic\ (01)} = -i\frac{e^{\phi^2 t/\pi}\eta(it)}{\vartheta_{11}(\nu, it)}, \qquad \phi = -i\zeta,\;\; \nu = i\phi t/\pi - \zeta t/\pi.$$

The fermionic part is

$$Z_{ab} = \frac{\vartheta_{ab}(\nu, it)}{e^{\phi^2 t/\pi}\eta(it)}.$$

In the rest of the direction, the D-branes are parallel, so all other angles are zero. Therefore, the fermionic piece is

$$Z_f = \frac{1}{2\eta^4(it)} e^{-\phi^2 t/\pi} \left[\vartheta_{00}(\nu, it)\vartheta_{00}^3(0, it) - \vartheta_{10}(\nu, it)\vartheta_{10}^3(0, it)\right.$$

$$\left. -\vartheta_{01}(\nu, it)\vartheta_{01}^3(0, it) - \vartheta_{11}(\nu, it)\vartheta_{11}^3(0, it)\right]$$

This may be computed by applying the generalized abstruse identity. We have

$$\phi_1 = \phi, \;\; \phi_2 = \phi_3 = \phi_4 = 0,$$

so

$$\phi_1' = \phi_2' = \phi_3' = \phi_4' = \frac{1}{2}\phi$$

and therefore

$$Z_f = \frac{1}{2\eta^4(it)} e^{-\phi^2 t/\pi} \vartheta_{11}^4(\frac{1}{2}\phi, it).$$

The bosonic piece in the other directions ($X^2, X^3, ..., X^9$ total of eight ... six of which are transverse) is

$$Z_{\substack{bosonic \\ 2,3,\ldots,9}} = V_p \left(\frac{1}{\sqrt{8\pi^2\alpha' t}}\right)^p e^{-ty^2/2\pi\alpha'} (\eta(it))^{-6}$$

Therefore the partition function is

$$Z = -iV_p \int_0^\infty \frac{dt}{t} (8\pi^2\alpha' t)^{-p/2} e^{-ty^2/2\pi\alpha'} \frac{\vartheta_{11}^4(\nu/2, it)}{\vartheta_{11}(\nu, it)} (\eta(it))^{-9}, \quad \nu = \zeta t/\pi$$

As the branes move the distance changes to $r^2 = y^2 + v^2\tau^2$. The potential may be extracted from

$$Z = -1 \int_{-\infty}^{\infty} d\tau V[r(\tau), v].$$

We easily obtain

$$V(r, v) = i\frac{2V_p v}{(\sqrt{8\pi^2\alpha'})^{p+1}} \int_0^\infty dt \; t^{(5-p)/2} e^{-tr^2/2\pi\alpha'} \frac{\vartheta_{11}^4(i\zeta/2\pi, i/t)}{\eta^9(i/t)\vartheta_{11}(i\zeta/\pi, i/t)},$$

where we used the modular properties of the ϑ and η functions.
Note: as $v \to 0, \; u \to 0$, so $V \to 0$.
Since

$$\vartheta_{11}(\nu, it) = -2q^{1/8} \sin\pi\nu \prod (1-q^m)(1-zq^m)(1-z^{-1}q^m), \quad \eta(it) = q^{1/24} \prod (1-q^m)$$

we have, as $v \to 0,\ \ \nu \to 0,\ \ Z \to 1.$

$$\frac{\vartheta_{11}^4(i\zeta/2\pi, i/t)}{\eta^9(i/t)\vartheta_{11}(i\zeta/\pi, i/t)} = \frac{8i\sinh^4(\zeta/2)}{\sinh(\zeta)} + ... = \frac{1}{2}v^3 + ..., \qquad \zeta \to v$$

So

$$\begin{aligned} V(r,v) &= -\frac{2V_p v^4}{(\sqrt{8\pi^2\alpha'})^{p+1}}\int_0^\infty dt\ t^{(5-p)/2}e^{-tr^2/2\pi\alpha'} + o(v^6) \\ &\sim -\frac{v^4}{r^{7-p}}\frac{V_p}{\alpha'^{p-3}} \end{aligned}$$

Problem: as $r \to 0, V \to \infty$! How can string theory claim finiteness at short distances (r is real distance - not bogus!)?
Answer: Let $r \to 0$ *before* expanding in v. r only appears in $e^{-tr^2/2\pi\alpha'}$. If we rescale $t \to t/r^2$, the $r \to 0$ corresponds to *large* t. If t is large in ϑ, η, then

$$\frac{\vartheta_{11}^4}{\eta^9\vartheta_{11}} \to \frac{\sinh^4\left(\frac{vt}{4}\right)}{\sinh(vt)}, \quad \zeta \sim v$$

From the exponential, $t \sim 2\pi\alpha'/r^2$ dominates. $ut \sim 2\pi\alpha' u/r^2$, so in the limit that $r \to 0$, ut becomes large and the integral oscillates rapidly. OScillation on a scale $ut \sim 1$, i.e., $2\pi\alpha' u \sim r^2$, i.e., $r \sim \sqrt{\alpha' v}$. This is the effective scale probed by the brane: $r_0 = \sqrt{\alpha' v}$. A slow brane $(v \to 0)$ probes scales smaller than the string scale! Moreover, we obtain an uncertainty in the position

$$\delta x \geq \sqrt{\alpha' v}.$$

The time it takes for this scattering process is

$$\delta t \sim \delta x / v$$

Therefore,

$$\delta x \delta t \geq \frac{\delta x}{v}\sqrt{\alpha' v} \simeq \frac{\alpha' v}{v} = \alpha'.$$

A new uncertainty principle! It implies that coordinates do **not** commute! It seems that Nature is described by noncommutative geometry. What can this possibly mean??
Consider two branes separated by a distance y. Strings ending on the same brane have a massless mode each, so we have two massless modes. A string stretched between the two branes has

$$L_0 = \frac{y^2}{4\pi^2\alpha'} + ...$$

This extra term makes $L_0 > 0$ i.e., there are no massless modes. At low energies, we only see two massless particles from the two branes. However, when $y \to 0$, the stretched string develops a massless mode, and there are two of

them. So when the two branes coincide, there are four massless modes. These four modes can be grouped into a matrix X_{ij} in an obvious notation.
Each X_{ij} is the position of the brane! When we develop a particle theory we need to treat the position of the brane as a 2×2 matrix. More generally, n distinct branes have n massless modes. The particle theory is just n copies of the same theory. When all n branes coincide, we have n^2 massless modes. Each massless mode corresponds to a symmetry of the theory (U(1)). With n^2 massless states the symmetry is enhanced to U(n) (n^2 generators).
Familiar Examples
Photon: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $U(1)$ symmetry ($A_\mu \rightarrow A_\mu + \partial_\mu \lambda$).
3 Photons: $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$, $U(1)^3$ symmetry.
Weak Bosons: Demand $SU(2)$ symmetry which has three generators, so $F^i_{\mu\nu} \neq \partial_\mu A^i_\nu = \partial_\nu A^i_\mu$. There is a correction, to obey the enhanced symmetry $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ (A_μ is a matrix ... $A_\mu = A^i_\mu \sigma_i$)
Gluons: Demand $SU(3)$ - eight gluons $(3^2 - 1)$.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad A_\mu = A^i_\mu \lambda_i$$

where λ_i represent the Gell-Mann matrices. The action is given by

$$S \sim \int d^4x \ \mathrm{Tr}\ F_{\mu\nu} F^{\mu\nu}.$$

If we only had eight copies of electromagnetism, we would have

$$S \sim \int d^4x \ \mathrm{Tr}\ F^i_{\mu\nu} F^{\mu\nu}_i.$$

Now we have interactions between gluons - enhanced symmetry (gluons and weak bosons, unlike photons have charge).
Potential: Set A_μ =constant, then

$$\mathrm{Tr}\ F_{\mu\nu} F^{\mu\nu} \sim \mathrm{Tr}\ [A_\mu, A_\nu]^2.$$

Back to D-branes: X^μ is like a^μ (that can be made precise - see Polchinski **8.6**). So the enhanced symmetry contains a potential

$$V \sim \mathrm{Tr}\ [X^\mu, X^\nu]^2$$

where μ, ν run over that dimension transverse to the branes. Expand around $X^\mu = 0$ in a Taylor series. There are no linear or quadratic terms in X^μ, so there is no mass term (which would come from $V(\phi) = V(0) + V'(0)\phi + \frac{1}{2}V''(0)\phi^2/m^2 + ...$)
So we have kn^2 massless modes, where k is the number of transverse dimensions. Also, $V = 0$ if and only if all $[X_m, X_n] = 0$, i.e., all X_m commute. This can be accomplished if we make them all diagonal. There are n diagonal elements, each corresponding to one of the D-branes. Thus this potential correctly describes n coincident non-interacting free D-branes.