String Theory I

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UNIT 1

A first look at strings

Following *"String Theory"* by J. Polchinski, Vol.I.
Notes written by students (work still in progress).
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1.1 Units

First, we must explain the unit convention we are going to use. Take the following two results from Quantum Mechanics and Special Relativity:

$$E = \hbar\omega \tag{1.1.1}$$

$$E = mc^2 \tag{1.1.2}$$

These two equations link energy to frequency and mass through some constant of proportionality. The question is, are these constants fundamental in nature or created by man? The answer is that they are artificial creations, existing purely because of the units we have chosen to work in. We could easily choose units such that $\hbar = c = 1$. By doing this, the number of fundamental units in the universe is reduced to 1, e.g., energy, all others being related to it:

$$[\text{energy}] = [1/\text{time}] = [\text{mass}] = [1/\text{length}] \tag{1.1.3}$$

1.2 Why Strings?

Our motivation behind the developement of String Theory is our desire to find a unified theory of everything. One of the major obstacles that previous theories have been unable to overcome is the formation of a quantum theory of gravity, and it is in this respect that String Theory has had notable success

(in fact, at present, String theory is the only theory which includes gravitational interactions). This leads us to believe that, while String Theory may not be the final answer, it is certainly a step in the right direction.
Let us first discuss the problems one runs into when trying to create a quantum theory of gravity using Quantum Field Theory as our guide. Take the Hydrogen atom, whose energy levels are given by:

$$E_n = -\frac{E_1}{n^2} \ , \ E_1 = \frac{\hbar^2}{2m_e a_0} \ , \ a_0 = \frac{\hbar^2}{m_e e^2} \tag{1.2.1}$$

where E_1 is the ground state energy and n labels the energy levels.
Suppose we had only the most basic knowledge of physics: what would we guess the energy of the Hydrogen atom to be? The parameters of the system are the mass of the electron m_e, the mass of the proton m_p, $\hbar$ and the electron charge e. We may neglect m_p as we are interested in the energy levels of the electron. We might guess the energy to be

$$E_0 = m_e c^2 \tag{1.2.2}$$

Equations (1.2.1) and (1.2.2) are clearly not the same, but if we take the ratio we obtain

$$\frac{E_1}{E_0} = \frac{e^4}{\hbar^2 c^2} = \left(\frac{1}{137}\right)^2 = \alpha^2 \tag{1.2.3}$$

This is a ratio, so is independent of our choice of units, so α is a fundamental constant that exists in nature independent of our attempts to decribe the world, and indicates some fundamental physics underlying the situation. In fact α is the fine structure constant and describes the probability for an electromagnetic interaction, e.g. proton - electron scattering (of which hydrogen is a special case in which the scattering results in a bound state).
From the diagram of an e-p interaction,

INSERT FIGURE HERE

each vertex contributes a factor e to the amplitude for the interaction, so that

$$\mathcal{A}_1 \equiv \text{Amplitude} \sim \text{e}^2 \tag{1.2.4}$$

$$\text{Probability} \sim |\text{Amplitude}|^2 \sim \text{e}^4 \sim \alpha^2 \tag{1.2.5}$$

Now according to classical analysis, this is the only amplitude we would get for the interaction, but in quantum mechanics there can be intermediate scattering events that cannot be observed: e.g.,

INSERT FIGURE HERE

contributes $o(\alpha^2)$ to the overall amplitude for the interaction. Inserting a complete set of states, we obtain the amplitude of this second-order process in terms of $\mathcal{A}_1$,

$$\mathcal{A}_2 \sim \int \frac{dE'}{E'} |\mathcal{A}_1|^2 \sim \alpha^2 \int \frac{dE'}{E'} \tag{1.2.6}$$

This is logarithmically divergent. However, all higher-order amplitudes have the same divergence and when we sum the series in α:

$$\text{Amplitude} = () + \alpha() + \alpha^2() + ... \tag{1.2.7}$$

it yields finite expressions for physical quantities.

Let us try it for the gravitational interaction between two point masses, each of mass M, separated by a distance r. The potential energy is:

$$V = \frac{GM^2}{r} \tag{1.2.8}$$

which shows upon comparison with electromagnetism that the "charge" of gravity is $e_g \sim \sqrt{G}M$. A gravitational "Hydrogen atom" will have energy levels

$$E_n \sim \frac{E_1}{n^2} \ , \ E_1 \sim \frac{Me_g^4}{2\hbar^2} \sim G^2M^5 \tag{1.2.9}$$

Comparing with $E_0 = Mc^2$, we obtain the ratio

$$\frac{E_1}{E_0} \sim G^2M^4 \sim e_g^4 \tag{1.2.10}$$

Immediately we see problems with using this charge to describe the gravitational interaction, because e_g is energy (mass) dependent. The classical scattering amplitude is

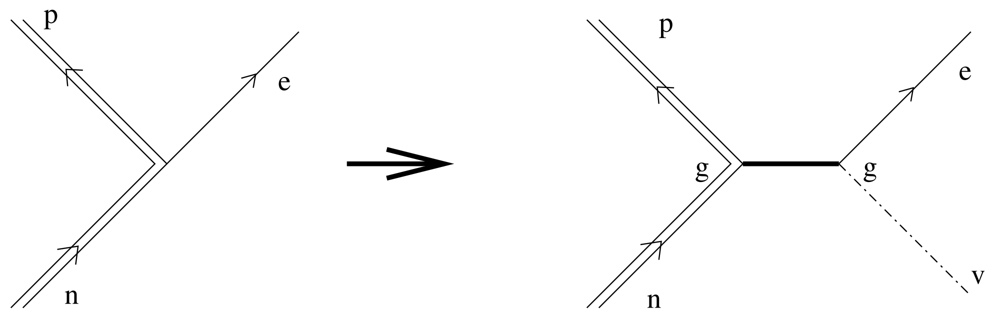
$$\mathcal{A}_1 \sim e_g^2 \sim GE^2 \tag{1.2.11}$$

where $E = Mc^2$ and the second-order contribution (exchange of two gravitons) is

$$\mathcal{A}_2 \sim \int \frac{dE'}{E'}|\mathcal{A}_1|^2 \sim G^2 \int dE'(E')^3 \tag{1.2.12}$$

which has a quartic divergence. Worse yet, higher-order amplitudes have worse divergences, making it impossible to make any sense of the perturbative expansion (1.2.7) (*non-renormalizability* of gravity).

We may see our way to a possible solution by considering the problem of beta decay: Initially it was treated as a three body problem with the proton - neutron - electron interaction occuring at one vertex. When the energies of the resultant electrons did not match experiment, the theory was modified to include a fourth particle, the neutrino, and the interaction was 'smeared' out: the proton and neutron interacted at one vertex, where a W boson was created, which traveled a short distance before reaching the electron - neutrino vertex.



So maybe we can solve our problems with quantum gravity by smearing out the interactions, so that the objects mediating the force are no longer point particles but extended one dimensional objects - strings.

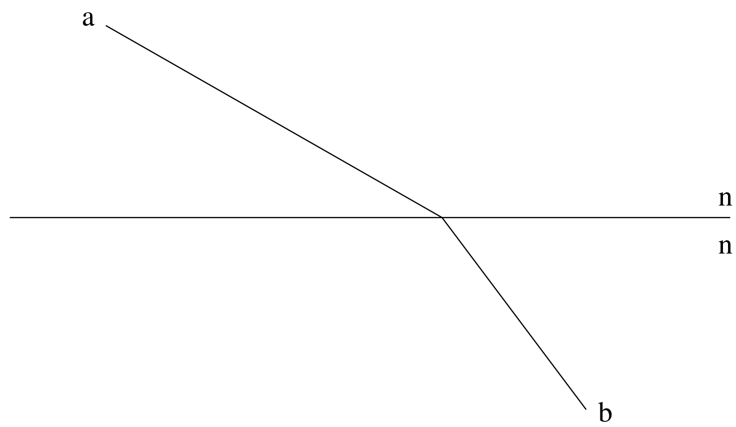
This is the general concept from which we will proceed. It is a difficult task - the gravitational interaction must obey a much larger symmetry than Lorentz invariance, it must be invariant under completely general co-ordinate transformations, and we must of course still be able to describe the weak, strong and electromagnetic interactions.

In this chapter we will take a first look at strings. Initially we examine the completely general equations of motion for a point particle using the method of least action, and then apply that method to the case of a general string moving in D dimensions. We will obtain the equations of motion for the string, and then attempt to quantize it and obtain its energy spectrum. This will highlight some basic results of string theory, as well as some fundamental difficulties.

1.3 Point particle

We begin by examining the case of a point particle, illustrating the method we will use for strings. The trajectory of a point particle in D-dimensional space is decribed by coordinates $X^{\mu}(\tau)$, where τ is a parameter of the particle's trajectory. For a massive particle, τ is its proper time. X^0 will be a timelike cordinate, the remaining $\vec{X}$ spanning space. Infinitesimal distances in spacetime:

$$-dT^2 = ds^2 = -(dX^0)^2 + (d\vec{X})^2 \tag{1.3.1}$$



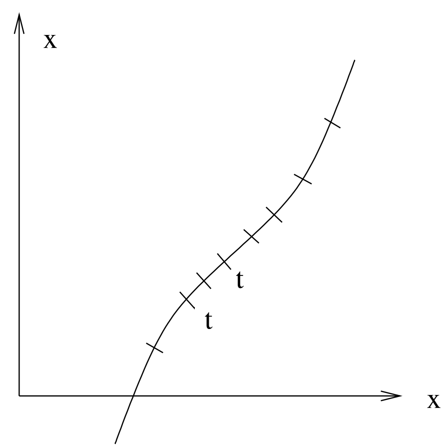
We wish to derive the equation of motion from an action principle. This is similar to Fermat's principle of minimizing time along a light ray. For the light ray joining points A and B, this yields Snell's Law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{1.3.2}$$

Along the trajectory of a free relativistic particle, proper time is maximized. Therefore, trajectories are obtained as extrema of the action

$$S = m \int_a^b dT \tag{1.3.3}$$

where we multiplied by the mass to obtain a dimensionless quantity (in units where $\hbar = 1$).



a and b are the fixed start and end points on the trajectory. There is a problem that when the mass $m = 0$, since the action is zero. This problem will be cleared up a little later.

We can write the action as

$$S = m\int_a^b d\tau \frac{dT}{d\tau} = m\int_a^b d\tau \sqrt{\left(\frac{dX^0}{d\tau}\right)^2 - \left(\frac{d\vec{X}}{d\tau}\right)^2} \tag{1.3.4}$$

(invariant under reparametrizations $\tau \to \tau'(\tau)$) from which we can define the Lagrangian for the system:

$$L = m\sqrt{\left(\frac{dX^0}{d\tau}\right)^2 - \left(\frac{d\vec{X}}{d\tau}\right)^2} \tag{1.3.5}$$

Using the convention $\dot{X} = dX/d\tau$, we write Lagrange's equations:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) = \frac{\partial L}{\partial X^\mu} \tag{1.3.6}$$

For this Lagrangian,

$$\frac{\partial L}{\partial X^\mu} = 0 \tag{1.3.7}$$

and we obtain the equation of motion for the point particle:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) = \frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{L}\right) = 0 \tag{1.3.8}$$

To see the physical meaning of this equation, switch from τ to X^0 (or set $\tau = X^0$ to "fix the gauge"). Then the 3-velocity is

$$v^i = \frac{dX^i}{dX^0} \tag{1.3.9}$$

and the equation of motion (1.3.8) reads

$$\ddot{u}^\mu = 0 \;,\; u^\mu = \gamma(1, \vec{v}) \;,\; \gamma = \frac{1}{L} = \frac{1}{\sqrt{1-\vec{v}^2}} \tag{1.3.10}$$

i.e., that the acceleration is constant, as expected.
We will now look at a better expression for the action: defining an extra field $\eta(\tau)$, which at the moment is arbitrary, we write a new Lagrangian:

$$L = \frac{1}{2\eta}\dot{X}^\mu \dot{X}_\mu - \frac{1}{2}\eta m^2 \tag{1.3.11}$$

This has the nice feature that it is still valid for $m = 0$. Lagrange's equation for η is

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\eta}}\right) = 0 = \frac{\partial L}{\partial \eta} \tag{1.3.12}$$

which implies

$$-\frac{1}{2\eta^2}\dot{X}^\mu \dot{X}_\mu - \tfrac{1}{2}m^2 = 0 \;\;\Rightarrow\;\; \eta = \sqrt{\frac{-\dot{X}^\mu \dot{X}_\mu}{m^2}} \tag{1.3.13}$$

Using this equation to eliminate η from the Lagrangian (1.3.11), we get back the original Lagrangian (1.3.5). Varying X^μ ,we obtain from (1.3.11)

$$\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\eta}\right) = 0 \tag{1.3.14}$$

which agrees with the previous eq. (1.3.8).
We will now examine the meaning of the field $\eta(\tau)$. The trajectory is parameterized by some co-ordinate of the system in terms of which an infinitesimal distance along the trajectory, ds, can be expressed. Let us suppose our trajectory is along the y axis. Then it would be easiest to parameterize the system with the co-ordinate y, and then $ds = dy$. We could, however, choose the parameter to be θ, the the angle between a line drawn from a fixed point on the x axis at a distance ℓ to a position on the y axis. Then our new distance would be

$$ds = \ell \left(\frac{1}{\cos^2\theta}\right) d\theta \tag{1.3.15}$$

The factor $\ell/cos^2\theta$ is our η^2. It represents the geometry of the system due to our choice of co-ordinates. In Minkowski space,

$$ds^2 = -\gamma_{\tau\tau} d\tau^2 \;\;,\;\; \gamma_{\tau\tau} = \eta^2 \tag{1.3.16}$$

where $\gamma_{\tau\tau}$ is the (single component) metric tensor. Under a reparametrization,

$$\tau \to \tau'(\tau) \;\;,\;\; \gamma_{\tau\tau} \to \left(\frac{d\tau}{d\tau'}\right)^2 \gamma_{\tau\tau} \tag{1.3.17}$$

i.e. $\gamma_{\tau\tau}$ transforms as a tensor (this follows from the invariance of ds^2). We can see that the action is invariant under this transformation: we have

$$\eta' = \frac{d\tau}{d\tau'}\eta \tag{1.3.18}$$

and $\dot{X}^\mu$ transforms as

$$\dot{X}^{\mu\prime} = \frac{dX^\mu}{d\tau'} = \frac{d\tau}{d\tau'}\frac{dX^\mu}{d\tau} \tag{1.3.19}$$

Thus the Lagrangian transforms as

$$L' = \frac{d\tau'}{d\tau} L \tag{1.3.20}$$

and the action transforms as

$$S = m\int_a^b d\tau L = m\int_a^b d\tau' L' \tag{1.3.21}$$

thus proving its invariance.
Now let us form the Hamiltonian for the system: the conjugate momenta to co-ordinate X^μ and the parameter η are

$$P_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{\dot{X}^\mu}{\eta} \ , \ P_\eta = \frac{\partial L}{\partial \dot{\eta}} = 0 \tag{1.3.22}$$

Then the Hamiltonian is:

$$H = P_\mu \dot{X}^\mu + P_\eta \dot{\eta} - L = \tfrac{1}{2}\eta(P^\mu P_\mu + m^2) \tag{1.3.23}$$

Here the role of η is that of a Lagrange multiplier; it is not a dynamical variable. From Hamilton's equation:

$$\frac{\partial H}{\partial \eta} = \dot{P}_\eta = 0 = P^\mu P_\mu + m^2 \tag{1.3.24}$$

which is Einstein's equation for the relativistic energy of a paricle of mass m.
Define χ as:

$$\chi = \frac{1}{2m} P^\mu P_\mu + \tfrac{1}{2} m = \frac{H}{\eta m} \tag{1.3.25}$$

Then $\chi = 0$ is a constraint which generates reparametrizations through Poisson brackets:

$$\delta X^\mu \sim \{X^\mu \, , \, \chi\} = \frac{P^\mu}{m} \ , \ \delta P^\mu \sim \{P^\mu \, , \, \chi\} = 0 \tag{1.3.26}$$

We may identify X^0 with time and solve for its conjugate momentum

$$P_0 = \sqrt{\vec{P}^2 + m^2} \tag{1.3.27}$$

This is the true Hamiltonian of the system. Equations of motion:

$$\dot{X}^i = \frac{\partial P_0}{\partial P_i} = \frac{P^i}{P_0} \ , \ \dot{P}_i = 0 \tag{1.3.28}$$

same equation as before, if we note $v^i = \dot{X}^i$, $1 - \vec{v}^2 = m^2/P_0^2$ and therefore,

$$\frac{\dot{v}^i}{\sqrt{1 - \vec{v}^2}} = \frac{P^i}{m} \tag{1.3.29}$$

This system may be quantized by

$$[P_i \, , \, X^j] = -i\delta_i^j \tag{1.3.30}$$

Eigenstates of the Hamiltonian:

$$H|\vec{k}\rangle = \omega|\vec{k}\rangle \ , \ \omega = \sqrt{\vec{k}^2 + m^2} \tag{1.3.31}$$

Alternatively, we may define light-cone coordinates in spacetime:

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^1) \qquad \vec{X}_T = (X^2, \dots, X^{D-1}) \tag{1.3.32}$$

The Lagrangian reads

$$L = \frac{1}{2\eta}\dot{X}^\mu \dot{X}_\mu - \frac{1}{2}\eta m^2 = -\frac{1}{\eta}\dot{X}^+\dot{X}^- + \frac{1}{2\eta}\dot{\vec{X}}_T^2 - \frac{1}{2}\eta m^2 \tag{1.3.33}$$

Let $X^+ = \tau$ play the role of time; then P_+ is the Hamiltonian. X^- and $\vec{X}_T$ are the coordinates and P_- and $\vec{P}_T$ are their conjugate momenta. The Lagrangian becomes:

$$L = -\frac{1}{\eta}\dot{X}^- + \frac{1}{2\eta}\dot{X}_i^2 - \frac{1}{2}\eta m^2 \tag{1.3.34}$$

yielding

$$P_- = \frac{\partial L}{\partial \dot{X}^-} = -\frac{1}{\eta} \tag{1.3.35}$$

$$P_i = \frac{\partial L}{\partial \dot{X}^i} = \frac{1}{\eta}\dot{X}_i \tag{1.3.36}$$

The Hamiltonian is

$$P_+ = \dot{X}^- P_- + \dot{X}^i P_i - L = \frac{\vec{P}_T^2 + m^2}{2P_-} \tag{1.3.37}$$

Note that there is no term $P_+\dot{X}^+$ because X^+ is not a dynamical variable in the gauge-fixed theory.
Quantization:

$$[P_i\,,\, X^j] = -i\delta_i^j \ , \ [P_-\,,\, X^-] = -i \tag{1.3.38}$$

Eigenstates of the Hamiltonian:

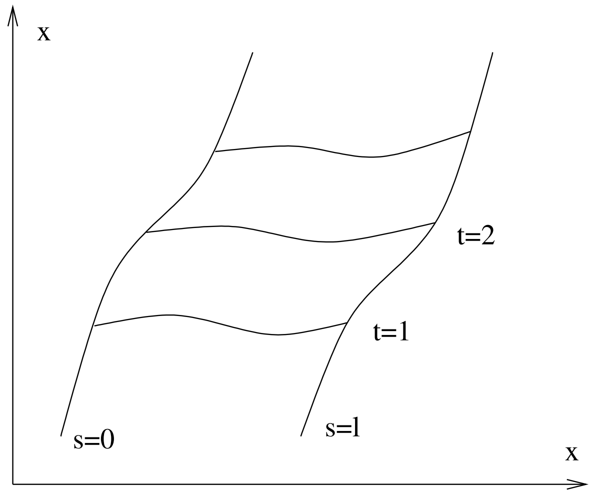
$$P_+|k_-, \vec{k}_T\rangle = \omega_+|k_-, \vec{k}_T\rangle \ , \ \omega_+ = \frac{\vec{k}_T^2 + m^2}{2k_-} \tag{1.3.39}$$

To compare with our earlier result, define ω and k_1 by

$$\omega_+ = \frac{1}{\sqrt{2}}\,(\omega + k_1) \ , \ k_- = \frac{1}{\sqrt{2}}\,(\omega - k_1) \tag{1.3.40}$$

Then $\omega^2 - \vec{k}^2 = \omega^2 - k_1^2 - \vec{k}_T^2 = 2\omega_+ k_- - \vec{k}_T^2 = m^2$.

1.4 Strings



Parametrize the string with $\sigma \in [0, \ell]$. In analogy to Fermat's minimization of time, we will minimize the area of the world-sheet mapped out by the string. To find an expression for the area, pick a point on the worldsheet and draw the tangent vectors

$$\vec{t}_\tau = \frac{\partial \vec{X}}{\partial \tau} = \dot{\vec{X}} \ , \ \vec{t}_\sigma = \frac{\partial \vec{X}}{\partial \sigma} = \vec{X}' \tag{1.4.1}$$

The infinitesimal parallelepiped with sides $\vec{t}_\tau d\tau$ and $\vec{t}_\sigma d\sigma$ has area

$$dA = |\vec{t}_\tau \times \vec{t}_\sigma| d\tau d\sigma \tag{1.4.2}$$

We obtain the total area by integrating over the worldsheet coordinates (τ, σ). The action to be minimized is the Nambu-Goto action

$$S_{NG} = T \int dA \tag{1.4.3}$$

where T is a constant that makes the action dimensionless (tension of the string). Using

$$(\vec{a} \times \vec{b})^2 = \vec{a}^2 \vec{b}^2 - (\vec{a} \cdot \vec{b})^2 = \begin{vmatrix} \vec{a}^2 & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b}^2 \end{vmatrix} \tag{1.4.4}$$

we deduce

$$S_{NG} = T \iint d\tau d\sigma \sqrt{-\det\ h_{ab}} \quad , \ h_{ab} = \partial_a X^\mu \partial_b X_\mu \tag{1.4.5}$$

where the minus sign in the square root is because we are working in Minkowski space. h_{ab} is the two-dimensional metric induced on the worldsheet,

$$h_{ab} = \begin{pmatrix} \dot{\vec{X}}^2 & \dot{\vec{X}} \cdot \vec{X}' \\ \dot{\vec{X}} \cdot \vec{X}' & \vec{X}'^2 \end{pmatrix} \tag{1.4.6}$$

and the Lagrangian density is

$$L = \sqrt{-\det h_{ab}} = \sqrt{(\dot{\vec{X}} \cdot \vec{X}')^2 - \dot{\vec{X}}^2 \vec{X}'^2} \tag{1.4.7}$$

where we ignored T (or set $T = 1$).

$$\sigma^a = (\tau, \sigma) \qquad a = 0, 1 \tag{1.4.8}$$

Lagrange Equation:

$$\partial_a \frac{\partial L}{\partial(\partial_a X^\mu)} = 0 \;\Rightarrow\; \left(\frac{\partial L}{\partial \dot{X}^\mu}\right)^{\!\cdot} + \left(\frac{\partial L}{\partial X^{\mu\prime}}\right)' = 0 \tag{1.4.9}$$

We have

$$\frac{\partial L}{\partial \dot{X}^\mu} = \frac{\dot{X}_\mu \vec{X}'^2 - X'_\mu \dot{\vec{X}} \cdot \vec{X}'}{L} \;,\; \frac{\partial L}{\partial X^{\mu\prime}} = \frac{X'_\mu \dot{\vec{X}}^2 - \dot{X}_\mu \dot{\vec{X}} \cdot \vec{X}'}{L} \tag{1.4.10}$$

Let us choose the worldsheet coordinates (τ, σ) so that the metric h_{ab} becomes proportional to the two-dimensional Mikowski metric,

$$h_{ab} \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.4.11}$$

Upon comparison with (1.4.6), we deduce the constraints

$$\dot{\vec{X}}^2 + \vec{X}'^2 = 0 \;,\; \dot{\vec{X}} \cdot \vec{X}' = 0 \tag{1.4.12}$$

which may also be cast into the form

$$(\dot{\vec{X}} \pm \vec{X}')^2 = 0 \tag{1.4.13}$$

Using the constraints, the Lagrange eq. (1.4.9) reduces to

$$\frac{\partial^2 X_\mu}{\partial \tau^2} = \frac{\partial^2 X_\mu}{\partial \sigma^2} \tag{1.4.14}$$

which is the wave equation. Introducing coordinates

$$\sigma^\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma) \tag{1.4.15}$$

so that

$$\partial_\pm = \frac{1}{\sqrt{2}}(\partial_\tau \pm \partial_\sigma) \tag{1.4.16}$$

the constraints and the wave equation become, respectively,

$$(\partial_\pm \vec{X})^2 = 0 \;,\; \partial_+ \partial_- X^\mu = 0 \tag{1.4.17}$$

The general solution to the wave equation is

$$X^\mu = f(\sigma^+) + g(\sigma^-) \tag{1.4.18}$$

where f and g are arbitrary functions. Let us write out the action explicitly with the simplifications made above. The parameter τ takes values from -∞ to +∞ and σ takes values between 0 and l, the length of the string. Anticipating future results, we write the constant T

$$T = \frac{1}{2\pi\alpha'} \tag{1.4.19}$$

where α' is called the Regge slope. Then, from equation (81),

$$S_{NG} = \frac{1}{4\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma(\dot{X}^2 - \acute{X}^2) \tag{1.4.20}$$

$$= \frac{1}{4\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma(\partial_a X^\mu \partial^a X_\mu) \tag{1.4.21}$$

We now examine the effects of boundary conditions which have yet to be taken into account in the equations of motion. We start off by varying the co-ordinates:

$$X^\mu \to X^\mu + \delta X^\mu \tag{1.4.22}$$

Starting from equation (93), the variation in the action is

$$\delta S_{NG} = \frac{1}{4\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma\left(\partial_a \delta X^\mu \partial^a X_\mu + \partial_a X^\mu \partial^a \delta X_\mu\right) \tag{1.4.23}$$

$$= \frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma\left(\partial_a \delta X^\mu \partial^a X_\mu\right) \tag{1.4.24}$$

And noting the total derivative

$$\partial_a(\delta X^\mu \partial^a X_\mu) = \partial_a \delta X^\mu \partial^a X_\mu + \delta X^\mu \partial_a \partial^a X_\mu \tag{1.4.25}$$

The last term is just the wave equation, which equals zero, so we are left with:

$$\delta S_{NG} = \frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^l d\tau d\sigma \partial_a(\delta X^\mu \partial^a X_\mu) \tag{1.4.26}$$

$$= \frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty} d\tau \left[\delta X^\mu \acute{X}_\mu\right]_0^l \tag{1.4.27}$$

We will now introduce two sets of boundary conditions that will get rid of this term and leave the equations of motion unchanged at the boundary:
Open string (Neumann) boundary conditions, which correspond to there being no forces at the boundary:

$$\acute{X}_\mu(\sigma = 0) = \acute{X}_\mu(\sigma = l) = 0 \tag{1.4.28}$$

Closed string boundary conditions, which means there is no boundary and the string co-ordinates are periodic:

$$X_\mu(\sigma = 0) = X_\mu(\sigma = l) \tag{1.4.29}$$

$$\acute{X}_\mu(\sigma = 0) = \acute{X}_\mu(\sigma = l) \tag{1.4.30}$$

We shall now look at deriving invariant quantities in the theory from symmetries using Noether's theorem. We will start with Poincare invariance, which is invariant under the transformation

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + Y^\mu \tag{1.4.31}$$

where Λ and Y are constant quantities. We construct the Noether current by applying this symmetry to the action. Taking the second term, we write the change in X^μ as

$$\delta X^\mu = Y^\mu \tag{1.4.32}$$

The change in the action is found by inserting this into equation (96):

$$\delta S_{NG} = \frac{1}{2\pi\alpha'} \int_{-\infty}^{+\infty} \int_0^l d\tau d\sigma \partial_a Y^\mu \partial^a X_\mu \tag{1.4.33}$$

The Noether current P^a_μ is defined by

$$\delta S = \int \int d\tau d\sigma \partial_a Y^\mu P^a_\mu \tag{1.4.34}$$

So in this case,

$$P^a_\mu = T\partial^a X_\mu \tag{1.4.35}$$

and

$$\partial_a P^a_\mu = T\partial_a \partial^a X_\mu = 0 \tag{1.4.36}$$

i.e. the Poincare symmetry has led to a conserved quantity in the Noether current.
Now let's do the same for the variation

$$\delta X^\mu = \epsilon \Lambda^\mu{}_\nu X^\nu \tag{1.4.37}$$

where ϵ is a small quantity. The variation in the action is now

$$\delta S_{NG} = \frac{1}{2\pi\alpha'} \int_{-\infty}^{+\infty} \int_0^l d\tau d\sigma \left(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu\right) \Lambda_{\mu\nu} \partial^a \epsilon \tag{1.4.38}$$

From this we can define the current as

$$J^{\mu\nu}_a = T(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu) \tag{1.4.39}$$

which is conserved:

$$\partial^a J_a^{\mu\nu} = T(\partial^a X^\mu \partial_a X^\nu - \partial^a X^\nu \partial_a X^\mu + X^\mu \partial^a \partial_a X^\nu - X^\nu \partial^a \partial_a X^\mu) = 0 \quad (1.4.40)$$

since the first two terms cancel and the last two terms are the wave equation, which equals zero.

Analogous to electromagnetism, we can define a charge. In EM, the charge is the integral over a volume of the zeroth (time) component of the current 4-vector j^μ, so here we define the charge for the current $P_\mu^a = (P_\mu^\tau, P_\mu^\sigma)$ as

$$P_\mu = \int_0^l d\sigma P_\mu^\tau \quad (1.4.41)$$

P_μ^τ can be seen to be the momentum of the string at a certain point, so $P_\mu, \mu > 0$ is the total momentum of the string, and P_0 is the total energy of the string.

Differentiating P_μ with respect to time:

$$\frac{dP_\mu}{d\tau} = \int_0^l d\sigma \dot{P}_\mu^\tau = \int_0^l d\sigma P_\mu^{\prime\sigma} = P_\mu^\sigma \bigg|_0^l = 0 \quad (1.4.42)$$

where the second equality follows from the wave equation and the last equality form the boundary conditions at 0 and l. This is simply conservation of momentum.

Similarly, for the current $J_{\mu\nu}^a$, which we interpret as the angular momentum of the string, we define the charge

$$J_{\mu\nu} = \int_0^l d\sigma J_{\mu\nu}^a \quad (1.4.43)$$

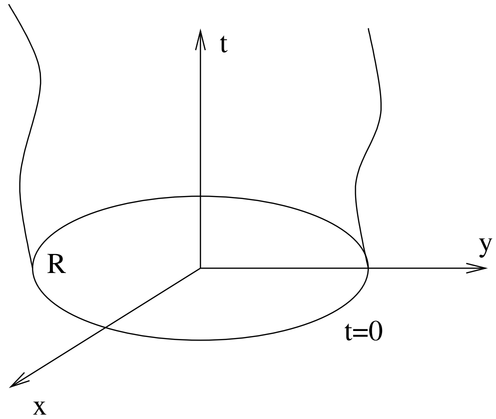
and we find

$$\frac{dJ_{\mu\nu}}{d\tau} = 0 \quad (1.4.44)$$

So from Poincare invariance we obtain conservation of momentum and angular momentum.

We now give two examples to demonstrate the above concepts:

Example 1



We take a closed string whose initial configuration is a circle centered on the x-y origin with radius R, and whose initial velocity is $\vec{v} = 0$. Then $X^\mu = (t, x, y)$. The solution to the wave equation satisfying these boundary conditions is:

$$x = R\cos\frac{2\pi\tau}{l}\cos\frac{2\pi\sigma}{l} \tag{1.4.45}$$

$$y = R\cos\frac{2\pi\tau}{l}\sin\frac{2\pi\sigma}{l} \tag{1.4.46}$$

$$t = \frac{2\pi R}{l}\tau \tag{1.4.47}$$

Let's check the constraints $\dot{X}^2 + \acute{X}^2 = 0$:

$$-\dot{t}^2 + \dot{x}^2 + \dot{y}^2 - \acute{t}^2 + \acute{x}^2 + \acute{y}^2 \tag{1.4.48}$$

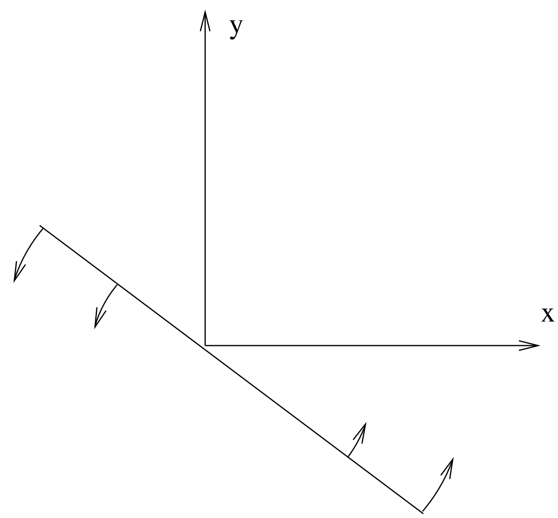
$$= -\left(\frac{2\pi R}{l}\right)^2 + \left(\frac{2\pi R}{l}\right)^2 \sin^2\frac{2\pi\tau}{l} + \left(\frac{2\pi R}{l}\right)^2 \cos^2\frac{2\pi\tau}{l} = 0 \tag{1.4.49}$$

The total energy of the string is given by

$$P_0 = \int_0^l d\sigma P_0^\tau = T\int_0^l d\sigma \partial_\tau X_0 = T\int_0^l d\sigma \partial_\tau t = \frac{2\pi RT}{l}\int_0^l d\sigma = 2\pi RT = E \tag{1.4.50}$$

The length of the string is $2\pi R$, so T can be identified as energy per unit length of the string, i.e. the tension.

Example 2



Now we consider an open string rotating in the x-y plane. The solution to the wave equation is

$$x = R\cos\frac{\pi\tau}{l}\cos\frac{\pi\sigma}{l} \tag{1.4.51}$$

$$y = R\cos\frac{\pi\tau}{l}\sin\frac{\pi\sigma}{l} \tag{1.4.52}$$

$$t = \frac{\pi R}{l}\tau \tag{1.4.53}$$

The speed of each point on the string is given by

$$\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \frac{l}{\pi R}\left(\frac{dx}{d\tau}, \frac{dy}{d\tau}\right) = \cos\frac{\pi\sigma}{l}\left(-\sin\frac{\pi\tau}{l}, \cos\frac{\pi\tau}{l}\right) \tag{1.4.54}$$

From which we see that

$$\vec{v}^2 = \cos^2\frac{\pi\sigma}{l} \tag{1.4.55}$$

Thus at the ends of the string $\sigma = 0, l$ we see that $\vec{v}^2 = 1$, i.e. the ends of the string travel at the speed of light. This is to be expected, since the string is massless and there are no forces on the ends of the string (Neumann boundary conditions). The intermediate points on the string don't travel at the speed of light because they experience the tension of the string.
The energy of the string is worked out exactly as before, and is found to be:

$$P_0 = TR\pi \tag{1.4.56}$$

Let us now work out the z component of the angular momentum of the string:

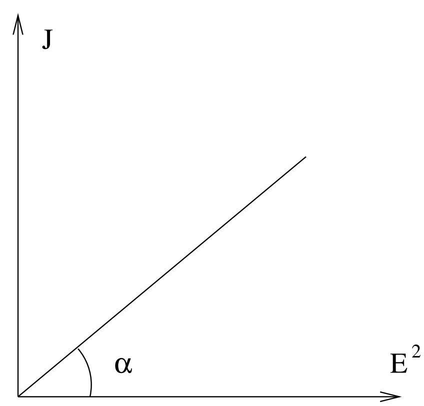
$$J_x y = T\int_0^l d\sigma(x\partial_a y - y\partial_a x) \tag{1.4.57}$$

$$= T\int_0^l d\sigma\frac{\pi R^2}{l}\left(\cos^2\frac{\pi\tau}{l}\cos^2\frac{\pi\sigma}{l} + \sin^2\frac{\pi\tau}{l}\cos^2\frac{\pi\sigma}{l}\right) \tag{1.4.58}$$

$$= \frac{T\pi R^2}{l}\int_0^l d\sigma\cos^2\frac{\pi\sigma}{l} = \tfrac{1}{2}T\pi R^2 \tag{1.4.59}$$

From the total energy and angular momentum we can form the quantity

$$\frac{J_{xy}}{E^2} = \frac{1}{2\pi T} = \alpha' \tag{1.4.60}$$



Regge Slope

Now if we want our strings to correspond to fundamental particles, their angular momenta correspond to the spins of the particles, which are either integer or half integer. Thus the above relation suggests that if we plot the spins of particles against their energies squared, we would observe a straight line. This was indeed observed for strongly interacting particles. In fact, string theory started out being a theory of the strong interaction, but then QCD came along. Now string theory become a theory of everything!

1.5 The Mode Expansion

We must first introduce the Hamiltonian formalism for future reference: we have the Lagrangian

$$L = \tfrac{1}{2}T(\dot{X}^2 - \acute{X}^2) \tag{1.5.1}$$

The conjugate momentum to the variable X is

$$\Pi_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = T\dot{X}_\mu \tag{1.5.2}$$

Then the Hamiltonian is given by

$$H = \int_0^l d\sigma(\Pi_\mu \dot{X}^\mu - L) \tag{1.5.3}$$

$$= \tfrac{1}{2}T\int_0^l d\sigma\left(\frac{\Pi^2}{T^2} + \acute{X}^2\right) \tag{1.5.4}$$

$$= \tfrac{1}{2}T\int_0^l d\sigma(\dot{X}^2 + \acute{X}^2) = 0 \tag{1.5.5}$$

So this is not a good Hamiltonian. We will find a good one later.
Now let us examine the mode expansion for open strings, which obey the Neumann boundary conditions. The general solution to the wave equation is a fourier expansion. Here we write such an expansion as follows:

$$X^\mu = x^\mu + 2\alpha'\frac{\pi}{l}p^\mu\tau + i\sqrt{2\alpha'}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\frac{1}{n}\alpha_n^\mu e^{\frac{-\pi i n\tau}{l}} cos\left(\frac{n\pi\sigma}{l}\right) \tag{1.5.6}$$

where x^μ is a constant and the first mode p^μ has been written out explicitly. The constants have been chosen on dimensional grounds
The fact that X^μ must be a real number yields the condition:

$$\alpha_n = (\alpha_{-n})^* \tag{1.5.7}$$

The conjugate momentum to X^μ is

$$\Pi^\mu = T\dot{X}^\mu = 2\alpha' T\frac{\pi}{l}p^\mu + \frac{\alpha' T}{l}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\alpha_n^\mu e^{\frac{-\pi i n\tau}{l}} cos\left(\frac{n\pi\sigma}{l}\right) \tag{1.5.8}$$

The centre of mass position of the string is given by

$$\bar{X}^\mu = \frac{1}{l}\int_0^l d\sigma X^\mu = x^\mu + 2\alpha'\frac{\pi}{l}p^\mu\tau \tag{1.5.9}$$

so the centre of mass moves in a straight line. The total momentum is

$$P^\mu = \int_0^l d\sigma\Pi^\mu = 2\alpha' l\frac{\pi}{l}p^\mu T = p^\mu \tag{1.5.10}$$

In both cases all the harmonic terms vanish on integration.
We now move to the light cone gauge mentioned before: we defined the transverse co-ordinates X^+ and X^- and fix the gauge by imposing the condition:

$$X^+ = x^+ + 2\alpha'\frac{\pi}{l}p^+\tau \qquad \alpha_n^+ = 0 \quad \forall n \tag{1.5.11}$$

and set

$$2\alpha'\frac{\pi}{l}p^+ = 1 \qquad (1.5.12)$$

so that

$$X^+ = x^+ + \tau \qquad (1.5.13)$$

which is the light cone gauge condition from before with an arbitrary constant x^+.
The centre of mass position can now be written

$$\bar{X}^\mu = x^\mu + \frac{p^\mu}{p^+}\tau \qquad (1.5.14)$$

The constraint from before:

$$(\dot{X} \pm \acute{X})^2 = 0 \qquad (1.5.15)$$

$$-2(\dot{X}^+ \pm \acute{X}^+)(\dot{X}^- \pm \acute{X}^-) + (\dot{X}^i \pm \acute{X}^i)^2$$

$$\begin{aligned} &= -((\dot{X}^0 + \dot{X}^1) \pm (\acute{X}^0 + \acute{X}^1))((\dot{X}^0 - \dot{X}^1) \pm (\acute{X}^0 - \acute{X}^1)) + (\dot{X}^i \pm \acute{X}^i)^2 \\ &= -((\dot{X}^0 \pm \acute{X}^0) + (\dot{X}^1 \pm \acute{X}^1))((\dot{X}^0 \pm \acute{X}^1) - (\dot{X}^1 \pm \acute{X}^1)) + (\dot{X}^i \pm \acute{X}^i)^2 \\ &= -(\dot{X}^0 \pm \acute{X}^0)^2 + (\dot{X}^1 \pm \acute{X}^1)^2 + (\dot{X}^i \pm \acute{X}^i)^2 = (\dot{X}^\mu \pm \acute{X}^\mu)^2 = 0 \end{aligned} \qquad (1.5.16)$$

$$\therefore 2(\dot{X}^+ \pm \acute{X}^+)(\dot{X}^- \pm \acute{X}^-) = (\dot{X}^i \pm \acute{X}^i)^2 \qquad (1.5.17)$$

$$\therefore 2(\dot{X}^- \pm \acute{X}^-) = (\dot{X}^i \pm \acute{X}^i)^2 \qquad (1.5.18)$$

The mode expansion of the world-sheet fields:

$$X^\mu = x^\mu + 2\alpha'\frac{\pi}{l}p^\mu\tau + i\sqrt{2\alpha'}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\frac{1}{n}\alpha_n^\mu e^{\frac{-\pi i n\tau}{l}}\cos\left(\frac{n\pi\sigma}{l}\right) \qquad (1.5.19)$$

$$\dot{x}^- \pm \acute{x}^- == 2\alpha'\frac{\pi}{l}p^- + \sqrt{2\alpha'}\frac{\pi}{l}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\alpha_n^- e^{\frac{-\pi i n}{l}(\tau\pm\sigma)} \qquad (1.5.20)$$

$$\therefore (\dot{x}^i \pm \acute{x}^i)^2 = \left[2\alpha'\frac{\pi}{l}p^i + \sqrt{2\alpha'}\frac{\pi}{l}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\alpha_n^i e^{\frac{-\pi i n}{l}(\tau\pm\sigma)}\right]^2 \qquad (1.5.21)$$

You can write α_n^- in terms of α_n^i:

$$(\dot{X}^i \pm \acute{X}^i)^2 = \frac{p^i p^i}{p^+ p^+} + \frac{\pi^2}{l^2} 2\alpha' \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i + 2\frac{\pi^2}{l^2}(2\alpha')^{3/2} p^i f(\sigma,\tau) \quad (1.5.22)$$

Concentrating on the zeroth mode:

$$2\frac{p^-}{p^+} = \frac{p^i p^i}{(p^+)^2} + \frac{1}{\alpha'(p^+)^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i \quad (1.5.23)$$

$$\therefore p^- = \frac{1}{2p^+}\left[p^i p^i + \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i\right] = H \quad (1.5.24)$$

From Einstein's equation:

$$p^\mu p_\mu + m^2 = 0 \quad (1.5.25)$$

$$m^2 = -p^\mu p_\mu = 2p^+ p^- - p^i p^i = \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^i \alpha_n^i \quad (1.5.26)$$

Now quantizing the system:

$$\left[p^i, x^j\right] = -i\delta^{ij} \qquad \left[\Pi^i(\sigma), X^j(\sigma')\right] = -i\delta^{ij}\delta(\sigma-\sigma') \quad (1.5.27)$$

$$\Pi_\mu = T\dot{X}_\mu = \frac{p^\mu}{l} + \sqrt{2\alpha'}\frac{\pi}{l}T \sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty} \alpha_n^\mu e^{\frac{-\pi i n\tau}{l}} \cos\left(\frac{n\pi\sigma}{l}\right) \quad (1.5.28)$$

$$X^\mu = x^\mu + 2\alpha'\frac{\pi}{l}p^\mu\tau + i\sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty} \frac{1}{n}\alpha_n^\mu e^{\frac{-\pi i n\tau}{l}} \cos\left(\frac{n\pi\sigma}{l}\right) \quad (1.5.29)$$

$$\left[\Pi^i(\sigma), X^j(\sigma')\right] = \frac{1}{l}\left[p^i, x^j\right] + \frac{i}{l} \sum_{m,n\neq 0} \frac{1}{n}\left[\alpha_m^i, \alpha_n^j\right] e^{\frac{-\pi i\tau}{l}(n+m)} \cos\left(\frac{n\pi\sigma}{l}\right)\cos\left(\frac{m\pi\sigma'}{l}\right) \quad (1.5.30)$$

$$\left[\alpha_m^i, \alpha_n^j\right] = m\delta^{ij}\delta_{m+n,0} \quad (1.5.31)$$

$$\therefore \quad \left[\Pi^i(\sigma), X^j(\sigma')\right] = -i\delta^{ij}\delta(\sigma-\sigma') \quad (1.5.32)$$

This is just the commutation relation for the harmonic oscillator operators with nonstandard normalization

$$a = \frac{1}{\sqrt{m}}\alpha_m^i \qquad a^\dagger = \frac{1}{\sqrt{m}}\alpha_- m^i \quad (1.5.33)$$

$$\therefore \quad \left[a^\dagger, a\right] = 1 \quad (1.5.34)$$

The state $|0,k\rangle$ is defined to be annihilated by the lowering operators and to be an eigenstate of the center-of-mass momenta

$$a|0\rangle = a|0,k\rangle = 0 \tag{1.5.35}$$

$$\Pi_{m,i}|0,k\rangle = |0,k\rangle \tag{1.5.36}$$

$$M^2 = \frac{1}{\alpha'}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\sum_{\substack{i=-\infty\\i\neq 0}}^{\infty}\alpha^i_{-n}\alpha^i_n = \frac{1}{\alpha'}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\sum_{\substack{i=-\infty\\i\neq 0}}^{\infty} m a^\dagger a = \frac{1}{\alpha'}N \tag{1.5.37}$$

$$M^2|0,k\rangle = \frac{1}{\alpha'}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{(D-2)n}{2}|0,k\rangle = \frac{(D-2)}{2\alpha'}\left(\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} n\right)|0,k\rangle \tag{1.5.38}$$

We have to perform the sum:

$$\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} n \tag{1.5.39}$$

To perform this sum, we will multiply by the sum by $e^{\frac{-2\pi n\epsilon}{l}}$ and then take the limit of $\epsilon \to 0$

$$\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} n e^{\frac{-2\pi n\epsilon}{l}} = \frac{\partial}{\partial C}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} n e^{-nC} = \frac{\partial}{\partial C}\left(\frac{1}{1-e^{-C}}\right) = \frac{e^{-C}}{(1-e^{-C})^2} = \frac{1}{C^2} - \frac{1}{12} \tag{1.5.40}$$

$$\text{where } \; C = \frac{2\pi\epsilon}{l} \tag{1.5.41}$$

$$\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} n e^{-nC} = \frac{1}{c^2} - \frac{1}{12} \tag{1.5.42}$$

$$M^2 = \frac{1}{\alpha'}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\sum_{\substack{i=-\infty\\i\neq 0}}^{\infty}\alpha^i_{-n}\alpha^i_n = \frac{(D-2)}{2\alpha'}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} n = 2p^+p^- - p^ip^i \tag{1.5.43}$$

The last equality was obtained using(64)

$$M^2 = \frac{(D-2)}{2\alpha'}\left(\frac{1}{c^2} - \frac{1}{12}\right) = \frac{(D-2)}{2\alpha'}\frac{l^2}{(2\pi\epsilon)^2} - \frac{(D-2)}{24\alpha'} = \frac{(D-2)}{(2\pi\epsilon)^2}\pi p^+ l - \frac{(D-2)}{24\alpha'} = 2p^+p^- - p^ip^i \tag{1.5.44}$$

The mass of each state is thus determined in terms of the level of excitation.

$$M^2 = \frac{1}{\alpha'}\left(N - \frac{D-2}{24}\right) \qquad N|0\rangle = 0 \tag{1.5.45}$$

This operator acting on the 0 ket yields:

$$M^2|0\rangle = -\frac{(D-2)}{24\alpha'}|0\rangle \tag{1.5.46}$$

The mass-squared is negative for $D > 2$. The state is a tachyon

The lowest excited states of the string are obtained by exciting one of the $n = 1$ modes once:

$$M^2(\alpha^i_{-1}|0\rangle) = \frac{1}{\alpha'}\left(1 - \frac{D-2}{24}\right)(\alpha^i_{-1})|0\rangle = \frac{1}{\alpha'}\left(\frac{26-D}{24}\right)|0\rangle \tag{1.5.47}$$

Lorentz invariance now requires that this state be massless, so the number of spacetime dimensions is $D = 26$

1.6 Closed strings

We must now look at the mode expansion and quantization os closed strings, which are required when looking at string interactions. The procedure is very similar to that of open strings, except now we have Dirichlet boundary conditions, i.e. X^μ is periodic. The mode expansion is now made up of left and right moving parts:

$$X^\mu = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma) \tag{1.6.1}$$

such that

$$X^\mu_R = \tfrac{1}{2}x^\mu + \alpha' p^\mu(\tau - \sigma) + i\sqrt{\tfrac{1}{2}\alpha'}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\frac{1}{n}\alpha^\mu_n e^{\frac{-2\pi i n(\tau-\sigma)}{l}} \tag{1.6.2}$$

$$X^\mu_L = \tfrac{1}{2}x^\mu + \alpha' p^\mu(\tau + \sigma) + i\sqrt{\tfrac{1}{2}\alpha'}\sum_{\substack{n=-\infty \\ x\neq 0}}^{\infty}\frac{1}{n}\tilde{\alpha}^\mu_n e^{\frac{-2\pi i n(\tau+\sigma)}{l}} \tag{1.6.3}$$

The sum of these is periodic. In the sum the integer n is in effect 2n so there are now twice as many modes as for the open string. Once again, the fact that X^μ is real means that

$$\alpha^\mu_n = (\alpha^\mu_{-n})^* \qquad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^* \tag{1.6.4}$$

Quantizing as before:

$$\left[\alpha^i_m, \alpha^j_n\right] = \left[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n\right] = m\delta^{ij}\delta_{m+n,0} \tag{1.6.5}$$

$$\left[\alpha^i_m, \tilde{\alpha}^j_n\right] = 0 \tag{1.6.6}$$

The mass operator is now

$$M^2 = 2p^+p^- - p^ip^i = \frac{2}{\alpha'}\left(N_R + N_L + \frac{2(D-2)}{24}\right) = \frac{2}{\alpha'}\left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - \frac{2(D-2)}{24}\right) \tag{1.6.7}$$

There are two symmetries in these expansions: the transformations

$$\tau \to \tau + \text{constant} \tag{1.6.8}$$

$$\sigma \to \sigma + \text{constant} \tag{1.6.9}$$

don't change the physics of the string. The first symmetry is shared with the open string, but the spatial translational symmetry is new.
The generator of the time translations is H, which we saw before to be

$$H = \int_0^l d\sigma(\dot{X}^2 + \acute{X}^2) \tag{1.6.10}$$

and the fact that H = 0 leads to the invariance under time translations. In the case of closed strings,

$$H = \int_0^l d\sigma[(\dot{X}_R + \dot{X}_L)^2 + (\acute{X}_R - \acute{X}_L)^2] \tag{1.6.11}$$

$$\sim \int_0^l d\sigma(\dot{X}_R^2 + \dot{X}_L^2) = H_R + H_L \tag{1.6.12}$$

where we have used the fact that we can write

$$\partial_\sigma \sim \partial_\tau \tag{1.6.13}$$

since τ and σ are interchangeable in the expansions of X_R and X_L to within a minus sign.
This generation of time translations comes from the constraint $\dot{X}^2 + \acute{X}^2 = 0$, so it is reasonable to suppose that spatial translations come from the other constraint $\dot{X} \cdot \acute{X} = 0$. Defining the operator D:

$$D = \int_0^l d\sigma \dot{X} \cdot \acute{X} \sim \int_0^l d\sigma \dot{X}^2 = \int_0^l d\sigma(\dot{X}_R^2 - \dot{X}_L^2) \sim N_R - N_L = 0 \tag{1.6.14}$$

So

$$N_R = N_L \tag{1.6.15}$$

which common sense tells us must be the case.

Now let us determinant the lowest states in the mass spectrum. For the vacuum state

$$M^2|0\rangle = -\frac{4(D-2)}{24\alpha'}|0\rangle \tag{1.6.16}$$

so the mass squared is negative for the vacuum state, as for the open string. The first excited state is $|\Omega^{ij}\rangle = \tilde{\alpha}^i_{-1}\alpha^j_{-1}|0\rangle$, where we must remember to keep $N_R = N_L$. We obtain

$$M^2|\Omega^{ij}\rangle = \frac{2}{\alpha'}\left(\frac{2-2(D-2)}{24}\right)|\Omega^{ij}\rangle \tag{1.6.17}$$

As before, we wish $M^2 = 0$ for the first excited state, so again we get $D = 26$. The situation is a bit more complicated than for the open string case, so let's look in a bit more detail.

The state $|\Omega^{ij}\rangle$ can be split into three parts: a symmetric, traceless part; an antisymmetric part and a scalar part:

$$|\Omega^{ij}\rangle = \left[\tfrac{1}{2}(|\Omega^{ij}\rangle + |\Omega^{ij}\rangle) - \frac{2}{D-2}\delta^{ij}|\Omega^{kk}\rangle\right] + \tfrac{1}{2}(|\Omega^{ij}\rangle - |\Omega^{ij}\rangle) + \frac{1}{D-2}\delta^{ij}|\Omega^{kk}\rangle \tag{1.6.18}$$

We call the three states $|G^{ij}\rangle, |B^{ij}\rangle, |\Phi\rangle$. The symmetric, traceless, spin 2 state $|G^{ij}\rangle$ can now be identified with the graviton, which didn't exist in the open string theory, so it seems some progress has been made. The spin 0 scalar state is called the dilaton.

Now we impose a further symmetry: invariance under the transformation

$$\sigma \to -\sigma \tag{1.6.19}$$

which is the condition for unoriented strings. This means

$$X_R \leftrightarrow X_L \tag{1.6.20}$$

$$\alpha_n \leftrightarrow \tilde{\alpha}_n \tag{1.6.21}$$

This condition immediately disallows the antisymmetric state, since under the transformation,

$$|B^{ij}\rangle \to -|B^{ij}\rangle \tag{1.6.22}$$

while $|G^{ij}\rangle$ and $|\Phi\rangle$ remain unchanged.

Next let us turn our attention to the fact that we have been working in light cone gauge, which is not Lorentz invariant. We would like to reassert lorentz invariance. We can do this by generalizing the commutation identity:

$$\left[\alpha^i_m, \alpha^j_n\right] = m\delta^{ij}\delta_{m+n,0} \quad \to \quad [\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{1.6.23}$$

This gives

$$\left[\alpha_m^0, \alpha_n^0\right] = -m\delta_{m+n,0} \tag{1.6.24}$$

Now let us define a state

$$|\phi\rangle = \alpha_{-1}^0|0\rangle \tag{1.6.25}$$

The norm of this state is

$$\langle\phi|\phi\rangle = \langle 0|\alpha_1^0\alpha_{-1}^0|0\rangle = \langle 0|[\alpha_1^0, \alpha_{-1}^0]|0\rangle + \langle|\alpha_{-1}^0\alpha_1^0|0\rangle = -\langle 0|0\rangle = -1 \tag{1.6.26}$$

Thus we have a negative norm state, which is physically meaningless. This tells us there is something wrong with our theory. We will come back to this in the next couple of chapters, and solve this problem.

1.7 Gauge invariance

Let us examine the open string state

$$|\phi\rangle = A_\mu(k)\alpha_{-1}^\mu|0,k\rangle \tag{1.7.1}$$

under the transformation

$$A_\mu \to A_\mu + k_\mu\omega(k) \tag{1.7.2}$$

where $\omega(k)$ is an arbitrary function. This is analogous to a gauge transformation in electromagnetism $\vec{A} \to \vec{A} + \nabla\omega$. The change in $|\phi\rangle$ is

$$|\delta\phi\rangle = k_\mu\omega\alpha_{-1}^\mu|0,k\rangle \tag{1.7.3}$$

and the norm is

$$\langle\delta\phi|\delta\phi\rangle = k_\mu k_\nu\omega^2\langle 0,k|\alpha_1^\mu\alpha_{-1}^\nu|0,k\rangle = k^2\omega^2 = 0 \tag{1.7.4}$$

since k = 0 (the mass is zero, and $m^2 = k^\mu k_\nu$). Thus the theory has produced gauge invariance! The equivalent state for the closed string is

$$|\phi\rangle = g_{\mu\nu}\tilde{\alpha}_{-1}^\mu\alpha_1^\nu|0,k\rangle \tag{1.7.5}$$

The gauge transformation is

$$g_{\mu\nu} \to g_{\mu\nu} + k_\mu\omega_\nu + k_\nu\omega_\mu \tag{1.7.6}$$

This time we find the norm state to be

$$\langle\delta\phi|\delta\phi\rangle \sim k^2 = 0 \tag{1.7.7}$$

Thus gauge invariance holds for the closed string state too. What's more, we can identify $g_{\mu\nu}$ with the gravitational potential, a sign that general relativity might be included in the theory.

UNIT 2

Conformal Field Theory

2.1 Massless scalars in two dimensions

We will start by looking at the Polyakov action with one change. The metric has been replaced with a Euclidean metric $\delta_{a,b}$ with signature (+,+).

$$S = \frac{T}{2} \int d\tau d\sigma (\dot{X}^{\mu} \dot{X}_{\mu} + X'^{\mu} X'_{\mu})$$

where T, $\dot{X}^{\mu}$, X'^{μ} represent the string tension, $\partial_{\tau} X^{\mu}$, $\partial_{\sigma} X^{\mu}$ respectively.
We can derive the equation of motion by varying the action with respect to the coordinate X. We find:

$$\delta_X S = 0 \rightarrow X''_{\mu} + \ddot{X}^{\mu} = \nabla^2 X^{\mu} = 0, \qquad \text{where} \qquad \nabla^2 = \partial^2_{\sigma_1} + \partial^2_{\sigma_2}$$

We can define z and $\bar{z}$ as linear combinations of σ_1 and σ_2. These will represent the new worldsheet coordinates

$$z = \sigma_1 + i\sigma_2, \qquad \bar{z} = \sigma_1 - i\sigma_2.$$

The bar denotes complex conjugate. We can also invert the coordinate transformation

$$\sigma_1 = \frac{z + \bar{z}}{2}, \qquad \sigma_2 = \frac{z - \bar{z}}{2}.$$

Define differentiation:

$$\partial_z = \partial = \frac{1}{2}(\partial_1 + i\partial_2) \qquad \text{and} \qquad \partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_1 - i\partial_2)$$

∇^2 can be written as $4\partial\bar{\partial}$, and the volume element is given by

$$d^2 z = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} d\sigma d\tau = 2 d\sigma d\tau.$$

Also define

$$\int d^2z\ \delta^2(z,\bar{z}) = 1$$

so that $\delta(\sigma^1)\delta(\sigma^2) = \delta^2(z,\bar{z})$. The action in complex coordinates is

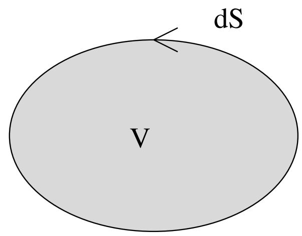
$$S = T\int d^2z \partial X^\mu \bar{\partial} X_\mu\ .$$

After varying the action with respect to the coordinate X_μ, we get the equation of motion

$$\partial\bar{\partial}X^\mu = 0.$$

This implies $\bar{\partial}X$ is a holomorphic function, ie a function of z. Also, ∂X is an antiholomorphic function, ie a function of $\bar{z}$.

Another useful result is the divergence theorem for complex coordinates. First, let's look at the divergence theorem for three dimensions i.e., electrostatics. The divergence theorem states that for any well behaved vector field $\mathbf{E}(\mathbf{x})$ defined within a volume V surrounded by the closed surface S the relation



$$\int_V d^3x\ \boldsymbol{\nabla}\cdot\boldsymbol{E} = \oint_S \boldsymbol{E}\cdot\boldsymbol{n}\ da$$

holds between the volume integral of the divergence of $\mathbf{E}$ and the surface integral of the outwardly directly normal component of $\mathbf{E}$.

In 2D:

$$\int d^2z\ (\partial E^z + \bar{\partial}E^{\bar{z}}) = i\oint_{\partial R}(E^z dz - E^{\bar{z}}d\bar{z}) \qquad \text{where} \qquad \hat{n} = (-d\bar{z}, dz).$$

We can now write the mode expansions in terms of the complex coordinates.

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{z-\bar{z}}{2}\frac{p^\mu}{p^+} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\{\alpha^\mu_n e^{\frac{2n\pi i z}{\ell}} + \tilde{\alpha}^\mu_n e^{\frac{-2n\pi i\bar{z}}{\ell}}\}$$

As one can see, this mode expansion can be broken into two pieces (left and right handed).

$$X^\mu(z,\bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z})$$

We see the left handed piece cooresponds to a holomorphic function and the right handed piece to a antiholomorphic function. We will never look at the mode expansion itself. Instead we will always look at various derivatives of the mode expansion.

$$\partial X_L^\mu = \frac{i}{2}\frac{p^\mu}{p^+} - \frac{\pi}{\ell}\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu \exp\left[\frac{in\pi z}{\ell}\right],$$

$$\bar{\partial} X_R^\mu = \frac{i}{2}\frac{p^\mu}{p^+} - \frac{\pi}{\ell}\sqrt{2\alpha'}\sum_{n\neq 0}\tilde{\alpha}_n^\mu \exp\left[\frac{in\pi \bar{z}}{\ell}\right]$$

We can absorb the p^μ by defining α_0^μ a certain way.

$$\text{Let } \alpha_0^\mu = -i\frac{\ell}{2\pi\sqrt{2\alpha'}}\frac{p^\mu}{p^+}$$

Now the derivatives on the fields simplify.

$$\partial X_L^\mu = -\frac{\pi}{\ell}\sqrt{2\alpha'}\sum_{all\ n}\alpha_n^\mu \exp\left[\frac{in\pi z}{\ell}\right],\ \ \bar{\partial} X_R^\mu = -\frac{\pi}{\ell}\sqrt{2\alpha'}\sum_{all\ n}\tilde{\alpha}_n^\mu \exp\left[\frac{in\pi \bar{z}}{\ell}\right]$$

2.2 Solution to the Boundary-Value Problem with Green function

The solution to the Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on the bounding surface S can be obtained by means of Greens theorem. In general, we want to solve the equation,

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -\mathbf{4}\pi\delta(\mathbf{x}-\mathbf{x}')$$

where G(x,x') is the potential and the delta function is a point source. The solution for G is given as

$$G(\mathbf{x},\mathbf{x}') = \frac{\mathbf{1}}{|\mathbf{r}-\mathbf{r_0}|} + \mathbf{F}(\mathbf{x},\mathbf{x}')$$

with F satisfying the Laplace equation inside the volume V:

$$\nabla'^2 F(\mathbf{x},\mathbf{x}') = \mathbf{0}$$

In two dimensions the Poisson equation is given by

$$\begin{aligned} \nabla^2 G(z,0;\bar{z},0) &= -2\pi\delta(z)\delta(\bar{z}) \qquad \text{where} \qquad \nabla^2 = \partial\bar{\partial} \\ &= -2\pi\delta^2(z,\bar{z}) \end{aligned}$$

where

$$G(z,\bar{z}) = \ln|z|^2 = \ln|z| + \ln|\bar{z}|$$

This is just the solution for the potential to a line charge in two dimensions. We can prove that G is a solution of the Poisson equation in problem 1.

$$\partial\bar{\partial}G = \partial\bar{\partial}\ln|z|^2 = \partial\frac{1}{\bar{z}} + \bar{\partial}\frac{1}{z} = 2\pi\delta^2(z,\bar{z}), \quad z = 0$$

Now that we see G is directly related to the potential, we can take the gradient to get the electric field.

$$E_z = \partial G, \quad E_{\bar{z}} = \bar{\partial}G$$

$$\int_V d^3x\ \nabla \cdot E = \oint_S E \cdot d\hat{n} = 2\pi R(\frac{1}{R}) = 2\pi$$

2.3 Amplitudes

We are interested in calculating vacuum expectation values

$$\langle 0|X^{\mu_1}(z_1)X^{\mu_2}(z_2)\cdots X^{\mu_n}(z_n)|0\rangle.$$

Let us start with the simplest nontrivial example, the two-point amplitude.

The two-point amplitude

We will focus only on the left-movers.

$$X^\mu = X^\mu_L = i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n \exp\left[\frac{-2\pi i n z}{\ell}\right].$$

$$\begin{aligned} A^{\mu\nu} &= \langle 0|X^\mu(z)X^\nu(z')|0\rangle \\ &= -\frac{\alpha'}{2}\sum_{n,m\neq 0}\frac{1}{nm}\exp\left[\frac{-2\pi i(n-m)z}{\ell}\right]\langle 0|\alpha^\mu_n\alpha^\nu_m|0\rangle \end{aligned}$$

In order evaluate the vacuum expectation value, break the sum into four pieces: $m,n>0$; $m>0,n<0$; $m<0,n>0$; $m,n<0$. Only the $m<0,n>0$ terms survive, because the others either kill the vacuum or produce an inner product of orthogonal states. To evaluate, we express the operators in terms of their commutator. This is possible because the second term in the commutator kills the vacuum.

$$[\alpha^\mu_n, \alpha^\nu_m] = n\eta^{\mu\nu}\delta_{m+m,0}.$$

After applying the Kroneker delta our sum reduces to

$$A^{\mu\nu} = \frac{\alpha'}{2}\sum_{n>0}\frac{1}{n}\exp\left[-\frac{2\pi i n(z-z')}{\ell}\right].$$

This is a sum of the form $\sum \frac{\omega^n}{n}$, but only converges for $|\omega| < 1$.

$$\left|\exp\left[-\frac{2\pi i(z-z')}{\ell}\right]\right| = \left|\exp\left[\frac{2\pi(z-z')}{\ell}\right]\right| < 1 \Rightarrow z_2 - z_2' < 0, z_2 < z_2'$$

So we see the Xs must be time ordered to give a correct result. Therefore, when we calculate any two-point function, we must use the time ordered product of the two operators

$$\langle 0|X(z)X(z')|0\rangle \Rightarrow \langle 0|T[X(z)X(z')]|0\rangle.$$

We define time ordering as

$$T[X^\mu(z)X^\nu(z')] = \begin{cases} X^\mu(z)X^\nu(z') & \text{for } z_2 < z_2' \\ X^\nu(z)X^\mu(z') & \text{for } z_2 > z_2' \end{cases},$$

or in terms of the step function the product becomes

$$T[X^\mu(z)X^\nu(z')] = \theta(z_2' - z_2)X^\mu(z)X^\nu(z') + \theta(z_2 - z_2')X^\nu(z)X^\mu(z').$$

Evaluating the sum:

$$A^{\mu\nu} = \frac{\alpha'}{2}\eta^{\mu\nu}\sum_n \frac{e^n}{n} = \frac{\alpha'}{2}\eta^{\mu\nu}\ln|1 - e^{-\beta}|$$

By applying the D'Alembertian to $A^{\mu\nu}$, we show it is a two-point Green function

$$\begin{aligned}
\partial\bar{\partial}T[X^\mu(z)X^\nu(z')] &= (\partial^2 + \bar{\partial}^2)T[X^\mu(z)X^\nu(z')] \\
&= T[\partial_1^2 X^\mu X^\nu] + T[\partial_1(-\delta(z_2 - z_2')X^\mu X^\nu + \delta(z_2 - z_2')X^\mu X^\nu)] \\
&= T[\partial_1^2 X^\mu X^\nu] + T[\partial_2\delta(z_2 - z_2')[X^\mu, X^\nu]] + T[\partial_2 X^\mu X^\nu)] \\
&= T[\partial_1^2 X^\mu X^\nu] + \delta(z_2' - z_2)[\partial_2 X^\mu, X^\nu] + T[\partial_2^2 X^\mu, X^\nu] \\
&= T[(\partial_1^2 X^\mu + \partial_2^2 X^\mu)X^\nu] + \delta(z_2' - z_2)[\partial_2 X^\mu, X^\nu] \\
&= \delta(z_2' - z_2)[\partial_2 X^\mu, X^\nu] \\
&= \pi\alpha'\delta^2(z - z')\eta^{\mu\nu}
\end{aligned}$$

$A^{\mu\nu}$ must be of the form of a Green function, $A^{\mu\nu} = \eta^{\mu\nu}G(z, z')$.
For future reference X^μ will imply only the holomorphic piece, unless specified otherwise. We may express the time-ordered product in terms of the normal ordered product minus the singularity.

$$T[X^\mu X^\nu] =: X^\mu X^\nu : -\frac{\alpha'}{2}\ln|z - z'|\eta^{\mu\nu}$$

As $z \to z'$:

$$X^\mu(z)X^\nu(z') = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z - z'| + \sum_{k>0}^{\infty}\frac{(z - z')^k}{k!} : X^\nu\partial_k X^\mu(z') :$$

$$\sum_k \frac{1}{k!}\left(-\frac{1}{2}\frac{\alpha'}{2}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right)^k = \exp\left[-\frac{\alpha'}{4}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right]$$

Define an operator $\mathcal{O}$ such that $\mathcal{O} = X^\mu X^\nu$.

$$:\mathcal{O} := \exp\left[-\frac{\alpha'}{4}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right]\mathcal{O}$$

This needs to be inverted.

$$\mathcal{O} = \exp\left[\frac{\alpha'}{4}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right] : \mathcal{O} :$$

Now $\mathcal{O}$ should have no singularities. We have to define the product between two $\mathcal{O}$s. This product will have singularities, unless the product of the two is normal ordered. The product of two time ordered operators, $:\mathcal{O}_1::\mathcal{O}_2:$, has singularities, whereas the time ordered product, $:\mathcal{O}_1\mathcal{O}_2:$, contains none.

$$:\mathcal{O}_1[X]\mathcal{O}_2[Y] := \exp\left[-\frac{\alpha'}{2}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\nu}\right] : \mathcal{O}_1 :: \mathcal{O}_2 :$$

invert

$$:\mathcal{O}_1[X] :: \mathcal{O}_2[Y] := \exp\left[\frac{\alpha'}{2}\int dzdz'\eta^{\mu\nu}\ln|z-z'|\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta Y^\nu}\right] : \mathcal{O}_1\mathcal{O}_2 :$$

Example: Let $\mathcal{O}_1 = \mathcal{O}_2 = \partial X^\mu(z)\partial X_\mu(z) = T(z)$

$$: T(z) :: T(z') :=: \partial X^\mu(z)\partial X_\mu(z) :: \partial' X^\nu(z')\partial' X_\nu(z') :$$

There are two possible double contractions and four possible single contractions,

$$\begin{aligned} : T(z) :: T(z') : &= 2 * \frac{\alpha'}{2}^2 \eta^{\mu\nu}\eta_{\mu\nu}(\partial\partial'\ln|z-z'|)^2 - 4 * \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'| : \partial X_\mu \partial' X_\nu : + : T(z)T(z') : \\ &= \frac{\alpha'^2}{2}\frac{d}{(z-z')^4} - \frac{2\alpha'}{(z-z')^2} : T : -\frac{\alpha'}{z-z'} : \partial' T(z') : + : T(z)T(z') : \end{aligned}$$

Example: Let $\mathcal{O}_1 =: e^{ik_1 X(z)} :$, $\mathcal{O}_2 =: e^{ik_2 X(z')} :$, $\frac{\delta}{\delta X^\mu}\mathcal{O}_n = ik_{n\mu}\mathcal{O}_n$

$$\begin{aligned} : \mathcal{O}_1\mathcal{O}_2 : &= \exp\left[\frac{\alpha'}{2}\ln|z-z'|\eta^{\mu\nu}(ik_{1\mu})(ik_{2\nu})\right] : \mathcal{O}_1 :: \mathcal{O}_2 : \\ &= \exp\left[-\frac{\alpha'}{2}k_1\cdot k_2\ln|z-z'|\right] : \mathcal{O}_1 :: \mathcal{O}_2 : \\ &= (z-z')^{-\frac{\alpha'}{2}k_1\cdot k_2} : \mathcal{O}_1 :: \mathcal{O}_2 : \end{aligned}$$

$$\begin{aligned} \Rightarrow : \mathcal{O}_1 :: \mathcal{O}_2 : &= (z-z')^{\frac{\alpha'}{2}k_1\cdot k_2} : \mathcal{O}_1\mathcal{O}_2 : \\ &= (z-z')^{-\frac{\alpha'}{2}k_1\cdot k_2} : e^{i(k_1+k_2)\cdot X}(1+\mathcal{O}(z-z')) : \end{aligned}$$

In this example we see the time-ordered product for two vertex operators representing tachyons.

2.4 Noether's Theorem

For every symmetry in a theory, there must be some conserved current, ie

$$\partial_\mu j^\mu = 0 \Rightarrow \text{Symmetry(S)}.$$

We can integrate over the zeroth component of the conserved current to get the charge.

$$Q = \int d^3x \partial_0 j^0 \rightarrow \frac{dQ}{dt} = \int_R d^3x j^0 = \int d^3x \nabla \cdot \vec{j} = \int d\vec{s} \cdot \vec{j} = 0$$

Q generates transformations.

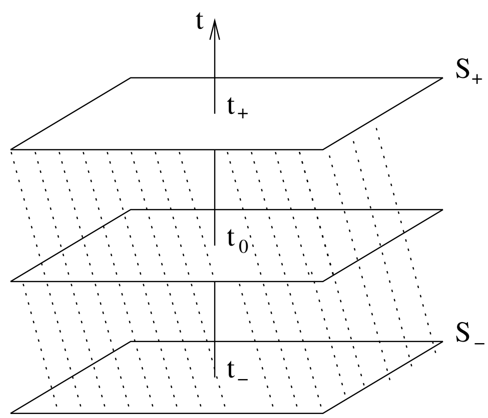
$$\delta A = i\epsilon[Q, A] = i\epsilon(QA - AQ) \qquad \epsilon << 1$$

Example: Let $Q = H$

$$\delta A = i\epsilon[H, A] = \dot{A}$$

or $Q = \vec{p}$

$$\vec{\nabla} A = i[\vec{p}, A]$$

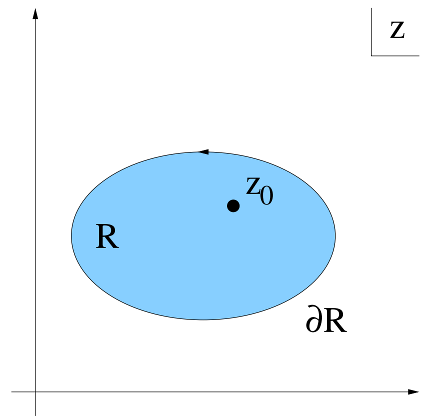


Look at $A(t_0)$:
Region R is bounded by $[S_+, S_-]$.

$$\begin{aligned}
\delta A &= i\epsilon(Q(t_+)A(t_0) - A(t_0)Q(t_-)) \\
&= i\epsilon\{\int_{S_+} d^3x\, j^0 A(t_0) - \int_{S_-} d^3x\, j^0 A(t_0)\} \\
&= i\epsilon \int_{\partial R} dS_\mu j^\mu A(t_0) \\
&= i\epsilon \int_{\partial R} dS_\mu T[j^\mu A(t_0)] \\
\delta A &= i\epsilon \int_R d^4x\, \partial_\mu T[j^\mu A(t_0)] \quad \text{"Ward Identity"}
\end{aligned}$$

Show the Ward Identity explicitly in two dimensions:

$$\begin{aligned}\delta A &= \frac{i\epsilon}{2\pi}\int_R d^2z\, \partial_a T[j^a A(z_0)], \quad j=(j_z, j_{\bar z}) \\ &= \frac{i\epsilon}{2\pi}\oint_{\partial R}(dz j_z - d\bar z j_{\bar z})A(z_0)\end{aligned}$$



$\partial_a j^a = \partial_{\bar z} j_z + \partial_z j_{\bar z} = 0$ for special case j_z is holomorphic, and $j_{\bar z}$ is antiholomorphic.

$$\oint \frac{dz}{2\pi} j_z A(z_0) = \lambda(z_0): \quad j_z A(z_0) = \ldots + \frac{\lambda(z_0)}{z-z_0} + \ldots$$

$$\delta A = \epsilon\lambda + \epsilon\bar\lambda = -\epsilon(\lambda - \bar\lambda)$$

Example: Define the transformation on X and the current density.

$$X^\mu \to x^\mu + \epsilon a^\mu : \quad j^\mu_a = \frac{i}{\alpha'}\partial_a A^\mu \quad j^\mu_z = \frac{i}{\alpha'}\partial X^\mu$$

$$\bar\partial j^\mu_z = 0 \qquad \text{remember X is holomorphic}$$

Let $A(z_0) =: e^{ik\cdot X}$:

$$\begin{aligned}\delta A &= \,:e^{ik\cdot(X+\epsilon a)}: \\ &= (1+i\epsilon k\cdot a)A(z_0)\end{aligned}$$

$$\begin{aligned}j^\mu_z A(z_0) &= \frac{i}{\alpha'}\frac{\alpha'}{2}\eta^{\mu\nu}\partial \ln|z-z_0|(ik_\nu) : A(z_0) : + \text{regular terms} \\ &= -\frac{1}{2}\frac{k^\mu}{z-z_0}A(z_0) \\ \Rightarrow & \quad \lambda = \frac{1}{2}k^\mu A(z_0) = -\bar\lambda : \quad \text{Residue}\end{aligned}$$

$$\epsilon(\lambda+\bar{\lambda})=\epsilon(k^{\mu}A(z_0))$$
$$\epsilon a^{\mu}(\lambda-\bar{\lambda})=\epsilon a^{\mu}k_{\mu}A(z_0)$$

Example: Let us look at translations.

$$\begin{aligned}\delta z=-\epsilon,\delta X^{\mu} &= X^{\mu}(z-\epsilon)-X^{\mu}(z)\\ &= -\epsilon\partial X_{\mu}\end{aligned}$$

Noether Current

$$S=-\frac{1}{2\pi\alpha'}\int\partial X^{\mu}\bar{\partial}X_{\mu}$$

$$\begin{aligned}\delta z=-\epsilon(z,\bar{z})\qquad \delta S &= \frac{1}{2\pi\alpha'}\int\partial(\epsilon\partial X^{\mu})\bar{\partial}X_{\mu}+\partial X^{\mu}\bar{\partial}(\epsilon\partial X_{\mu})\\ &= \frac{1}{\pi\alpha'}\int\bar{\partial}\epsilon(\partial X^{\mu}\partial X_{\mu})\end{aligned}$$

2.5 Conformal Invariance

$$\begin{aligned}T &= \frac{1}{\pi\alpha'}:\partial X^{\mu}\partial X_{\mu}: \quad \bar{\partial}T=0 \quad \text{conserved, T is holomorphic}\\ \bar{T} &= \frac{1}{\pi\alpha'}:\bar{\partial}X^{\mu}\bar{\partial}X_{\mu}: \quad \partial\bar{T}=0 \ \ \bar{\text{T}}\text{ is antiholomorphic}\end{aligned}$$

$$T_{\tau\tau}=-\frac{1}{2\alpha'}:\dot{X}^2+x'^2:=T_{\sigma\sigma}$$
$$T_{\tau\sigma}=T_{\sigma\tau}=-\frac{1}{\alpha'}\dot{X}\cdot X': \quad \text{Traceless} \quad T^a_a=0.$$

For an arbitrary function $v(z)$:

$$j(z)=iv(z)T(z),\qquad \bar{\partial}j=0.$$

There are an infinite number of conservation laws. We may calculate the operator product expansion for the stress tensor with the field X^{μ}

$$\begin{aligned}T(z)X^{\mu}(z') &= :\partial X^{\nu}\partial X_{\nu}:X^{\mu}(z')=\frac{1}{\alpha'}\eta^{\mu\nu}\partial\ln|z-z'|\alpha'\partial X_{\nu}(z')+...\\ &= \frac{1}{z-z'}\partial X^{\mu}(z')+...\\ \bar{T}(\bar{z})X^{\mu}(\bar{z}') &= \frac{1}{\bar{z}-\bar{z}'}\bar{\partial}X^{\mu}(\bar{z}')+...\end{aligned}$$

From $j=iv(z)T(z)$, $\partial^a j_a=0$

$$\bar{\partial}j=0 \text{ or } \partial\bar{\text{j}}=0$$

$$\bar{\partial}(v(z)T(z)) = 0$$

$v(z)T(z)$ is a conserved current, where

$$vTx^\mu \sim \frac{v\partial x^\mu}{z - z'}.$$

Let z transform as $\delta z = z + \epsilon V$, then

$$x^\mu \longrightarrow x^\mu - \epsilon v \partial x^\mu$$

and $T(z)A(z')$ can be expanded as Laurent series

$$T(z)A(z') = \frac{a_{-1}}{z - z'} + \frac{a_{-2}}{(z - z')^2} + ... + \text{regular terms},$$

then λ is

$$\begin{aligned}
\lambda &= \oint \frac{dz}{2\pi} iv(z)T(z)A(z') \\
&= \oint \frac{dz}{2\pi} i \left[\frac{a_{-1}v(z)}{z - z'} + \frac{a_{-2}v(z)}{(z - z')^2} + .. \right] \\
&= ia_{-1}v(z') + ia_{-2}\partial v(z') + \frac{i}{2!}a_{-3}\partial^2 v(z') + \cdots. \text{ ie.} \\
\delta A &= -\epsilon a_{-1}v - \epsilon a_{-2}\partial v - \frac{\epsilon}{2!}a_{-3}\partial^2 v - \cdots, \\
&= -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} a_{-n-1}\partial^n v.
\end{aligned}$$

Therefore, to find a_{-n}, for all n, we need to find how $A(z)$ transforms under the conformal transformation $z \to z + \epsilon v(z)$. The simplest case is a scaling: $v(z) = z,\ z \to z + \delta z = (1 + \epsilon)z$. Find $A(z)$ that has a simple scaling property (eigenfunction), $\delta A = -h\epsilon A$, for finite ζ.

$$\begin{aligned}
A' &= (1 + \epsilon)^{-h} A \\
&= \zeta^{-h} A.
\end{aligned}$$

or $A(z, \tilde{z})$ transforms as

$$A(z, \bar{z}) \to \zeta^{-h} \bar{\zeta}^{-\tilde{h}} A(z, \bar{z}').$$

For $\zeta = re^{i\theta}$,

$$A'(z', \bar{z}') \to r^{-(h+\tilde{h})} e^{-(h-\tilde{h})\theta} A(z, z')$$

$h+\tilde{h}$ is the dimension of A, and determines the transformation under scaling. $h - \tilde{h}$ determines the transformation under spin. If A is order h, then ∂A is order $h + 1$, i.e.

$$\begin{aligned}
\frac{\partial A}{\partial z} &= \frac{\partial z'}{\partial z} \frac{\partial A}{\partial z'} \\
&\to (1 - \epsilon z)(1 - h\epsilon z)\partial A \\
&\to (1 - (h + 1)\epsilon z)\partial A.
\end{aligned}$$

or

$$\bar{\partial}A, \quad \tilde{h} \to \tilde{h}+1$$

Compare the coefficient of ∂v with the equations for δA. This implies $a_{-2} = hA$. For simplicity, let $v(z) = 1$.

Special Case: Do a translation transformation on $z,\ z \to z+\epsilon$.Then $\delta A = A(z-\epsilon) - A(z) = \epsilon\partial A \to a_{-1} = \partial A$. For an arbitary $v(z),\ z \to z+\epsilon v(z),\ \delta A$ is given by

$$\begin{aligned} \delta A &= -h\epsilon\partial v A \\ A' &= (\tfrac{\partial\zeta}{\partial z})^{-h}(\tfrac{\partial\bar{\zeta}}{\partial\bar{z}})^{-\tilde{h}}A \\ A' &= (1+\epsilon\partial v)^{-h}A. \end{aligned}$$

If δA has only two singularity terms in the form below, call A a primary field, ie.

$$T(z)A(z') = \frac{\partial A}{z-z'} + \frac{h\partial A}{(z-z')^2}.$$

Some examples for $T(z)A(z')$

$$\begin{aligned} T(z)X^\mu(z') &= -\frac{1}{\alpha'} :\partial X^\nu\partial X_\nu: X^\mu(z') = \frac{1}{2}\eta^{\mu\nu}\partial\ln(z-z')\partial X_\nu \\ &= \frac{1}{z-z'}\partial X^\mu,\ \text{ie. } h=\tilde{h}=0 \\ T(z)\partial^2X^\mu &= \frac{1}{2}\eta^{\mu\nu}\partial\partial^2\ln(z-z')\partial X_\nu \\ &= \frac{2}{(z-z')^3}\partial X^\mu(z) \\ &= \frac{2}{(z-z')^3}[\partial X^\mu(z') + (z-z')\partial^2X(z') + \frac{1}{2!}(z-z')^2\partial^3X^\mu(z') + ... \\ &= \frac{2}{(z-z')^3}\partial X^\mu + \frac{2}{(z-z')^2}\partial^2X^\mu(z') + \frac{1}{z-z'}\partial\partial^2X^\mu + ... \\ T(z):e^{ikX}: &= \frac{1}{\alpha'}\left(\frac{\alpha'}{2}\eta^{\mu\nu}k_\nu\partial\ln(z-z')\right)^2 :e^{ik\cdot X}: + \eta^{\mu\nu}k_\nu\partial\ln(z-z') :\partial X_\mu e^{ik\cdot X}: \\ &= \frac{\alpha'}{4}\frac{k^2}{(z-z')^2}:e^{ik\cdot X}: + \frac{1}{z-z'}:\partial e^{ik\cdot X} \end{aligned}$$

For $A = e^{ik\cdot X},\ T(z)A(z')$ implies $h = \frac{\alpha' k^2}{4}$ and $\bar{T}(\bar{z})A(\bar{z}')$ implies $\tilde{h} = \frac{\alpha' k^2}{4}$.Therefore A has weight $(h,\tilde{h}) = (\frac{\alpha' k^2}{4},\ \frac{\alpha' k^2}{4})$. If a translation is applied to z $(z \to z+\epsilon v)$, then $e^{ik\cdot X(z)} \to e^{ik\cdot X(z-\epsilon v)}$ and $\delta(e^{ik\cdot X(z)}) = -\epsilon v\partial e^{ik\cdot X(z)}$, which means that $h=0$. We have just shown that h is nonzero, and arrives from normal ordering (quantum effect).

$$\begin{aligned} \partial X e^{ik\cdot X} &\to h = 1 + \tfrac{\alpha' k^2}{4} \\ \partial^2 X e^{ik\cdot X} &\to h = 2 + \tfrac{\alpha' k^2}{4} \\ \partial^m X e^{ik\cdot X} &\to h = m + \tfrac{\alpha' k^2}{4} \\ \partial^{m_n}X^{\mu_n}...\partial^{m_2}X^{\mu_2}\partial^{m_1}X^{\mu_1}e^{ik\cdot X} &\to h = m_n + ... + m_2 + m_1 + \tfrac{\alpha' k^2}{4} \end{aligned}$$

It looks like these make a good *basis* for operators.

$$\begin{array}{cccc}
X^\mu & h=0 & \tilde{h}=0 & (0,0) \\
\partial X^\mu & h=1 & \tilde{h}=0 & (1,0) \\
\bar{\partial} X^\mu & h=0 & \tilde{h}=1 & (0,1) \\
\partial^2 X^\mu & h=2 & \tilde{h}=0 & (2,0) \\
e^{ik\cdot X} & h=\frac{\alpha' k^2}{4} & \tilde{h}=\frac{\alpha' k^2}{4} & (\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}) \\
\partial X e^{ik\cdot X} & h=1+\frac{\alpha' k^2}{4} & \tilde{h}=1+\frac{\alpha' k^2}{4} & (1+\frac{\alpha' k^2}{4}, 1+\frac{\alpha' k^2}{4}) \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

$T(z)$ has weight $h=2$, but is not a primary field. The TT OPE is given by:

$$T(z)T(z') = \frac{D}{2(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{2}{\alpha'(z-z')} : \partial^2 X_\mu \partial X^\mu :$$

where $\partial^2 X_\mu \partial X^\mu$ can be written as $\frac{1}{2}(\partial(\partial X_\mu \partial X^\mu))$ if $T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : + V_\mu \partial^2 X^\mu$

2.6 Free CFTs

In this section we will explore various different conformal field theories. We may classify all CFTs by knowing their central charges and three-point functions. We will find the central charges for the following free conformal field theories. We will leave the three-point functions for the reader.

Linear dilaton

The TT OPE is given by

$$\begin{aligned}
T(z)T(z') &\sim \frac{D}{2(z-z')^4} + V_\mu V^\mu \frac{\alpha'}{2} \partial^2 \partial'^2 \ln(z-z') + \ldots \\
&= \frac{D}{2(z-z')^4} + \frac{6 V_\mu V^\mu \alpha'}{2(z-z')^4} + \ldots \\
&= \frac{c}{2(z-z')^4} + \ldots,
\end{aligned}$$

where $c = D + 6\alpha' V_\mu V^\mu$. Therefore the central charge for the Linear Dilaton theory can be any number. When the number of dimensions is one or two the theory is *exactly* solvable. Also, if we compactify some dimensions V^μ can live in the compact subspace, because there is no need for Lorentz invariance there.

$$\begin{aligned}
T(z)X^\mu(z') &\sim \partial \ln(z-z') \partial X^\mu + V^\mu \frac{\alpha'}{2} \partial^2 \ln(z-z') + \ldots \\
&= \frac{1}{z-z'} \partial X^\mu + \frac{V^\mu \alpha'}{2(z-z')^2}, \quad h=0 \text{ and } X^\mu \text{ is not a primary field,}
\end{aligned}$$

$$v(z)T(z)X^\mu(z') \sim \frac{V^\mu\alpha'}{2(z-z')^2}[v(z') + (z-z')\partial v + ...] + \frac{v}{z-z'}\partial X^\mu + ...$$

For $\delta X^\mu = -\epsilon,\ \lambda = \frac{1}{2}V^\mu\alpha'\partial v + v\partial X^\mu$.

bc theory

Let b and c be anticommuting fields, i.e. spinors.The action can be writen as

$$S = \frac{1}{2\pi}\int d^2z b\bar\partial c$$

The equations of motion are given by: $\bar\partial c = 0,\ \bar\partial b = 0$, where b and c are holomorphic. If we let $\ell = 2\pi$, then we can write b and c as

$$b(z) = i\sum b_n e^{inz},\ c(z) = i\sum c_n e^{inz},$$

where

$$\begin{aligned}
\{b(z), c(z')\} &= \delta(\sigma-\sigma')_{\text{equaltime}},\ z_2 = z_2' \text{ and } 0<\sigma<2\pi \text{ or} \\
\{b_m, c_n\} &= \delta_{m+n,0} \\
\langle 0|b(z)c(z')|0\rangle &= \frac{1}{1-e^{i(z-z')}} \\
&\sim \frac{1}{z-z"} + \text{regular terms},\ \text{then} \\
:b(z)c(z'): &= b(z)c(z') - \frac{1}{z-z'}
\end{aligned}$$

If b has weight $h_b = \lambda$, then $h_c = 1-\lambda$.This is known since the action has weight 0 and the volume element has weight $(-1,-1)$. From the transformation $\delta z = \epsilon(z)$, b will change to

$$\begin{aligned}
b' &= \left(\frac{\partial z'}{\partial z}\right)^\lambda b(z-\epsilon) \\
&= (1-\lambda\partial\epsilon)(b-\epsilon\partial b),\ \text{then} \\
\delta b &= -\lambda\partial b - \epsilon\partial b,\ \text{and} \\
\delta c &= -(1-\lambda)\partial c - \epsilon\partial c \\
\delta S &= \int \bar\partial\epsilon((\partial b)c - \lambda\partial(bc)),
\end{aligned}$$

where $(\partial b)c - \lambda\partial(bc)$ is the Noether current in this case, then T is

$$\begin{aligned}
T &= :(\partial b)c: -\lambda\partial(:bc:),\ \text{and} \\
c(z)b(z') &= \frac{1}{z-z'} + ... \\
b(z)c(z') &= \frac{1}{z-z'} + ...
\end{aligned}$$

$$\begin{aligned} T(z)T(z') &= -\left(\frac{1}{z-z'}\right)^2 - 2\lambda\partial\left(\frac{1}{z-z'}\partial\frac{1}{z-z'}\right) + \lambda^2\partial\partial'\frac{1}{(z-z')^2} \\ &= \frac{-1+6\lambda-6\lambda^2}{2(z-z')^4} + ... \\ &= \frac{c}{2(z-z')^4} \\ c &= -2+12\lambda-12\lambda^2 = 1-3(2\lambda-1)^2. \end{aligned}$$

From the Linear Dilaton theory $c = D + 6\alpha V^2$. Let the charges from the two theories be equal, ie. $D + 6\alpha V^2 = 1 - 3(2\lambda - 1)^2$ and solve for V. For the case where $D = 1$, one will obtain

$$V = \frac{1}{\sqrt{2\pi}}(2\lambda - 1).$$

Can the bosons be equivalent to the fermions? We will see later. Let us explore a special case, $\lambda = \frac{1}{2}$. We find $V = 0,\ c = 1$ and, $b,\ c$ can be writen as a linear combination of scalar fields Ψ_1 and Ψ_2

$$b(z) = \frac{1}{\sqrt{2}}(\Psi_1 + i\Psi_2), \quad c(z) = \frac{1}{\sqrt{2}}(\Psi_1 - i\Psi_2).$$

The action may be expressed in terms of the Ψs

$$S = \frac{1}{4\pi}\int d^2z(\Psi_1\bar{\partial}\Psi_1 + \Psi_2\bar{\partial}\Psi_2),$$

with a stress tensor

$$T = -\frac{1}{2}\Psi_i\partial\Psi_i, \quad i = 1, 2.$$

Another interesting case is for $\lambda = 2$ and V=0. Then the central charge, c becomes $c = -26$ from $c = 1 - 3(2\lambda - 1)^2$. This is the result obtained in chapter 1.

$\beta\gamma$ theory

The next example, the bosonic case, let β and γ be commuting scalar fields. The $\beta\gamma$ action is given by:

$$S = \frac{1}{\sqrt{2\pi}}\int d^2z\beta\bar{\partial}\gamma,$$

where

$$\bar{\partial}\beta = \bar{\partial}\gamma = 0$$

The procedure is the same as for the spinor case.

$$\begin{aligned} \beta(z)\gamma(z') &= \frac{1}{z-z'} + ... \\ \gamma(z)\beta(z') &= -\frac{1}{z-z'} + ... \\ c &= -1 + 3(2\lambda-1)^2 \end{aligned}$$

Note that if $\lambda = \frac{3}{2}$, then $c = 11$. If we combine the central charges for the four theories, the number of physical dimensions reduces from twenty-six to ten.

$$\left.\begin{array}{cccc} X^\mu & D & (b,c) & -26 \\ \Psi^\mu & \frac{d}{2} & (\beta,\gamma) & 11 \end{array}\right\} D + \frac{d}{2} - 26 + 11 = 0 \Longrightarrow D = 10.$$

From

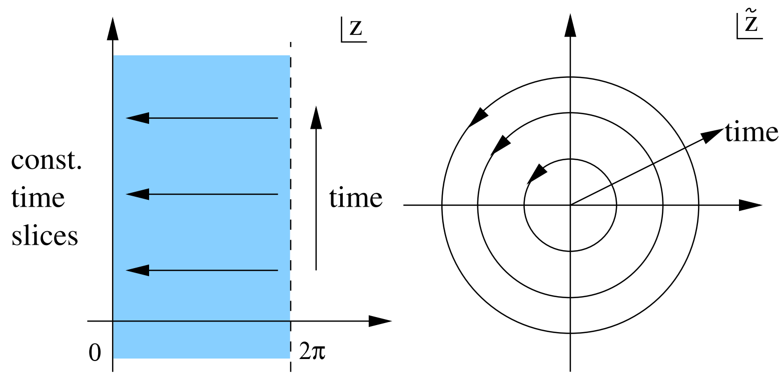
$$X^\mu_L(z) = x^\mu - \frac{\alpha'}{2} p^\mu z + i\sqrt{\frac{\alpha'}{2}} \sum \frac{1}{n} \alpha^\mu_n e^{inz}$$

$$\begin{aligned} \langle X_L(z), X_L(z')\rangle &= \ln(1 - e^{i(z-z')}) \\ &\sim \ln(z - z') + ... \end{aligned}$$

2.7 Virasoro Algebra

The worldsheet of a free closed string moving through space-time looks like a cylinder whose radius may fluctuate depending on the excitation mode. We can map this cylinder to the complex plane. This map would be equivalent to sqeezing one end of the string so it looks like a cone.Then smash the cone into a plane. Now the worldsheet coordinates can be expressed as complex coordinates, where r represents τ and the phase, θ, represents the position on the string, σ ($z \mapsto \tilde{z}$, cylinder $\mapsto$ plane). For closed strings $\lambda = 2\pi, 0 < \sigma < 2\pi$

$$\tilde{z} = e^{-iz}$$



Instead of expanding in Fourier modes, do a Laurent Expansion.

$$X^\mu \sim \sum \frac{1}{n} \alpha^\mu_n \tilde{z}^{-n}$$

What happens to $T(z)$:

$$\begin{aligned} T(z) &= \frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \\ &= \sum T_m e^{imz} \\ &\stackrel{?}{=} \sum T_m \tilde{z}^{-m} \end{aligned}$$

$T(z)$ doesn't transform as easily as guessed. Subtract the term with central charge to make it a primary field (tensor). Then transform each coordinate.

$$T(z)T(z') = \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{1}{z-z'}\partial T(z') + \ldots$$

multiply both sides by $v(z)$

$$v(z)T(z)T(z') = v(z)[\ldots]$$

$$\delta T = -\epsilon\lambda = \epsilon\frac{c}{2}\frac{1}{3!}\partial^3 v - 2\epsilon\partial v T - \epsilon v\partial T$$

$$z \to z + \epsilon v(z) \;\; z \to \tilde{z} = e^{-iz}$$

$$T(z) = (\frac{\partial\tilde{z}}{\partial z})^2 T(\tilde{z}) + \frac{c}{12}\{\tilde{z}, z\},$$

where $\{\tilde{z}, z\}$ is known as a Schwarzian derivative, defined as

$$\{\tilde{z}, z\} = \frac{2\tilde{z}'''\tilde{z}' - 3(\tilde{z}'')^2}{2(\tilde{z}')^2},$$

which equals $1/2$ for our example.

$$T = -\tilde{z}^2\tilde{T} + \frac{c}{24}$$

We can invert this and solve for $\tilde{z}$,

$$\begin{aligned} \tilde{T}(\tilde{z}) &= -\tilde{z}^{-2}\tilde{T} + \frac{c}{24}\tilde{z}^{-2} \\ &= \sum L_m z^{-m-2}, \qquad\qquad L_m = -T_m + \frac{c}{24}\delta_{m,0} \end{aligned}$$

Invert the equation and solve for L.

$$L_m = \oint \frac{d\tilde{z}}{2\pi i}\tilde{z}^{m+1}\tilde{T}$$

The Hamiltonian $\int_0^{2\pi}\frac{d\sigma}{2\pi}T$ can now be written in terms of L. Including left and right movers, the Hamiltonian is

$$H = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24},$$

and the number operator is $N = L_0 - \bar{L}_0$. We can see that $\bar{\partial}L_m = 0$ since $\bar{\partial}[\tilde{z}^{m+1}\tilde{T}] = 0$. This implies all L_ms generate symmetries.

Commutators

Recall the Ward identity for any operator $A: \ \delta A = i\epsilon[Q, A]$ implies Q can be written as an integral of the current, j. $Q = \oint \frac{dz}{2\pi i} j(z)$ The OPE of T with some primary field of weight h is given by:

$$T(z)A(z') = \frac{h}{(z-z')^2}A(z') + \frac{\partial A(z')}{z-z'} + \dots$$

Look at the variation of A:

$$\delta A = -\epsilon h \partial v A - \epsilon v \partial A$$

choose $v = \tilde{z}^{m+1} \quad j = \tilde{z}^{m+1}\tilde{T} \quad Q = \oint j \sim L_m$

$$\Rightarrow \delta A = i\epsilon[L_m, A]$$

$$[Q, A] = h(m+1)\tilde{z}^m + \tilde{Z}^{m+1}\partial A = [L_m, A]$$

expand A in a Laurent expansion:

$$\begin{aligned} A &= (\frac{\partial \tilde{z}}{\partial z})^{-h} \sum A_m \tilde{z}^{-m} \\ &= \sum A_m \tilde{z}^{-m-h} \end{aligned}$$

This is the expansion for a primary field in the $\tilde{z}$ coordinates. Look at the commutator of L with A in these cordinates.

$$\begin{aligned} [L_m, A_n] &= h(m+1)A_{n+m} - (n+m+h)A_{n+m} \\ &= [(h-1)m - n]A_{m+n} \end{aligned}$$

We got an algebra from an OPE. We know the algebra, but what are the representations of the algebra.

$$\delta T = \text{expected} + \frac{c}{12}\partial^3 v$$

$$[L_m, T] = \text{expected} + \frac{c}{12}(m+1)m(m-1)\tilde{z}^{m-2}$$

$$\begin{aligned} [L_m, L_n] &= \text{expected} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\ &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}. \end{aligned}$$

This is the **Virasoro algebra**. Let us focus on the special case $m = 0$. The action of the Hamiltonian L_0:

$$[L_0, A_n] = -nA_n$$

even if $A_n = L_n$. If

$$L_0|\psi\rangle = E|\psi\rangle, \qquad |\psi'\rangle = L_n|\psi\rangle$$

$$\begin{aligned} L_0|\psi'\rangle &= [L_0, L_n]|\psi\rangle + L_n L_0|\psi\rangle \\ &= -nL_n|\psi\rangle + EL_n|\psi\rangle \\ &= (E-n)L_n|\psi\rangle \\ &= (E-n)|\psi'\rangle \end{aligned}$$

For $n > 0 \Rightarrow L_n|\psi\rangle$ has lower *energy* then $|\psi\rangle$. Is the spectrum of L_0 unbounded? This needs to be fixed.
Let us look at the $n = -1, 0, 1$ Virasoro subalgebra. Is this analogous to the raising and lowering operators for angular momentum.

$$[L_0, L_1] = -L_1, \ \ [L_0, L_{-1}] = L_{-1}, \ \ [L_1, L_{-1}] = 2L_0$$

We notice that this closed algebra is independent of c, therefore for every CFT we should get this subalgebra.This is a Lie algebra SL(2,$\mathbb{R}$) and is not compact. Look at Quantum Mechanics:

$$[L_0, L_+] = L_+, \ \ [L_0, L_-] = -L_-, \ \ [L_+, L_-] = 2L_0$$

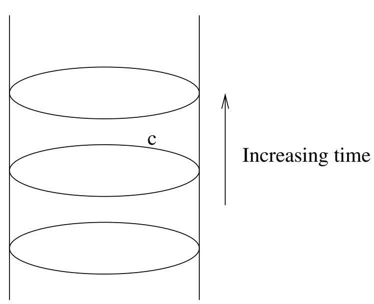
This is close to the algebra above, but not the same.The difference is actually very important!

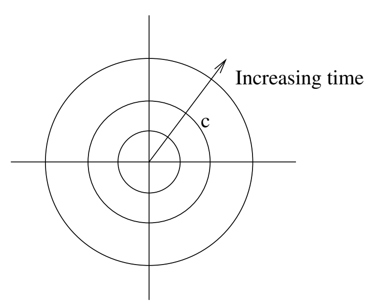
2.8 Mode Expansions

Free Scalars

Now that we know how to transform to these $\tilde{z}$ coordinates, we can look at the same calculations and look for similarities.The mode expansion is given by,

$$X_L^\mu = x^\mu - i\frac{\alpha'}{2}p^\mu \ln\tilde{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m>0}\frac{1}{m}\alpha_m^\mu \tilde{z}^{-m}.$$





The two-point function which has to be radially ordered is given by,

$$\begin{aligned}\langle X_L^\mu(\tilde z)X_L^\nu(\tilde z')\rangle &= \frac{\alpha'}{2}\sum\frac{1}{m}(\frac{\tilde z'}{\tilde z})\eta^{\mu\nu}|\tilde z'|<|\tilde z| \quad \rightarrow \text{ time ordering...i.e. radial ordering}\\ &= \frac{\alpha'}{2}\ln|1-\frac{\tilde z'}{\tilde z}|\eta^{\mu\nu}.\end{aligned}$$

The normal ordered product is given by,

$$:XX:=XX=\frac{\alpha'}{2}\ln|z-z'| \text{ in z picture.}$$

Now we can compare our definition for :: to switching a and $a^\dagger$ around.

$$\begin{aligned}:X_L^\mu(\tilde z)X_L^\nu(\tilde z'): &= :(X_L^+ + X_L^-)^\mu(X_L^+ + X_L^-)^\nu:\\ &= X_L^\mu(\tilde z)X_L^\nu(\tilde z') + [X^{(-)}(z'),X^{(+)}(z)]\\ &= X_L^\mu(\tilde z)X_L^\nu(\tilde z') - \frac{\alpha'}{2}\sum_n\frac{1}{n}\eta^{\mu\nu}\frac{\tilde z'}{\tilde z}^n\end{aligned}$$

$$[x^\mu,-i\frac{\alpha'}{2}p^\mu\ln\tilde z]=XX+\frac{\alpha'}{2}\eta^{\mu\nu}\ln|\tilde z-\tilde z'|.$$

From normal ordering

$$X^\mu(z)X^\nu(z')=\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-z')+:X^\mu(z)X^\nu(z'):,$$

the operator product can be writen as product = singularity + normal ordering product From

$$L^m=\oint\frac{dz}{2\pi i}z^{m+1}T(z), \qquad \text{charge}$$

where $z^{m+1}T(z)$ is a conserved current and

$$\begin{aligned}\partial X^\mu &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\alpha_m^\mu z^{-m-1}\\ P^\mu &= \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu\\ X^\mu &= x^\mu+p^\mu\ln z+i\sum_{m\neq0}\frac{1}{m}\alpha_m^\mu z^{-m}\\ T(z) &= \frac{1}{\alpha'}:\partial X^\mu\partial X_\mu:\\ &= \frac{1}{2}\sum_{n_1,n_2}z^{-n_1-1}z^{-n_2-1}:\alpha_{n_1}^\mu\alpha_{n_2\mu}:\\ L_m &= \frac{1}{2}\sum_n:\alpha_{m-n}^\mu\alpha_{n\mu}:\end{aligned}$$

The Hamiltonian (L_0) is given by:

$$L_0 = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu}$$

What about the equal-time commutator? That means $|z| = |z'|$. To calculate $X^\mu(z)X^\nu(z')$, we need to approach $|z| \to |z'|$ from $|z| > |z'|$. To calculate $X^\nu(z)X^\mu(z')$, we need to approach $|z| \to |z'|$ from $|z| < |z'|$. Thus, the commutation relations are

$$\begin{aligned}
[X^\mu(\sigma), \Pi^\nu(\sigma')] &= 2\pi i \delta(\sigma - \sigma') \\
[X^\mu(\sigma), X^\nu(\sigma')] &= 0, \text{ where } \mathrm{X} = \mathrm{X_L(z)} + \mathrm{X_R(\bar{z})} \\
[X^\mu_L(z), X^\nu_L(z')] &= X^\mu_L(z)X^\nu_L(z') - X^\nu_L(z')X^\mu_L(z) \\
&= \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z - z') - \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z - z') \\
&= \frac{\pi i}{2}\alpha'\eta^{\mu\nu}\frac{d}{dz}(\text{step function}) \\
&= \frac{\pi i}{2}\alpha'\eta^{\mu\nu}\delta(z - z'),
\end{aligned}$$

where differentiating a step function, a delta function is obtained.
N.B. The commutator $[X_L, X_L] \neq 0$ means X_L is **not** a coordinate. It is a combination of a coordinate and momentum.
Another interesting example

$$\begin{aligned}
e^A e^B &= e^{[A,B]} e^B e^A, \text{ then} \\
: e^{ik_1 X_1} :: e^{ik_2 X_2} : &= e^{\pm i\pi \frac{\alpha'}{2} k_1 \cdot k_2} : e^{ik_2 X_2} :: e^{ik_1 X_1} :,
\end{aligned}$$

For the special case $D = 1$, $X^\mu = \frac{\alpha'}{2}\Psi$, $k^\mu_1 = \pm\sqrt{\frac{2}{\alpha'}} = k^\mu_2$ and let $\mathcal{O}_1 = e^{\pm i\psi}$ and $\mathcal{O}_2 = e^{\pm i\psi}$, then

$$\mathcal{O}_1\mathcal{O}_2 = e^{\pm i\pi}\mathcal{O}_2\mathcal{O}_1 = -\mathcal{O}_2\mathcal{O}_1, \quad (\text{Quantum group!})$$

or

$$\begin{aligned}
\{\mathcal{O}_1\mathcal{O}_2, \mathcal{O}_2\mathcal{O}_1\} &= 0 \\
\psi(z)\psi(z') &\sim \ln(z - z'), \text{ then} \\
: e^{i\psi}(z) :: e^{-i\psi}(z') : &\sim e^{-\ln(z-z')} : e^{i\psi}(z)e^{-i\psi}(z') : \\
&\sim \frac{1}{z - z'} : e^{i\psi}(z)e^{-i\psi}(z') : \\
: e^{i\psi}(z) :: e^{i\psi}(z') : &\sim e^{\ln(z-z')} : e^{i\psi}(z)e^{i\psi}(z') : \\
&\sim (z - z') : e^{i\psi}(z)e^{i\psi}(z') : .
\end{aligned}$$

Example: bc **CFT**

We can write b and c in terms of ψ: $b =: e^{i\psi}$: and $c =: e^{i\psi}$:. The stress tensor is given as

$$T_\psi =: \partial\psi\partial\psi : + V\partial^2\psi.$$

From $b(z) = \sum b_m z^{-m-\lambda}$ and $c(z) = \sum c_m z^{-m-1+\lambda}$, we may calculate the OPE, $b(z)c(z') \sim \frac{1}{z-z'}$. The anticommutator between b and c is

$$\{b_m, c_n\} = \delta_{m+n,0}.$$

b_n and c_n are annihilation operators for $n > 0$. For the zero modes, $m, n = 0$ the anticommutator is

$$\{b_0, c_0\} = 1$$

If we let $|\Psi\rangle$ be a null state of b, ie. $b_0|\psi\rangle = 0$ and $c_0|\psi\rangle = |\chi\rangle$, then

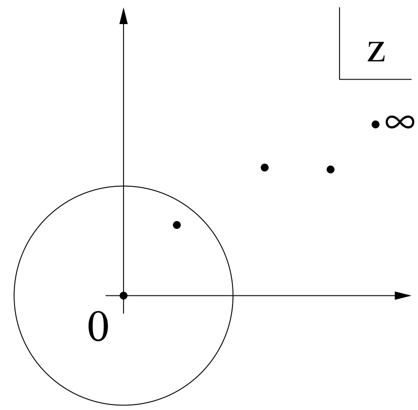
$$\begin{aligned}
c_0|\chi\rangle &= c_0c_0|\psi\rangle = 0, \qquad \{b_m, b_n\} = \{c_m, c_n\} = 0 \\
b_0|\chi\rangle &= b_0c_0|\psi\rangle = \{b_0, c_0\}|\psi\rangle = |\psi\rangle, \text{ then} \\
\langle\psi|\psi\rangle &= \langle\chi|b_0b_0|\chi\rangle = 0 \\
\langle\chi|\chi\rangle &= \langle\psi|b_0b_0|\psi\rangle = 0, \text{ and} \\
\langle\psi|\chi\rangle &\neq 0.
\end{aligned}$$

So $|\psi\rangle$ and $|\chi\rangle$ are independent *vacua*. Hilbert space: act on $|\psi\rangle,\ |\chi\rangle$ with the creation operators $b_{-n},\ c_{-n},\ n > 0$. By convention we will group b_0 and c_0 with creation and annihilation operators respectively. Then $|\psi\rangle$ is the vacuum, $|\psi\rangle = |0\rangle$.
N.B. Define $\langle 0| = \langle\chi|$ and $\langle\psi|\chi\rangle = \langle\chi|\psi\rangle = 1$.

2.9 Vertex operators

Vertex operators are one-to-one corresponding to their states, $I \simeq |0>$ for an operator ∂X^μ and momentum operator p^μ

$$\begin{aligned}
\partial X^\mu(z) &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_m^\mu z^{-m-1} \\
p^\mu|0\rangle &= 0, \\
p^\mu|0;k\rangle &= k^\mu|0;k\rangle, \text{ then} \\
\partial X^\mu(z)|0\rangle &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=1}^{\infty} \alpha_{-m}^\mu z^{-m-1}|0\rangle \\
&= -i\sqrt{\frac{\alpha'}{2}}(\alpha_{-1}^\mu + \alpha_{-2}^\mu z + \alpha_{-3}^\mu z^2 + ...)|0\rangle \text{ letting } z \to 0 \\
\partial X^\mu(0) &= -i\sqrt{\frac{\alpha'}{2}}\alpha_{-1}^\mu|0\rangle
\end{aligned}$$



From $|A\rangle \simeq A(z)$ or $|A\rangle = A(0)|0\rangle,\ \partial X^\mu$ operating at a bra is

$$\begin{aligned}\langle 0|\partial X^\mu(z) &= -i\sqrt{\frac{\alpha'}{2}}\langle 0|\sum_{m=1}^{\infty}\alpha^\mu_{-m}z^{-m-1},\ \text{letting } z\to\infty \\ \langle 0|\partial X^\mu(\infty) &= -i\sqrt{\frac{\alpha'}{2}}\langle 0|\alpha^\mu_{-1},\end{aligned}$$

then the product can be consider as

$$\underbrace{\langle 0|}_{\text{time at }+\infty}\quad \text{time order stuffed}\quad \underbrace{|0\rangle}_{\text{time at }-\infty}$$

From the equation of $\partial X^\mu,\ \alpha^\mu_1$ is

$$\alpha^\mu_{-1} = \oint\frac{dz}{2\pi i}\ \underbrace{z^{-1}\partial X^\mu(z)}_{\text{conservedcurrent}}\ ,$$

$$z\to e^{-iz}$$

$\alpha^\mu_{-2}|0\rangle$ also can be obtained from

$$\begin{aligned}\partial^2 X^\mu(z)|0\rangle &= -i\sqrt{\frac{\alpha'}{2}}(\alpha^\mu_{-2}+2\alpha^\mu_{-3}z+...)|0\rangle,\ \text{for any } \alpha^\mu_{-\mathrm{m}} \\ \partial^m X^\mu(0) &= -i\sqrt{\frac{\alpha'}{2}}((m-1)!\alpha^\mu_{-m}+O(z))|0\rangle, \text{letting z}\to 0\end{aligned}$$

$$\alpha^\mu_{-m}(0)\simeq i\sqrt{\frac{2}{\alpha'}}\frac{1}{(m-1)!}\partial^m X^\mu(z).$$

$\alpha^\mu_{-m}\alpha^\nu_{-n}|0\rangle$ can be obtained from

$$:\partial^m X^\mu\partial^\nu X^\nu:|0\rangle = -\frac{\alpha'}{2}(m-1)!(n-1)!\alpha^\mu_{-m}\alpha^\nu_{-n}+\underbrace{...}_{\text{go to}}0\text{ in the infinite past}|0\rangle$$

Some examples of vertex operators are

$$\begin{aligned} e^{ik\cdot X}|0\rangle &= |0;k\rangle \\ \lim_{z\to 0} :e^{ik\cdot X(z)}:|0\rangle &= e^{ik\cdot X}|0\rangle, \text{ then} \\ :\partial X^{m_1}\partial X^{m_2}...e^{ik\cdot X}: &\simeq \alpha_{-m_1}\alpha_{-m_2}....|0;k\rangle \end{aligned}$$

2.10 Primary fields

A primary field $A,\ A \to |A\rangle, A(0)|0\rangle = |A\rangle$, with a state $|m_1 + m_2 + ...; k >$ can be writen as

$$\begin{aligned} |A\rangle &= \alpha{-m_1}^{\mu_1}\alpha_{-m_2}^{\mu_2}...|0;k\rangle\text{where} \\ A(z) &= \frac{1}{(n_1-1)!}\frac{1}{(n_2-1)!}...:\partial^{n_1}X^{\mu_1}\partial^{n_2}X^{\mu_2}...e^{ik\cdot X}:. \end{aligned}$$

If we let $\lambda = 2$ for the bc theory:

$$\begin{aligned} b_{-m}|\psi\rangle &\longrightarrow \frac{1}{(m-2)!}\partial^{m-2}b,\ b_0|\psi\rangle = 0 \\ c_{-m}|\psi\rangle &\longrightarrow \frac{1}{(m+1)!}\partial^{m+1}c \\ T_{bc}(z) &= :(\partial b)c: -\lambda\partial:(bc):, \end{aligned}$$

where

$$\begin{aligned} b(z) &= \sum b_m z^{-m-\lambda} \\ c(z) &= \sum c_m z^{-m+\lambda} \end{aligned}$$

The Virasoro operator in this case is

$$\begin{aligned} L_m^{bc} &= \oint \frac{dz}{2\pi i} z^{m+1} T_{bc} \\ &= \sum_n -(n+\lambda)b_n c_{m-n} + \lambda(m+1)b_n c_{n-m} + a\delta_{n,0} \\ &= \sum_n (m\lambda - n):b_n c_{n-m}: + a\delta_{m,0} \\ T_{bc}(z)T_{bc}(z') &\sim \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2}T_{bc} + \frac{1}{z-z'}\partial T_{bc} \end{aligned}$$

where $c = 1 - 3(2\lambda - 1)^2$, and the commutation relations for L are given by:

$$\left[L_m^{bc}, L_n^{bc}\right] = (n-m)L_{n+m}^{bc} + \frac{c}{12}(m^3 - n)\delta_{n+m,0}$$

For $m = 1$ and $n = -1$, the commutator is

$$\begin{aligned} \left[L_1^{bc}, L_{-1}^{bc}\right]|\Psi\rangle &= 2L_0^{bc}|\Psi\rangle \\ L_1 L_{-1}|\Psi\rangle - L_{-1}L_1|\Psi\rangle &= \lambda b_0 c_1 (1-\lambda) b_{-1} c_0 |\Psi\rangle \\ &= \lambda(1-\lambda) c_1 b_{-1}|\Psi\rangle \\ &= \lambda(1-\lambda)\{c_1, b_{-1}\}|\Psi\rangle \\ &= \lambda(1-\lambda)|\Psi\rangle \end{aligned}$$

2.11 Operator product expansion

Consider the commutator between L_m and A

$$\begin{aligned} [L_m, A] &= z^{m+1}\partial A + h(m+1)z^m A, \text{ where} \\ A(z) &= \sum A_n z^{-n-h} \end{aligned}$$

For $m = 0$,

$$\begin{aligned} [L_0, A] &= z\partial A + hA \\ [L_0, A_n] &= -nA_n \\ \text{as } z \to 0,\ [L_0, A(0)] &= hA(0) \\ L_0|A\rangle &= [L_0, A(0)]\,|0\rangle + A(0)L_0|0\rangle,\ L_0|0\rangle = 0, \\ &= hA(0)|0\rangle = h|A\rangle \\ \text{as } z \to 0,\ [L_m, A] &= \text{for } m > 0, \text{ then} \\ L_m|A\rangle &= 0, m > 0, \end{aligned}$$

where L_0 is bounded from below.

2.12 Unitary CFTs

Define an inner product $\langle ...|...\rangle$ such that $L_m^\dagger = L_{-m}$. We want an inner product of positive norm.

$$\langle\psi|L_m|\chi\rangle = \langle L_m^\dagger \psi|\chi\rangle$$

Example: For X^μ: $\langle 0; k|0; k'\rangle = 2\pi\delta(k - k')$.
$[\alpha_m^\mu, \alpha_n^\nu] = \eta^{\mu\nu}\delta_{m+n,0}, \quad \alpha_m^{\mu\dagger} = \alpha_{-m}^\mu$.
$\parallel \alpha_{-1}^\mu|0; k\rangle \parallel^2 < 0$ for $\mu = 0$ and $\parallel \alpha_{-1}^\mu|0; k\rangle \parallel^2 > 0$ for $\mu = i \neq 0$.
This can be corrected by letting $\Phi \to X'$. This conformal field theory is unitary and its action is

$$S = \frac{1}{2\pi\alpha'}\int d^2 z \partial\phi\bar{\partial}\phi.$$

Theorem: For highest weight A, $h \geq 0$,.
Proof: $2h\langle A|A\rangle = \langle A|\,[L_1, L_{-1}]\,|A\rangle = \parallel L_{-1}|A\rangle \parallel^2 \geq 0$.

Corollary: Any eigenstate of L_0 has $h \geq 0$ (it has energy $\geq h$ h.w.s. energy).
Theorem: $c > 0$: $\frac{c}{12}(m^3 - m) = \langle A| \left[L_m, L_{-m}\right] |A\rangle - 2m\langle A|L_0|A\rangle \geq 0$ for $L_0|A\rangle = 0$.
And we have gotten what we wanted; a positive norm for the highest weight state.

UNIT 3

BRST Quantization

BRS&T: Becchi-Ronet-Stora & Tyutin.

3.1 Point particle

Recall

$$S = \frac{1}{2}\int d\tau \left(\frac{1}{\eta}\dot{X}^\mu \dot{X}_\mu - \eta m^2\right) = \int d\tau \,(p_\mu \dot{x}^\mu - \eta\chi)$$

where

$$\chi = \frac{1}{2}(p^\mu p_\mu + m^2).$$

The constraint $\chi = 0$ generates the transformation

$$\delta X^\mu = \epsilon\{X^\mu, \chi\} = \epsilon p^\mu, \;\; \delta p^\mu = 0.$$

Quantization: $|\vec{k}\rangle, \;\; H = p_0 = \sqrt{\vec{p}^2 + m^2}, \;\; H|\vec{k}\rangle = \sqrt{k^2 + m^2}|\vec{k}\rangle$. where

$$\omega = \sqrt{k^2 + m^2}$$

is the dispersion relation.
Sexier approach: Let ϵ be anticommuting, say $\epsilon \to \epsilon c$, where ϵc are commuting and anticommuting respectively. Promote c to coordinate status. Let b be its conjugate momentum, so

$$S_{bc} = \int d\tau b\dot{c}.$$

The action $S' = S + S_{bc} = \int d\tau(p_\mu \dot{x}^\mu + b\dot{c})$ is invariant under

$$\begin{aligned} \delta_B X^\mu &= \epsilon c p^m u, \;\; \delta_B p^\mu = 0 \\ \delta_B b &= -\epsilon(\chi - m^2) \end{aligned}$$

Check:

$$\delta S' = \int d\tau \left[\epsilon(c\dot{p}^\mu)p_\mu - \epsilon(\chi - m^2)\dot{c}\right] = \int d\tau \epsilon \frac{d}{d\tau}(\frac{1}{2}c(p^\mu p_\mu + m^2)) = 0.$$

Generated by $Q_B = c\chi$ Nilpotent:

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = 0$$

Quantization: $\{b, c\} = 1, \ [P_\mu, X^\nu] = -i\eta^{\mu\nu}$. The b, c theory is much like b_0, c_0 in strings. States $|\psi\rangle, \ |\chi\rangle$

$$b_0|\psi\rangle = 0, \ c_0|\chi\rangle, \ c_0|\psi\rangle = |\chi\rangle, \ b_0|\chi\rangle = |\psi\rangle.$$

Include momentum, $|k\rangle \otimes |\psi\rangle = |k, \psi\rangle \ (p^\mu|k\rangle = k^\mu|k\rangle)$

$$Q_B|k, \psi\rangle = \frac{1}{2}(k^2 + m^2)|k, \chi\rangle, \quad Q_B|k, \chi\rangle = 0,$$

where $|k, \psi\rangle$ is closed for $k^2 + m^2 = 0$ and $|k, \chi\rangle$ is closed. Set of closed states $|k, \psi\rangle, \ k^2 + m^2 = 0$ is a set of physical states, in agreement with the analysis above (gauge fixed).

3.2 Strings

Recall $S = \frac{1}{2\pi\alpha'}\int d^2z \partial X^\mu \bar{\partial} X_\mu$. Constraint $T = \frac{1}{\alpha'}\partial X^\mu \partial X_\mu = 0$ (and similarly for $\bar{T}$) generates conformal transformations

$$\delta X^\mu = \epsilon v \partial X^\mu$$

Now make v anticommuting, $v \to c$, then b conjugate momentum bc system we already studied.

$$S_{bc} = \frac{1}{2\pi}\int d^2 z b \bar{\partial} c.$$

Let us guess that $S' = S + S_{bc}$ is invariant under the transformations

$$\delta_B X^\mu = i\sigma c \partial X^\mu, \quad \delta_B b = i\epsilon T ...$$

This does not quite work. We need $T \to T + T_{bc}$ and then $\delta_B c \neq 0$. The correct transformations are

$$\delta_B X^\mu = i\sigma c \partial X^\mu, \quad \delta_B b = i\epsilon(T + T_{bc}), \quad \delta_B c = i\epsilon c \partial c.$$

Then $\delta S' = 0$. The corresponding Noether current is

$$j_B = cT + \frac{1}{2} : cT_{bc} : + \frac{3}{2}\partial^2 c.$$

Require: j_B have weight $h = 1$, so the charge $Q_B = \oint \frac{dz}{2\pi i} j_B$ has $h = 0$ (conformally invariant scalar operator). If we look at the cT part in j_B we see $h_T = 2$ therefore $h_c = -1$. Therefore the bc system must have $\lambda = 2$ and $h_b = 2$.

3.3 Mode Expansion

$$Q_B = \sum_n c_n L_{-n} + \frac{1}{2}\sum_{m,n}(m-n) : c_m c_n b_{-m-n} : - c_0 = \sum_n : c_n \left(L_{-n} + \frac{1}{2} L^{bc}_{-n} - \delta_{n0} \right) :$$

where the minus sign in front of c_0 comes from $\frac{l(1-\lambda)}{2} = -1$ and is in disagreement between mode and conformal normal ordering.

3.4 Nilpotency

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = \frac{1}{2}\sum \left([L_m^{TOT}, L_n^{TOT}] - (m-n)L_{m+n}^{TOT} \right) c_{-m} c_{-n}$$

where $L_m^{TOT} = L_m + L_m^{bc} - \delta_{m,0}$. The right hand side is $\frac{1}{12}(D + c_{bc})(m^3 - m)$ where $c_{bc} = 1 - 3(2\lambda - 1)^2 = -26$. Our BRST charge is then

$$Q_B^2 \sim \frac{1}{12}(D - 26) = 0$$

if and only if $D = 26$. Conversely, suppose $Q_B^2 = 0$. Define $L_m^{TOT} = \{Q_B, b_m\}$. Then

$$\begin{aligned} [L_m^{TOT}, L_n^{TOT}] &= [L_m^{TOT}, \{Q_B, b_m\}] = \{Q_b, [L_m, b_n] \\ &= \{Q_B, (m-n)b_{m+n}\} = (m-n)L_{m+n}. \end{aligned}$$

Physical states are annihilated by $Q_B, \quad (Q_B|phys\rangle = 0)$. Note that $|phys\rangle + Q_B|\psi\rangle$ is also physical. they represent the same system (like A_μ and $A_\mu + \partial_\mu \psi$ in QED). Therefore, $|phys\rangle$=equivalence class $|A\rangle + Q_B|\psi\rangle$. Cohomology of the conformal group
Note:

$$(\langle A| + \langle \psi|Q_B)(|B\rangle + Q_B|\psi\rangle) = \langle A|B\rangle$$

for physical $|A\rangle, \ |B\rangle \ (Q_B|A\rangle = Q_B|B\rangle = 0)$ where we assume $Q_B^\dagger = Q_B$. In particular, $\langle \psi|Q_B|B\rangle = 0$ therefore $\langle \psi|Q_B = 0$.

3.5 A note on BRST cohomology

Given a group with symmetry group G generated by the algebra

$$[L_i, L_j] = i f_{ij}^k L_k.$$

Introduce ghosts b_i, c^i such that

$$\{c^i, b_j\} = \delta^i_j, \ \ \{c^i, c^j\} = \{b_i, b_j\} = 0.$$

Define the BRST charge

$$\begin{aligned} Q_B &= c^i L_i - \frac{i}{2} f_{ij}^k c^i c^j b_k \\ &= c^i \left(L_i + \frac{1}{2} L_i^{bc} \right), \quad L_i^{bc} = -i f_{ij}^k c^j b_k \end{aligned}$$

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = i c^i c^j f_{ij}^k L_k - i f_{lm}^k c^l c^m \{c^i, b_k\} L_i - \frac{1}{2} f_{ij}^k f_{kl}^m c^i c^j c^l b_m = 0.$$

due to the Jacobi identity

$$\begin{aligned} [[L_i, L_j], L_k] + [[l_j, L_k], L_i] + [[L_k, L_i], L_j] &= 0 \\ i f_{ij}^m [L_m, L_k] + i f_{jk}^m [L_m, L_i] + i k_{ki}^m [L_m, L_j] &= 0 \\ -f_{ij}^m f_{mk}^l L_l - f_{jk}^m f_{mi}^l L_l - f_{ki}^m f_{mj}^l L_l &= 0 \end{aligned}$$

For strings, $[L_m, L_n] = (m-n) L_{m+n}$ and $f_{mn}^k = (m-n)\delta_{k,m+n} \quad c^m = c_{-m}$, so

$$Q_B = c_{-m} L_m - \frac{1}{2}(m-n) c_{-m} c_{-n} b_{m+n}.$$

3.6 BRST Cohomology for open strings

Open strings are easier than closed strings, but they are entirely similar. We introduce a vacuum $|\psi\rangle$ such that $b_0|\psi\rangle = 0$ and $|\chi\rangle = c_0|\psi\rangle$. Then $\langle\psi|\psi\rangle = 0$, but $\langle\chi|\psi\rangle \neq 0$ so, we define the inner product by $\langle\psi|c_0|\psi\rangle$.
Let $|\psi\rangle \otimes |k\rangle = |\psi; k\rangle, \quad \langle k|k'\rangle = (2\pi)^D \delta^D(k-k')$. Physical states will be constructed from $|\psi\rangle$, so $b_0|phys\rangle = 0$. Then $L_0|phys\rangle = \{Q_B, b_0\}|phys\rangle = 0$.

$$L_0 = \alpha' p^2 + \sum_n n b_{-n} c_n + \sum_n \alpha_{-n}^\mu \alpha_{n\mu} - 1$$

$$\hat{H} = \{|\psi\rangle, \ b_0|\psi\rangle = 0, \ L_0|\psi\rangle = 0\}$$

$$Q_B|\psi\rangle = |Z\rangle, \ b_0|Z\rangle = L_0|\psi\rangle = 0, \ L_0|Z\rangle = [L_0, Q_B]|\psi\rangle = 0.$$

Therefore $Q_B : \hat{H} \to \hat{H}$. In $\hat{H}$, $|k\rangle$ is specified by $|\vec{k}\rangle$, because k^0 is given in terms of $\vec{k}$ through $L_0 = 0$. Therefore we can define the inner product

$$\langle\vec{k}|\vec{k}'\rangle = 2k^0 (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}').$$

which is a Lorentz invariant defintion.
Example: $|\psi; \vec{k}\rangle$

$$L_0|\psi; \vec{k}\rangle = (\alpha' p^2 - 1)|\psi; \vec{k}\rangle = 0 \Rightarrow k^2 = \frac{1}{\alpha'}.$$

$$Q_B|\psi; \vec{k}\rangle = 0, \quad |\psi; \vec{k}\rangle \neq Q_B|Z\rangle$$

Therefore $|\psi;\vec{k}\rangle$ are all the cohomology classes. Same as in the light-cone quantization.
Example: $|\psi\rangle = \left(A_\mu(\vec{k})\alpha^\mu_{-1} + \beta(\vec{k})b_{-1} + \gamma(\vec{k})c_{-1}\right)|\psi;\vec{k}\rangle$

$$\langle\psi|\psi\rangle = (A^*_\mu A^{\mu *} + \beta^*\gamma + \gamma^*\beta)\langle\psi;\vec{k}|\psi;\vec{k}\rangle$$

There are 26 positive-norm states: $A_i, \beta = \gamma \ \ (\alpha^i_{-1}|\psi;\vec{k}\rangle,\ (b_{-1}+c_{-1})|\psi;\vec{k}\rangle)$, 2 negative-norm states: $A_0,\ \beta = -\gamma,\ \ (\alpha^0_{-1}|\psi;\vec{k}\rangle,\ (b_{-1}-c_{-1})|\psi;\vec{k}\rangle)$.

$$Q_B|\psi\rangle = 0 \Rightarrow (c_{-1}k\cdot\alpha_1 + c_1 k\cdot\alpha_{-1})|\psi\rangle = 0 \Rightarrow (k_\mu A^\mu c_{-1} + \beta k_\mu \alpha^\mu_{-1})|\psi;\vec{k}\rangle = 0$$

Therefore $k\cdot A = 0$ and $\beta = 0$. This gets rid of negative-norm states $k^0 A_0 \neq 0$ for all $k^0 \neq 0$ and $\beta = \gamma = 0$ is the other negative-norm state. 26 states remain: 2 have zero-norm:

$$k_\mu\alpha^\mu_{-1}|\psi;\vec{k}\rangle, \quad c_{-1}|\psi;\vec{k}\rangle.$$

They are orthogonal to all physical states $\langle ...|\psi\rangle = 0$.
$c_{-1}|\psi;\vec{k}\rangle$ is **exact**.
Proof: Let $|Z\rangle = \tilde{A}_\mu\alpha^\mu_{-1}|\psi;\vec{k}\rangle,\ \ k\cdot\tilde{A}\neq 0$. Then $Q_B|Z\rangle = k\cdot\tilde{A}c_{-1}|\psi;\vec{k}\rangle$. Therefore $c_{-1}|\psi;\vec{k}\rangle = \frac{1}{k\cdot\tilde{A}}Q_B|Z\rangle$.
$k\cdot\alpha_{-1}|\psi;\vec{k}\rangle$ is **exact**.
Proof: Let $|Z\rangle = b_{-1}|\psi;\vec{k}\rangle$, then $Q_B|Z\rangle = k\cdot\alpha_{-1}|\psi;\vec{k}\rangle$. In each BRST cohomolgy class there is a gauege equivalence $A^\mu = A^\mu + \alpha k^\mu$.

No-Ghost Theorem

In the light-cone gauge, we considered the space $\mathcal{H}^\perp$ is a Hilbert space. **No-Ghost Theorem:** BRST cohomology is isomorphic to $\mathcal{H}^\perp$.
Definition: $\alpha^\pm_m = \frac{1}{\sqrt{2}}(\alpha^0_m \pm \alpha^1_m)$.
Therefore

$$[\alpha^+_m, \alpha^-_n] = -m\delta_{m+n,0}, \quad [\alpha^+_m,\alpha^+_n] = [\alpha^-_m,\alpha^-_n] = 0.$$

Q_1 is the part of Q proportional to the α^-_{-m} oscillators.

$$Q_1 = -\sqrt{2\alpha'}k^+\sum_m a^+_{-m}c_m.$$

$$L_m = \alpha^+_0\alpha^-_m + \sum_n \alpha^+_n\alpha^-_{m-n} + \frac{1}{2}\sum_n \alpha^i_n\alpha^i_{m-n}, \ \ Q = \sum_m L_{-m}c_m + ...$$

$$Q_1^2 = 0.$$

Definition: $R = \frac{1}{\sqrt{2\alpha'}k^+}\sum_m a^+_{-m}b_m$

$$S = \{Q_1, R\} = \sum_m (mb_{-m}c_m + mc_{-m}b_m - \alpha^+_{-m}\alpha^-_m - \alpha^-_{-m}\alpha^+_m)$$

N.B.:

$$[Q_1, S] = [Q, \{Q, R\}] = Q_1 R Q_1 - Q_1 R Q_1 = 0$$

Theorem: $|\psi\rangle \in Large(S) \cup Kernel(Q_1) \Rightarrow |\psi\rangle$ is **exact**.
Proof: $|\psi\rangle = S|Z\rangle, \quad Q_1|\psi\rangle = Q_1|Z\rangle = 0$. Therefore, $|\psi\rangle = \{Q, R\}|Z\rangle = Q_1 R|Z\rangle$.
Corollary: Cohomology of $|\psi\rangle$ non-trivial only if $|\psi\rangle \in ker(S)$, i.e., $S|\psi\rangle = 0$.
Now $|\psi\rangle \in \mathcal{H}^\perp \Rightarrow Q_1|\psi\rangle = 0$ (trivial from the definition of Q_1). From the definition of S in terms of the oscillators, $S|\psi\rangle = 0$ only if $|\psi\rangle$ has no $\alpha^{\pm}_{-m},\ b_{-m},\ c_{-m}$ excitations, therefore $|\psi\rangle \in \mathcal{H}^\perp$. Therefore the Q_1 cohomology is $\mathcal{H}^\perp$.
The no-ghost theorem for Q_1: Let us go back to Q_B.
Define $U = \{Q_B - Q_1, R\} = \{Q_B, R\} - S$, therefore $\{Q_B, R\} = S + U$. Map $|\psi\rangle \in ker(S) \mapsto |Z\rangle = (1 - S^{-1}U + S^{-1}US^{-1}U + ...)|\psi\rangle$. S^{-1} makes sense for all $U|\psi\rangle$ contains $\alpha^{\pm}_{-m}$ excitations therefore, $U|\psi\rangle \notin ker(S)$ Clearly, $(S+U)|\psi\rangle = S|\psi\rangle = 0$. This establishes an isomorphism

$$ker(S) \cong ker(S + U).$$

We can show the cohomology of $Q_B \cong ker(S + U)$ just like we showed cohomology of $Q_1 \cong ker(S)$. Therefore, $coh(Q_B) \cong ker(S + U) \cong ker(S) \cong coh(Q_1) \cong \mathcal{H}^\perp$. Q.E.D.
Inner products: $\langle Z_1|Z_2\rangle = \langle \psi_1|\psi_2\psi\rangle$ (positive definite).

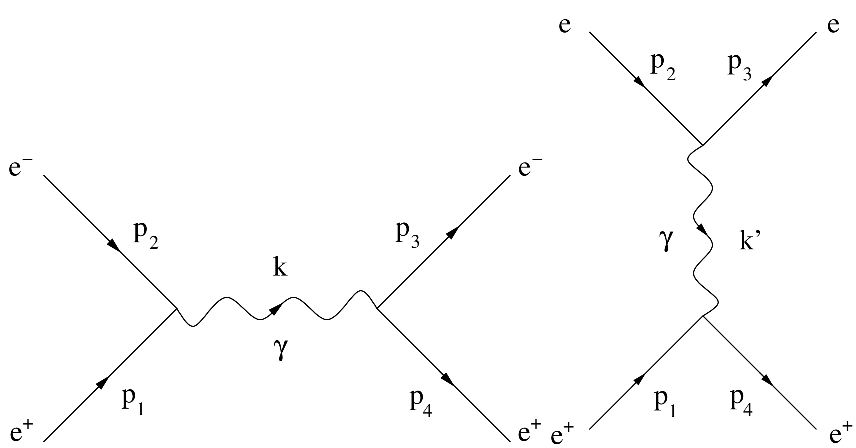
UNIT 4

Tree-level Amplitudes

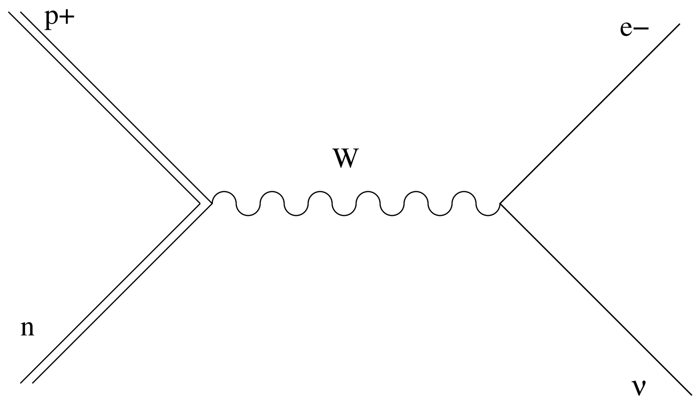
4.1 String Interactions

In particle theory, we need to introduce a multi-particle space (Fock space) where creation and annhilation are possible. In string theory, the tools we have developed for one string are sufficient for the description of multi-string states and interactions! The entire quantum theory of strings is based on these tools!

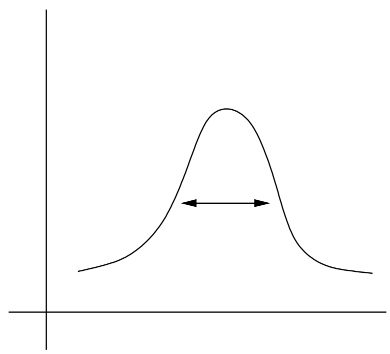
Example of particle interactions



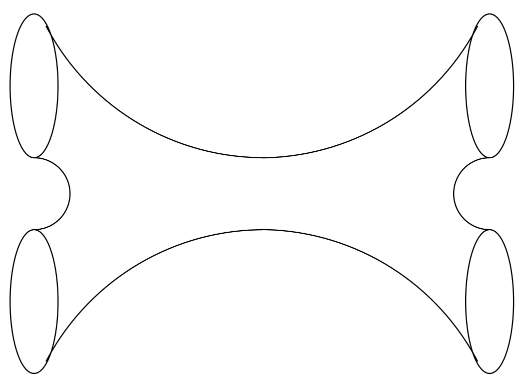
$1/k^2$: inverse of the Klein-Gordon operator $\Box\phi = 0$, $\Box^{-1} \sim 1/k^2$ There is a pole at $k^2 \sim 0$, e.g., β-decay



Amplitude$\sim \frac{1}{k^2 - m_W^2}$, pole at $k^2 = m_W^2$, resonance.

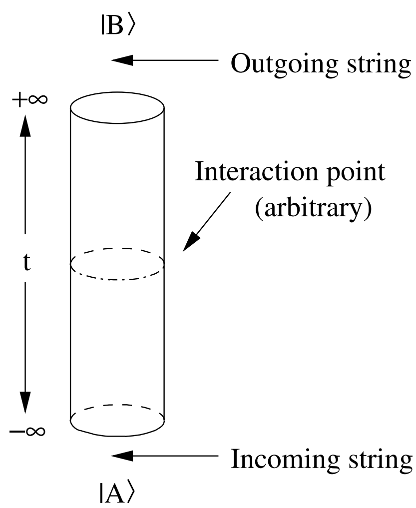


Strings

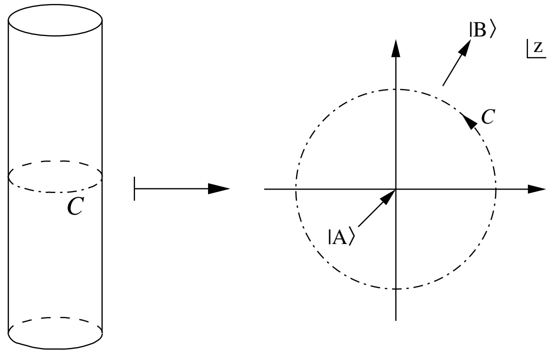


The interaction consists of strings joining and splitting. Where do they join? This is a stupid question. It depends on the time slicing. Therefore this is a *fuzzy* interaction. Moreover, the shape (geometry) of the surface is not important, only the topology is important. There is one diagram for *all* tree diagrams.

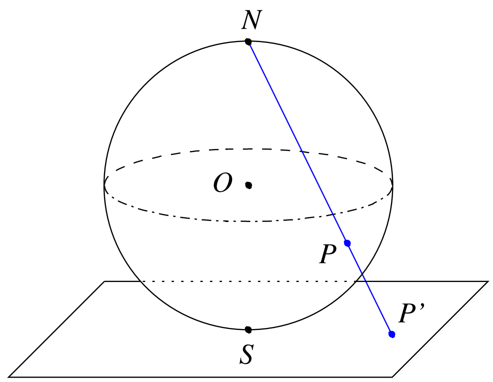
Example



There is an arbitrary interaction point. The amplitude is constructed by joining two semi-infinite cylinders. Map the cylinders to a plane:



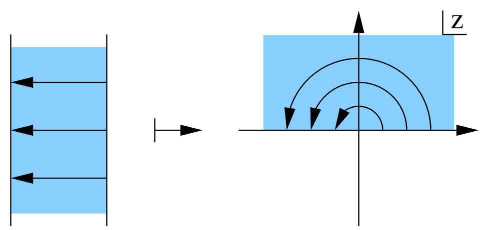
cylinder $\to \mathbb{C}\bigcup\{\infty\} = S^2$ (sphere). This is done through *stereographic* projection (sphere=fat cylinder).



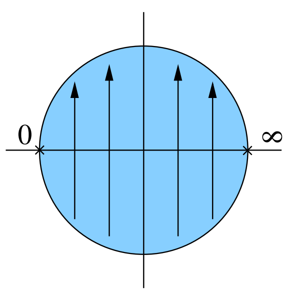
Amplitude: sphere with states (operator insertions) at North and South poles. Notice the equivalence of the two poles (clinder $z \to \frac{1}{z}$).

Open Strings

Make a strip by cutting the cylinder in half along the axis.



We then map the strip to the upper-half plane which can then be mapped to the unit circle via the mapping $z \to \frac{z-i}{z+i}$.



Each string is a semi-infinite cylinder (or strip), which is mapped to a disk. When we put two on a sphere, they were simply represented by insertion of $A(z)$ at $z = 0,\ z = \infty$.
Guess: For scattering of N strings we can do the same, i.e., on a sphere select points $z_1, z_2, ..., z_N$ and insert operators $A_i(z_i)$. Then the amplitude is

$$A \sim \langle 0|A_1(z_1)A_2(z_2)...A_N(z_N)|0\rangle.$$

Now $A(0)$ is equivalent to $\oint_{\mathcal{C}} \frac{dz}{2\pi i} A(z)$.
For conformal invariance, we require that all A_i have dimension $h_i = 1$ so that $\int dz A_i(z)$ have zero dimension (conformally). Then we should define

$$Amp \sim \int dz_1 ... dz_N \langle 0| A_1(z_1) ... |0\rangle.$$

In fact, the measure should read $\int d^2 z_1 ... d^2 z_N$, but we will not be writing the $\bar{z}$ piece explicitly. The proper dimension of $A_i(z, \bar{z})$ should be $h_i = 1, \ \bar{h}_i = 1$. In general,

$$A(z) \sim : \partial^{m_1} X \partial^{m_2} X ... e^{ik\cdot X} :,$$

where $h = m_1 + m_2 + ... \alpha' k^2 = 1$. We shall work with the simplest case $A(z) = e^{ik\cdot X}, \ k^2 = \frac{1}{\alpha'}$. The rest is similar.
Complication: The amplitude is conformally invariant: $z \to z + \epsilon v(z)$ where $v(z)$ is analytic. $v(z)$ should be analytic everywhere in $\mathbb{C} \cup \{\infty\}$. We need to check that the transformation is analytic at infinity. So let $z \mapsto \frac{1}{z} = z'$.

$$\delta z' = -\frac{1}{z^2}\delta z = -\epsilon \frac{1}{z^2} v(z) = -\epsilon z'^2 v\left(\frac{1}{z'}\right).$$

therefore $v(z) = a + bz + cz^2$ so that $z'^2 v\left(\frac{1}{z'}\right)$ is analytic. This is a six-parameter family of transformations. It includes SO(3) (rotation group). Special Cases:
• $z \mapsto z + \epsilon a$ generated by L_{-1}. Recall $[L_m, A] = z^{m+1}\partial A + h(m+1) z^m A$ where $h = 1$ for BRST invariance. So $[L_{-1}, A] = \partial A - \frac{1}{z} A$ i.e., L_{-1} generates translations in z.
Finite transformation: $z \mapsto z + a$,

$$A(z) \to e^{aL_{-1}} A(z) e^{-aL_{-1}} = A(z+a).$$

•$z \mapsto z + \epsilon b z = (1 + \epsilon b) z$ generated by L_0.

$$[L_0, A] = z\partial A + A.$$

$$A(z) \to e^{bL_0} A(z) e^{-bL_0} = A(e^b z).$$

Finite transformation: $z \to e^b z$.
•$z \mapsto z + \epsilon c z^2$ generated by L_1.

$$[L_1, A] = z^2 \partial A + zA.$$

Finite transformations: $z \to \frac{z}{1 - cz} = z'$

$$A(z) \to e^{cL_1} A(z) e^{-cL_1} = A\left(\frac{z}{1 - cz}\right).$$

Combination of all three: $z \mapsto \frac{az+b}{cz+d}, \ \ ad - bc = 1$ defines the group SL(2, $\mathbb{C}$) whose algebra is

$$[L_1, L_{-1}] = 2L_0, \quad [L_1, L_0] = L_1, \quad [L_{-1}, L_0] = -L_{-1}.$$

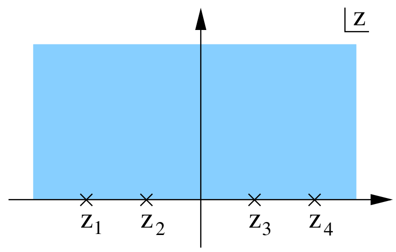
This is a closed algebra (no constant term) and is common in *all* conformal field theories.
How come a matrix entered acting on a number z? **Answer**: Consider the vector

$$(z_1,\ z_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + dz_2 \\ cz_1 + dz_2 \end{pmatrix}.$$

Let $z = z_1/z_2$. Then

$$z \mapsto \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{az + b}{cz + d}.$$

For **open strings**: the Real axis is a boundary, and the group of symmetries becomes SL$(2, \mathbb{R})$, $a, b, c, d \in \mathbb{R}$. Then under $z \rightarrow \frac{az+b}{cz+d}$, ∂ is invariant. The upper-half plane maps to itself.
Amplitude for open strings:



$$Amp \sim \langle V(z_1)V(z_2)...V(z_N)\rangle, \quad z_i \in \mathbb{R},$$

where the product is time ordered and thus the z_is are ordered. How do we integrate over z_i? Due to SL$(2, \mathbb{R})$ symmetry, we have redundency, so naive integral would be proportional to the volume of SL$(2, \mathbb{R})$ which is infinite! We need to fix the gauge by choosing three points. Easiest to fix them to $(0, 1, \infty)$. This is an arbitrary choice, but all choices are equivalent by the SL$(2, \mathbb{R})$ symmetry. We will integrate over the rest of the parameters.

Example 1: Three tachyons

Consider three tachyons, $V_i(z) =: e^{ik_i \cdot X(z)}$: The amplitude is given by

$$A \sim \langle 0|V_1(z_1)V_2(z_2)V_3(z_3)|0\rangle,$$

where

$$X^\mu(z) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}a_m^\mu\left(z^{-m} + \bar{z}^{-m}\right).$$

Since $z \in \mathbb{R}$, X^μ reduces to

$$X^\mu(z) = x^\mu - i\alpha' p^\mu \ln|z| + i\sqrt{2\alpha'}\sum_{m\neq 0}\frac{1}{m}a_m^\mu z^{-m}.$$

Since we can fix three points, let us choose ($z_1=\infty,\ z_2=1,\ z_3=0$), then

$$V_3(z_3=0)|0\rangle=|0;k_3\rangle,\quad \langle 0|V_1(z_1=\infty)=\langle 0;-k_1|.$$

The amplitude becomes

$$A\sim\langle 0;-k_1|V_2(z_2=1)|0;k_3\rangle=\langle 0;-k_1|0;k_2+k_3\rangle=\delta^D(k_1+k_2+k_3).$$

One can derive this for arbitrary z_1,z_2,z_3, due to the SL(2,$\mathbb{R}$) symmetry.

Example 2: Two tachyons and one vector

Consider two tachyons, $V_i(z)=:e^{ik_i\cdot X(z)}:$ and a vector, $V_j(z)=:A^\mu\partial X_\mu e^{ik_j\cdot X(z)}:$, where $k_j^2=0$. We may act the vertex operators on the vacuum states

$$\begin{aligned} A &\sim \langle 0;-k_1|V_2(1)A^\mu\partial X_\mu|0;k_3\rangle\sim\langle 0;-k_1|e^{ik_2\cdot x}e^{\sqrt{\frac{\alpha'}{2}}a_1\cdot k_2}A_\mu\alpha^\mu_{-1}|0;k_3\rangle \\ &\sim \sqrt{2\alpha'}A\cdot k_2\delta^D(k_1+k_2+k_3). \end{aligned} \tag{4.1.1}$$

A is transverse to it's momentum ($A\cdot k_3=0$) therefore, the amplitude is

$$A\sim\sqrt{\frac{\alpha'}{2}}A\cdot(k_2-k_1)\delta^D(k_1+k_2+k_3),$$

where the dot product represents the coupling of the electromagnetic potential to the charged scalar. We may check the gauge invariance of the amplitude. Using the gauge transformation $A^\mu\ \rightarrow\ A^\mu+\omega k_3^\mu$, the amplitude becomes

$$\delta(A)\sim k_3\cdot(k_2-k_1)=k_2^2-k_1^2=0.$$

Example 3: Four tachyons

This is the first nontrivial amplitude. Due to the SL$(2,\mathbb{R})$ symmetry, we may fix three operators. Now we have an extra operator we can not fix. We must integrate over its parameter. After we operate vertex operators on the vacuum states, the amplitude is given by

$$A\sim\langle 0;-k_1|:e^{ik_2\cdot X}(1)::e^{ik_3\cdot X}(z):|0;k_3\rangle.$$

This is a time-ordered product and we must integrate over z from $[0,1]$. The amplitude becomes

$$A\sim\int_0^1 dz\langle 0;-k_1|:e^{ik_2\cdot X}(1)::e^{ik_3\cdot X}(z):|0;k_4\rangle.$$

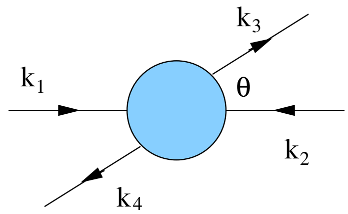
Using the mode expansion of X^μ,

$$A\sim\int_0^1 dz\langle 0;-k_1|e^{ik_2\cdot x}e^{\sqrt{2\alpha'}\sum_{m>0}k_2\cdot\alpha_m/m}e^{ik_3\cdot x}e^{i\frac{\alpha'}{2}k_3\cdot p\ln|z|}e^{\sqrt{2\alpha'}\sum_{n>0}k_3\cdot\alpha_{-n}/n}|0;k_4\rangle.$$

Using the Hausdorff formula, $e^A e^B = e^{[A,B]} e^B e^A$,

$$\begin{aligned} A &\sim \int_0^1 dz \langle 0; -k_1 | z^{2\alpha' k_3 \cdot k_4} e^{-2\alpha' k_2 \cdot k_3 \sum z^m/m} | 0; k_3 + k_4 \rangle \\ &\sim \int_0^1 dz z^{2\alpha' k_3 \cdot k_4} (1-z)^{-2\alpha' k_2 \cdot k_3} \delta^D (k_1 + k_2 + k_3 + k_4) \end{aligned}$$

Define the Mandelstam variables



$$s = (k_1 + k_2)^2 = (k_3 + k_4)^2 = -2k_3 \cdot k_4 - \frac{2}{\alpha'}, \quad t = -(k_2 + k_3)^2, \quad u = -(k_2 + k_4)^2,$$

and

$$s + t + u = -\frac{4}{\alpha'}.$$

The amplitude expressed in terms of Mandelstam variables becomes

$$A \sim \int_0^\infty dz z^{-\alpha' s - 2} (1-z)^{-\alpha' t - 2} \delta^D (k_1 + k_2 + k_3 + k_4) \sim B(-\alpha' s - 1, \alpha' t - 1),$$

where B is the Euler-beta function with the property

$$B(x, y) = \frac{\Gamma(y)\Gamma(y)}{\Gamma(x + y)}.$$

This is known as the *Veneziano* amplitude. Note, there are poles at $-\alpha' s - 1 = 0$ and $\alpha' t - 1 = 0$. Let us focus on the first pole $(-\alpha' s - 1 = 0)$.

$$\begin{aligned} A &\sim \Gamma(-\alpha' s - 1) = \frac{\Gamma(-\alpha' s)}{\alpha' s + 1} + \ldots \quad (\Gamma(x+1) = x\Gamma(x)) \\ &\sim -\frac{1}{-\alpha' s + 1} + \ldots \end{aligned}$$

The pole is due to an intermediate tachyon $(s = -1/\alpha')$. Unitarity requires This checks, since The next pole is at $\alpha' s = 0$.

$$\Gamma(-\alpha' s - 1) = -\frac{\Gamma(-\alpha' s)}{\alpha' s + 1} = \frac{\Gamma(-\alpha' s + 1)}{(\alpha' s + 1)(\alpha' s)} = \frac{1}{\alpha' s} + \ldots$$

The amplitude becomes

$$A \sim \frac{1}{\alpha' s} \frac{\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha' t - 2)} = \frac{\Gamma(-\alpha' t - 2)}{\alpha' s} + \ldots = \frac{u - t}{2s} + \ldots$$

where we used the condition $s+t+u=-4/\alpha'$.
Check unitarity: The amplitude is gauge invariant. Summing over the polarizations $\sum \epsilon^\mu \epsilon^\nu = \eta^{\mu\nu}$ gives the amplitude

$$A \sim \alpha' \frac{(k_1-k_2)(k_3-k_4)}{2k^2} = \frac{u-t}{2s}.$$

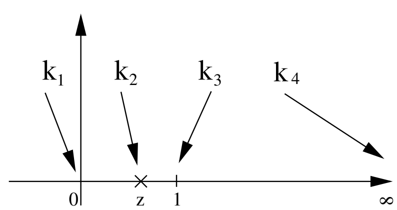
All the poles in $\alpha' s$: $\alpha' s = -1, 0, 1, 2, ...$ which are the masses of the open string states. ($\alpha' s^2 = N-1$ from $L_0 - 1 = 0$). Curious Result: same structure of poles we obtain for $\alpha' t$, since the amplitude is symmetric in s and t. This would also be true of a field theory amplitude.
Alternate derivation of the poles: It is instructive to find the poles without performing the integral for two reasons. (a) We can not always do the integral. (b) We can see what type of world-sheet contributes to the pole (physical picture for an effective field theory).
Let $z \to 0$

$$A \sim \int_0 dz z^{-\alpha' s-2} + ... = \left. \frac{z^{-\alpha' s=1}}{-\alpha' s-1} \right|_0 + ... = -\frac{1}{\alpha' s+1} + analytic.$$

Taylor expansion:

$$A \sim \int_0 z^{-\alpha' s-2}(1-z)^{-\alpha' t-2} = \int_0 dz z^{-\alpha' s-2}(1+(\alpha' t+2)z+...),$$

where the first and second terms in the expansion represent the $\alpha' s = -1$ and $\alpha' s = 0$ poles respectively. The other poles are acquired through higher order terms in the expansion. Poles in $\alpha' t$ are obtained from $z \to 1$.



There is no reason to restrict $\int dz$ to $\int_0^1 dz$. We would like to extend the integral to $\int_\infty^\infty dz$. The integral becomes $\int_{-\infty}^0 + \int_0^1 + \int_1^\infty$.
$\int_{-\infty}^0$: ordering $(k_1+k_2+k_3+k_4)$ which is $\int_0^1$ with $k_2 \leftrightarrow k_1$. The effect is switching t and u. This can be seen through the transformation $z \mapsto 1-\frac{1}{z}$ which maps $(0,1) \mapsto (-\infty, 0)$. Therefore, if $\int_0^1 = I(s,t)$ then $\int_{-\infty}^0 = I(t,u)$. Similarly, $\int_1^\infty = I(s,u)$. Therefore, the integral becomes

$$\int_{-\infty}^{\infty} = I(s,t) + I(s,u) + I(t,u).$$

Now the amplitude is completely symmetric in s, t, u.

BRST invariance

If $V(z)$ has weight $h=1$, then $\int dzV(z)$ has weight $h=0$. It is BRST invariant. Let us check this.

$$\begin{aligned}[Q,V(z)] &= \sum c_{-n}[L_n,V(z)] = \sum c_{-n}(z^{n+1}\partial V(z)+(n+1)z^nV(z)) \\ &= c(z)\partial V(z)+\partial c(z)V(z) = \partial(c(z)V(z))\end{aligned}$$

Therefore

$$[Q,\int V] = \int \partial(c(z)V(z)) = 0.$$

What happens with the three Vs that we fixed? To turn them into $h=0$ operators, we multiply them by $c(z)$. Then $c(z)V(z)$ has the weight $h=0$.

$$\{Q,cV\} = \{Q,c\}V - c[Q,V] = c\partial cV - cc\partial V - c\partial cV = 0.$$

Now in the amplitude, we have three $c(z)$s, $z_i=0,1,\infty$. The amplitude must be defined with respect to the SL$(2,\mathbb{R})$ invariant vacuum. Recall:

$$b_0|\psi\rangle = 0, \quad |\chi\rangle = c_0|\psi\rangle.$$

$$L_m^{bc} = \sum_n (2m-n) : b_n c_{m-n} : -\delta_{m,0}$$

So,

$$L_0^{bc} = \sum_n n : b_{-n}c_n : -1,\ L_1^{bc} = \sum_n (2-n) : b_n c_{-n} :,\ L_{-1}^{bc} = \sum_n (-2-n) : b_n c_{-n-1} : .$$

The operators act on the states

$$L_0^{bc}|\psi\rangle = -|\psi\rangle, \quad L_1^{bc}|\psi\rangle = 0, \quad L_{-1}^{bc}|\psi\rangle = b_{-1}|\chi\rangle,$$

So $|psi\rangle$ is *not* invariant. Let $|0\rangle = b_{-1}|\psi\rangle$.

$$[L_0^{bc},b_{-1}] = b_{-1},\ \ [L_1^{bc},b_{-1}] = 2b_0,\ \ [L_{-1}^{bc},b_{-1}] = 0.$$

Therefore,

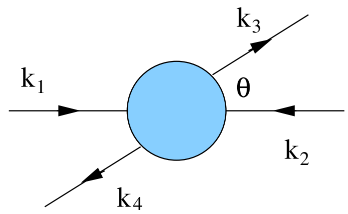
$$L_0^{bc}|0\rangle = b_{-1}|\psi\rangle - b_{-1}|\psi\rangle = 0,\ L_{-1}^{bc}|0\rangle = b_{-1}b_{-1}|\psi\rangle = 0,\ L_{-1}^{bc}|0\rangle = b_{-1}b_{-1}|\chi\rangle = 0.$$

So, $|0\rangle$ *is* SL$(2,\mathbb{R})$ invariant.
The ghost contribution is

$$\langle 0|c(\infty)c(1)c(0)|0\rangle, \quad c(z) = \sum_n c_n z^{-n+1}.$$

$$c(0)|0\rangle = c_1|0\rangle = |\psi\rangle, \quad \langle 0|c(\infty) = \langle\psi|, \quad \psi|c(1)|\psi\rangle = \langle\psi|c_0|\psi\rangle = 1.$$

High Energy



$$k_1^\mu = (E/2, \vec{p}),\ \ k_2^\mu = (E/2, -\vec{p}),\ \ k_3^\mu = (-E/2, -\vec{p'}),\ \ k_4^\mu = (-E/2, \vec{p'}).$$

where $\left(\frac{E}{2}\right)^2 - \vec{p}^2 = m^2,\ \ |\vec{p'}| = p$. The Mandelstam variables become

$$s = -(k_1+k_2)^2 = E^2,\ t = -(k_1+k_3)^2 = (4m^2-E^2)\sin^2\frac{\theta}{2},\ u = -(k_1+k_4)^2 = (4m^2-E^2)\cos^2\frac{\theta}{2}.$$

The high energy limit is equivalent to the small angle limit, where $s \to 0$ and t is fixed. The gamma function is approximated by

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}.$$

The amplitude is

$$A \approx \frac{\Gamma(-\alpha' s - 1)\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha' s - \alpha' t - 2} \approx \frac{s^{-\alpha' s - 1}}{s^{-\alpha' s - \alpha' t - 2}} e^{\alpha' t + 1}\Gamma(-\alpha' t - 1) \sim s^{\alpha' t + 1}\Gamma(-\alpha' t - 1).$$

This is the Regge behavior. At the poles $\alpha' t - 1 \sim -n$, the amplitude goes as $A \sim s^n$ which is the exchange of a particle of spin n.
For a fixed angle,θ=fixed: $s, t \to \infty,\ s/t =$ fixed. The amplitude becomes

$$\begin{aligned} A &\sim \frac{s^{-\alpha' s - 1} t^{-\alpha' t - 1}}{(s+t)^{-\alpha' s - \alpha' t}} \sim \frac{s^{-\alpha' s} t^{-\alpha' t}}{u^{\alpha' u}} \sim e^{-\alpha'(s\ln s + t\ln t + u\ln u)} \\ &\approx e^{-\alpha'(s\ln(s/s) + t\ln(t/s) + u\ln(u/s))} \\ &\approx e^{-\alpha' s(\frac{t}{s}\ln\frac{t}{s} + \frac{u}{s}\ln\frac{u}{s}} \\ &\approx e^{-\alpha' s\left(-\sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2}\right)} \\ &\approx e^{-Cs},\quad C > 0. \end{aligned}$$

unlike in field theory, where the amplitude goes as $A \sim s^{-n}$. Therefore the underlying smooth extended object of size $\sqrt{\alpha'}$.

4.2 A Short Course in Scattering Theory

We define the $\langle in|$ state in the *real* infinite past ($t \to -\infty$), and the $|out\rangle$ state in the infinite future ($t \to \infty$). These states are both described by free particles. There is an isomorphism

$$|in\rangle = S|out\rangle,\quad S = \lim_{t\to\infty} e^{iHt/\hbar}..$$

To conserve probabilities, S must be unitary, $S^\dagger S = 1$ (c.f. unitarity of evolution operator, $U = e^{iHt/\hbar}$). The transition probability ($S = I + iT$) is

$$|\langle i_{-\infty}|f_\infty\rangle|^2 = |\langle i|T|f\rangle|^2,$$

where $|i>$ and $|f\rangle$ represent states in the same Hilbert space. We will discard the I because it represents $|i\rangle \to |i\rangle$ (forward scattering i.e., along the beam: undetectable).
Unitarity

$$S^\dagger S = I = I + i(T - T^\dagger) + T^\dagger T.$$

Therefore

$$\langle i|T|f\rangle - \langle i|T^\dagger|f\rangle^* = i\langle i|T^\dagger T|f\rangle.$$

Insert complete sets of physical states

$$\begin{aligned} \langle i|T|f\rangle - \langle i|T^\dagger|f\rangle^* &= i\sum_n \langle i|T^\dagger|n\rangle\langle n|T|f\rangle, \\ 2Im < i|T|f\rangle &= \sum_n < i|T|n\rangle\langle f|T|n\rangle^*. \end{aligned} \tag{4.2.1}$$

Viewed as a function of s, $\langle i|T|f\rangle$ has poles in s. Away from the pole, $\langle i|T|f\rangle$ is real, so the left hand side vanishes.
Near the pole, we obtain a behavior $\sim \frac{1}{s+m^2}$ (pole at $s = -m^2$). to find the imaginary part, first *regulate* the amplitude

$$\frac{1}{s+m^2} \to \frac{1}{s+m^2+i\epsilon}$$

for small ϵ. Then

$$Im\frac{1}{s+m^2} \to Im\frac{1}{s+m^2+i\epsilon} = \frac{-\epsilon}{(s+m^2)^2+\epsilon^2} = -\pi\delta(s+m^2).$$

Therefore, for

INSERT FIGURE HERE

the imaginary part is

INSERT FIGURE HERE

This is in agreement with unitarity.

4.3 N-point open-string tree amplitudes

$$Amp \sim \langle : e^{ik_1\cdot X}(z_1) : ... : e^{k_2\cdot X}(z_n) :\rangle = A(z_1, ..., z_n)$$

Consider

$$\partial_1 A(z_1, ..., z_n) \sim \langle \partial_{z_1} : e^{ik_1 \cdot X}(z_1) : ... : e^{k_2 \cdot X}(z_n) : \rangle$$

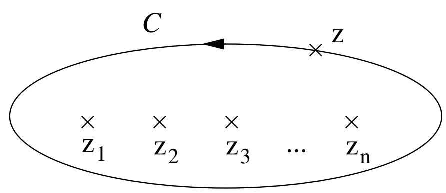
To evaluate this, consider the OPE

$$ik \cdot \partial X(z) : e^{ik_1 \cdot X}(z_1) := \frac{\alpha' k_1^2}{2(z - z_1)} e^{ik_1 \cdot X(z_1)} : + \partial_1 : e^{ik_1 \cdot X}(z_1) : + ...$$

So, first replace $\partial_1 : e^{ik_1 \cdot X}(z_1) :$ by $\partial X(z) : e^{ik_1 \cdot X}(z_1) :$ in Amp and define

$$f^\mu(z) = \langle \partial X^\mu(z) : e^{ik_1 \cdot X}(z_1) : ... : e^{ik_n \cdot X}(z_n) : \rangle$$

The singularity structure of $f^\mu(z)$ can be deduced from OPEs



$$\partial X(z) : e^{ik_1 \cdot X}(z_1) := -\frac{i\alpha' k_1^\mu}{2(z - z_1)} e^{ik_1 \cdot X(z_1)} : + ...$$

Therefore,

$$f^\mu(z) = -\frac{i\alpha'}{2} A(z) \sum_{i=1}^{n} \frac{k_1^\mu}{z - z_i} + ...$$

Behavior at $z \to \infty \; : z' = \frac{1}{z}$

$$\partial X^\mu = \frac{\partial z'}{\partial z} \partial' X^\mu = -\frac{1}{z^2} \partial' X^\mu$$

which implies

$$f^\mu(z) = -\frac{1}{z^2} \langle \partial' X^\mu + ... \rangle.$$

Therefore, as $z \to \infty$, $f^\mu(z) \sim \frac{1}{z^2} \to 0$ $(\langle \partial' X^\mu : ... : \rangle$ analytic at $\infty)$ Therefore the holomorphic piece vanishes and

$$f^\mu(z) = -\frac{i\alpha'}{2} A \sum_{i=1}^{n} \frac{k_i^\mu}{z - z_i}.$$

Now define a contour $\mathcal{C}$ surrounding all z_i's. There are two ways to evaluate the contour integral, $\oint \frac{dz}{2\pi i} f(z)$. Cauchy $\Rightarrow \oint \frac{dz}{2\pi i} f^\mu(z) = -\frac{i\alpha'}{2} A \sum_{i=1}^n \frac{k_i^\mu}{z - z_i}$, or in the transformed coordinate $\frac{z'=1}{z}$, $\mathcal{C}$ encircles $z' = 0$ where $f^\mu(z)$ is analytic. Therefore

$$\oint \frac{dz}{2\pi i} f^\mu(z) = 0 \Rightarrow \sum_{i=1}^{n} k_i^\mu = 0$$

The momentum is conserved.
Now consider $ik \cdot f$ and compare with the OPE

$$ik \cdot \partial x(z) : e^{ik \cdot X}(z_1) := \frac{\alpha' k_1^2}{2(z - z_1)} : e^{ik_1 \cdot X}(z_1) : + \partial_1 : e^{ik_1 \cdot X}(z_1) : + ...$$

which implies

$$ik \cdot f = \frac{\alpha'}{2} \frac{k_1^2}{z - z_1} A + \frac{\alpha'}{2} \sum_{i \neq 1} \frac{k_1 \cdot k_i}{z - z_i} A$$

Therefore

$$\partial_1 A = \frac{\alpha'}{2} \frac{k_1^2}{z - z_1} A + \frac{\alpha'}{2} \sum_{i \neq 1} \frac{k_1 \cdot k_i}{z - z_i} A.$$

Therefore

$$\partial_1 \ln A = \frac{\alpha'}{2} \sum_{i \neq 1} \frac{k_1 \cdot k_i}{z_1 - z_i} A.$$

Repeating for other points,

$$\partial_j \ln A = \frac{\alpha'}{2} \sum_{i \neq j} \frac{k_j \cdot k_i}{zi - z_j} A.$$

By integrating we obtain $\ln A = \sum_{i<j} \ln |z_i - z_j|^{k_i \cdot k_j} + const$ where we added the $\bar{z}$ piece. Therefore,

$$A \propto \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j}.$$

For open strings, $\alpha' \to 2\alpha'$, so

$$A \propto \prod_{i<j} |z_i - z_j|^{2\alpha' k_i \cdot k_j}.$$

SL($2,\ \mathbb{R}$) Invariance

$$z \to z' = \frac{az + b}{cz + d}, \qquad czz' + dz' = az + b \to z = \frac{dz' - b}{a - cz'}, \quad ad - bc = 1.$$

Therefore,

$$z_i - z_j = \frac{dz_i' - b}{a - cz_i'} - \frac{dz_j - b}{a - cz_j'} = \frac{z_i' - z_j'}{(a - cz_i')(a - cz_j')}. \tag{4.3.1}$$

Therefore,

$$A \propto \prod_{i<j} |z_i - z_j|^{2\alpha' k_i \cdot k_j} = \prod_{i<j} |z_i' - z_j'|^{2\alpha' k_i \cdot k_j} \prod (a - cz_i')^{2\alpha' k_i^2}, \quad k_i^2 = \frac{1}{\alpha'}. \tag{4.3.2}$$

$$dz_i = \frac{dz_i}{(a - cz_i')^2}.$$

If we let $z_j \rightarrow z_i$ in (4.3.1), we find that the amplitude is invariant under SL(2, $\mathbb{R}$) transformations. The measure is given by

$$\prod dz_i = \prod dz_i' \prod (a - cz_i')^{-2},$$

however, the last factor cancels with the overall factor in (4.3.2).

4.4 Closed Strings

For open strings we found four tachyons,

$$\begin{aligned} A_{open} &\sim \int_{-\infty}^{\infty} dz\ z^{2\alpha' k_3 \cdot k_4} (1-z)^{2\alpha' k_2 \cdot k_3} \delta^D (k_1 + k_2 + k_3 + k_4) \\ &= \int_{-\infty}^{0} + \int_0^1 + \int_1^{\infty} \end{aligned}$$

where

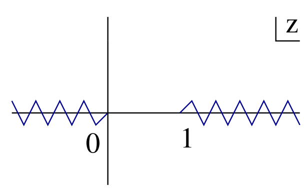
$$\int_0^1 = I(s,t) = \int_0^1 dz\ z^{-\alpha' s - 2} (1-z)^{-a' t - 2} \delta^D (k_1 + k_2 + k_3 + k_4)$$

$$\int_{-\infty}^{0} = I(t,u), \quad \int_1^{\infty} = I(s,u), \ \ z \in \mathbb{R}.$$

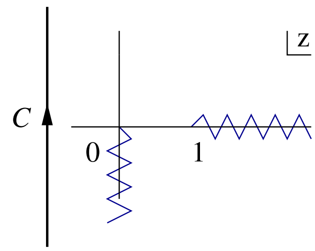
For closed strings, z is the entire $\mathbb{C}$ and we need to multiply the holomorphic and anti-holomorphic pieces, so

$$A_{closed} \sim \int d^2 z\ |z|^{-\alpha' s/2 - 4} |1-z|^{-\alpha' t/2 - 4}.$$

Note: $s \rightarrow s/4$ is due to the different expansion of the $X^\mu s$. The tachyon mass is $m^2 = -\frac{4}{\alpha'}$, whereas for the open string it is, $m^2 = -\frac{1}{\alpha'}$.
To calculate the amplitude for the closed string, treat z and $\bar{z}$ as independent variables and deform the contour of integration until it coincides with the real axis. Then $z, \bar{z} \in \mathbb{R}$. We must take care with the branch cuts.
There are three cases.
(i) $\bar{z} < 0$: Contour for z:



$\mathcal{C}$ has branch cuts on the same side and therefore contributes nothing.
(ii) $\bar{z} > 1$:
There is no contribution for the same reason as in (i).
(iii) $0 < \bar{z} < 1$:



$$A_{closed} \sim \oint dz\ z^{-\alpha' s/4-2}(1-z)^{-\alpha' t/4-2} \times \int_0^1 d\bar{z}\ \bar{z}^{-\alpha' s/4-2}(1-\bar{z})^{-\alpha' t/4-2}$$

Contribution from the upper side of $\mathcal{C}$ is

$$\int_1^\infty d\eta |\eta|^{-\alpha' s/4-2} e^{-i\pi(\alpha' t/4+2)} |1-\eta|^{-\alpha' t/4-2} \times I(s/4, t/4).$$

The lower side gives

$$\int_1^\infty d\eta\ |\eta|^{-\alpha' s/4-2} e^{+i\pi(\alpha' t/4+2)} |1-\eta|^{-\alpha' t/4-2} \times I(s/4, t/4).$$

Therefore the amplitude for the closed string is

$$A_{closed} \sim \sin\frac{\pi\alpha' t}{4} I(t/4, u/4) I(s/4, t/4).$$

This can be cast in a symmmetric form by using the transformation properties of the Gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad for \frac{-\alpha' t}{4} - 1$$

So, since $s + t + u = 4m^2 = -16/\alpha'$

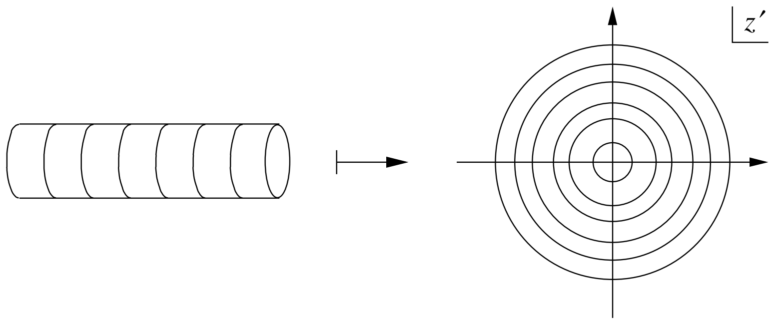
$$\Gamma(-\alpha' t/4 - 1)\Gamma(2 + \alpha' t/4) = \frac{\pi}{\sin(\alpha' t\pi/4)}.$$

Therefore the amplitude is given by

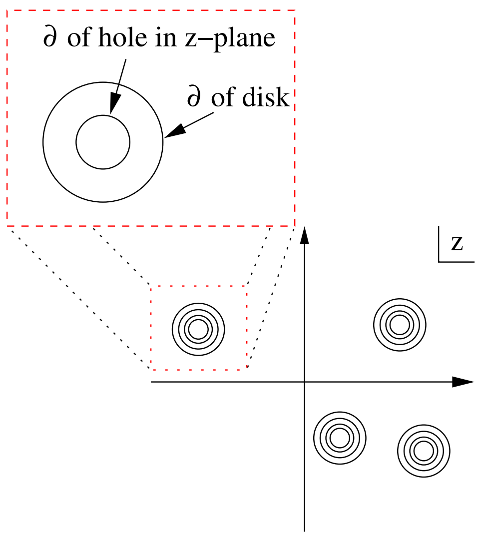
$$A_{closed} \sim \pi \frac{\Gamma(-\alpha' s/4-1)\Gamma(-\alpha' t/4-1)\Gamma(-\alpha' u/4-1)}{\Gamma(-\alpha' s/4-\alpha' t/4-1)\Gamma(-\alpha' t/4-\alpha' u/4-1)\Gamma(-\alpha' u/4-\alpha' s/4-1)}.$$

4.5 Moduli

Build closed-string four-point amplitude as follows. In the z-plane, drill holes. This will represent the diagram on the left with amputated legs. Now attach the legs by telescopically collapsing each semi-infinite tube to a disc:



Next, patch the discs on the z-plane. This produces a sphere with four punctures. There will be regions of overlap where $z' = f(z)$.



By conformal transformations, I can fix three points (due to SL$(2, \mathbb{C})$ symmetry). The fourth point cannot be fixed. Call it z. Punctured spheres with two different z's, are **not** related by a conformal transformation. There are inequivalent surfaces.
They are parametrized by **two** parameters, z_1 and $\bar{z}_1 \in \mathbb{C}$. These parameters are called moduli and their space, moduli space (although it should be called modulus space) (c.f. vector space). They are also called Teichmuller parameters. They label conformally inequivalent surfaces. To calculate amplitudes, we need to integrate over the moduli.

E.g., the four-point amplitude, $\langle V_1(\infty)V_2(1)V_3(z_1)V_4(0)\rangle$ need to integrate over $z_1 \to \int d^2z_1$. In general, N-point amplitudes integrate $\int d^2z_1...d^2z_{N-3}$ at $z = \infty,\ 1,\ 0$ we specified $V \sim c\tilde{c} : e^{ik\cdot X}$:. We can do the same for the unfixed V's to put them all on equal footing.
Thus, let $V_i = c\tilde{c} : e^{ik_i\cdot X}$:, $\forall i$. Since we introduced an extra $c,\ \tilde{c}$, we need to compensate for it with a $b,\ \tilde{b}$ insertion.
To do this work as follows. Shift $z_1 \to z_1 + \delta z_1$. This is implemented in the z'-plane by a coordinate transformation

$$z' \to z' + \delta z_1 v^z(z', \bar{z}').$$

where v^z is of course **not** conformal (depends on z as well as $\bar{z}$). Introduce the Beltrami differential.

$$\psi = \partial_{\bar{z}} v^z$$

There is a similar differential for the complex conjugate

$$\bar{\psi} = \partial_z v^{\bar{z}}$$

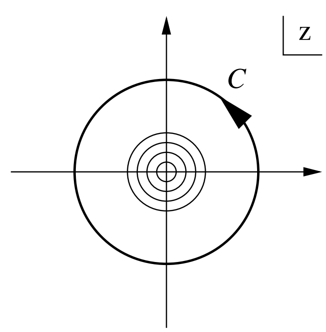
If v^z represents a conformal transformation, then $\psi,\ \bar{\psi} = 0$. Thus ψ encodes information about conformally inequivalent surfaces.
We will insert $\frac{1}{2\pi}\int d^2z'(p\psi + \tilde{b}\bar{\psi})\times$ anti-holomorphic in the amplitude. We integrate over the patch that we will use so

$$Amp \sim \int d^2z_1 \langle V_1V_2V_3V_4 \left(\frac{1}{2\pi} \int d^2z'(p\psi + \tilde{b}\bar{\psi}) \times (anti) \right) \rangle$$

Since $\partial_{\bar{z}} b = 0$, the integral is $\sim \int d^2z' \left(\partial_{\bar{z}}(bv^z) + \partial_z(\tilde{b}v^{\bar{z}}) \right)$. Therefore it can be written as (divergence theorem)

$$B_1 = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(dz' b v^z - d\bar{z}' \tilde{b} v^{\bar{z}} \right)$$

where $\mathbb{C}$ is in the overlap region of z and z'.



Explicitly,

$$v^z = \frac{\partial z'}{\partial z_1}.$$

In the overlap region, $z = z' + z_1$, so $v^z + 1 = 0$, therefore $v^z = -1$. Therefore

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} = dz' b v^z = -b_{-1}, \quad b(z) = \sum b_n z^{-n-2}, c = \sum c_n z^{-n+1}.$$

$$\int dz_1 b_{-1} V_3 = \int dz_1 b_{-1} c\tilde{c} : e^{ik_3 \cdot X}(z_1) := \tilde{c} : e^{ik_3\cdot X}(z_1)$$

where

$$b_{-1}c = \oint_{\mathcal{C}} dz' b(z')c(z_1) = 1.$$

$$\int dz_1 b_{-1}\tilde{c} : e^{ik_3\cdot X}(z_1) :=: e^{ik_3\cdot X}(z_1) :$$

so the b-insertions kill $c\tilde{c}$ from V_3 and the amplitude is as before.

4.6 BRST Invariance

$$\{Q_B, B_1\} = \frac{1}{2\pi i}\oint_c dz' \left(T v^z - d\bar{z}\bar{T} v^{\bar{z}}\right)$$

Recall

$$T(z')V_3(z_1) = \frac{h}{(z'-z_1)^2} + \frac{1}{z'-z_1}\partial V_3 + \ldots, \quad h = 0!$$

Therefore $\{Q_B, B_1\}V_3 \sim \oint dz' \partial V_3 = 0$ unless the moduli space has ∂ (not true here, but argument is general and sometimes $\partial \neq 0$).

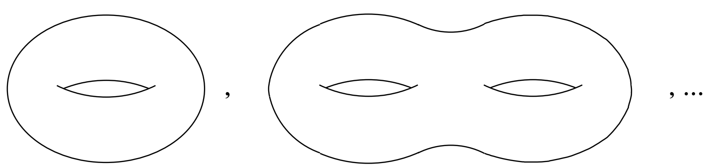
UNIT 5

Loop Amplitudes

5.1 One-loop Amplitudes

For closed string amplitudes, we consider a sphere which has six isometries (parametrized by three complex numbers $a, b, c \sim SL(2, \mathbb{C})$), so we had to put four punctures to get a modulus (conformally inequivalent surfaces).

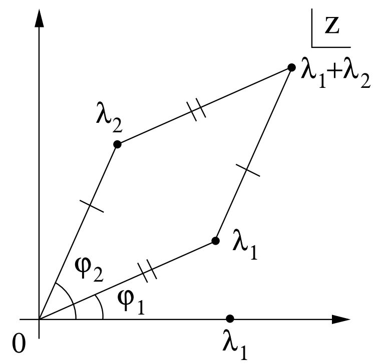
For loop amplitudes (containing *virtual* strings - quantum mechanical corrections, only present due to Heisenberg's uncertainty principle) we need to consider higher-genus Riemann surfaces (fortunately they have all been classified).



We will start with the torus. Unlike the sphere, there exist tori that are conformally inequivalent, without any punctures. So we need to study the torus by itself first. What does it represent? A virtual string that lives in the *vacuum*. Or nothing creating a pair of strings which then annihilate to produce nothing again.

Are we about to study nothing? You bet! There is energy associated with *nothing* and this energy is observable if gravity is present! It is the **Cosmological Constant**.

First let us study the geometry



$$z \approx z + \lambda_1 \approx z + \lambda_2 \therefore z \approx z + n\lambda_1 + m\lambda_2 .$$

Different choices of λ_1, λ_2 lead to conformally inequivalent surfaces.
Example: $z \to \zeta z$ (rescaling) $\Rightarrow\ \lambda_1 \to \frac{1}{\zeta}\lambda_1,\ \lambda_2 \to \frac{1}{\zeta}\lambda_2$ However $\tau = \frac{\lambda_1}{\lambda_2}$ =invariant!
Now change λ_1, λ_2 thusly:

$$\begin{pmatrix} \lambda_2' \\ \lambda_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}, \quad a, b, c, d, \in \mathbb{Z}, \quad ad - bc = 1$$

Then

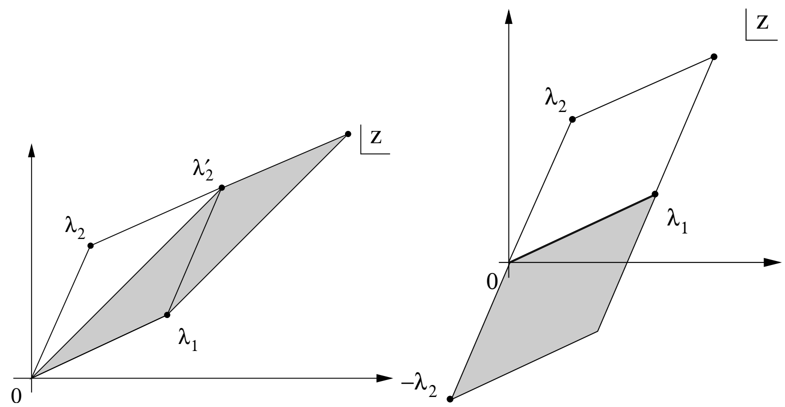
$$\begin{aligned} z &\approx z + n'\lambda_1' + m'\lambda_2' \\ &\approx z + (n'\ m') \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \\ &\approx z + n\lambda_2 + m\lambda_1 \end{aligned}$$

where

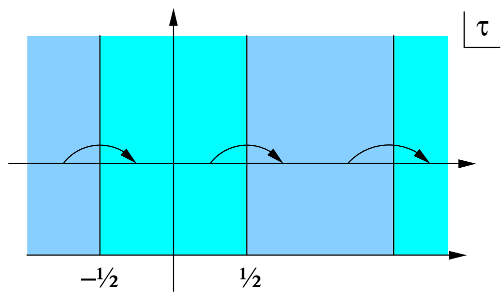
$$\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n' \\ m' \end{pmatrix}$$

Therefore it is the same torus. But $\lambda_1', \lambda_2' \to \tau' = \frac{\lambda_2'}{\lambda_1'} = \frac{a\tau + b}{c\tau + d}$.
Conclusion: $\tau,\ tau'$ define the **same** torus. τ up to SL(2,$\mathbb{Z}$) transformations uniquely labels conformally inequivalent tori. Therefore τ is a *modulus.*
We need to integrate over τ. Find the integration region. SL(2,$\mathbb{Z}$) is generated by two transformations $(S,\ T)$.

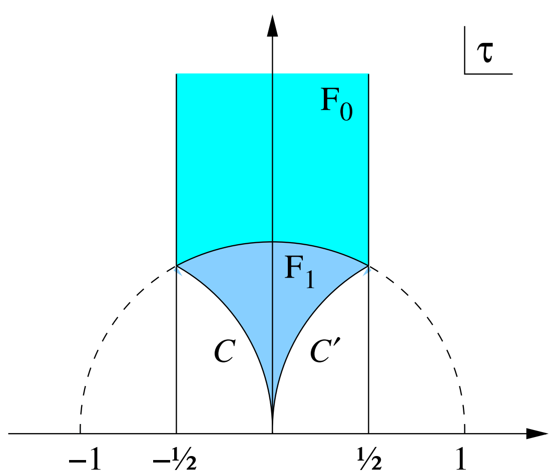
$$T:\ \tau' = \tau + 1, \quad S:\ \tau' = \frac{1}{\tau}$$



In the τ-plane, T divides it into inequivalent regions (mapping one region into the adjacent region). So concentrate on $-\frac{1}{2} \leq Re(\tau) \leq \frac{1}{2}$.

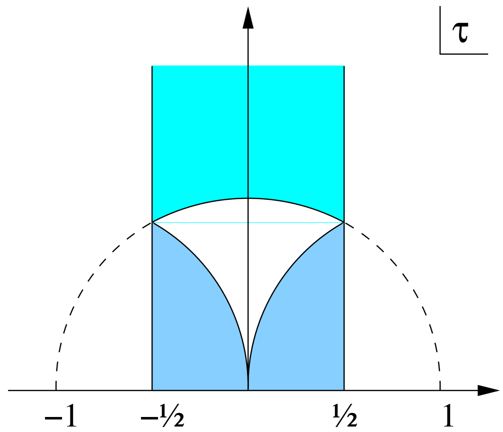


The $Im\tau$ axis, $\tau = it$, acting with S: $it' = -\frac{1}{it}$, so $t' = \frac{1}{t}$ mapping $(1, \infty) \leftrightarrow (0, 1)$. the point i is fixed and so the entire arc $\tau = e^{i\theta}$.



The region above the unit circle and in between $|Re(\tau)| = \frac{1}{2}$ is "irreducible", called the fundamental region,F_0.

S maps $\tau = \frac{1}{2} + it$ onto $\tau' = -\frac{1}{\frac{1}{2}+it} = \frac{-\frac{1}{2}+it}{\frac{1}{4}+t^2}$, $\quad \infty \to 0, \quad t = \frac{\sqrt{3}}{2} \to -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
So $F_0 \to F_1$.



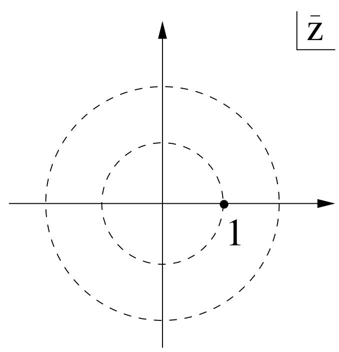
All Fundamental regions are equivalent. Integration should be over **one** fundamental region (does not matter which one).

5.2 String on a Torus

Set $\lambda_1 = 2\pi$ (without l.o.g.). If $\tau = it$, then

INSERT FIGURE HERE

This is a cylinder (closed string propogating for time t and then coming back to where it started from). In general, $\tau = \tau_1 + i\tau_2$, to $t = \tau_2$. τ_1 represents an angle $(0 \leq \tau_1 \leq 2\pi)$. The string is twisted by τ_1 before it gets identified with the initial string. This is the "cylinder" picture. We can map it onto the "sphere" picture as before, $\tilde{z} = e^{-iz}$.



$$z \approx z + 2\pi, \quad trivial$$

$$z \approx z + 2\pi\tau$$

$\tilde{z} \approx \tilde{z}e^{-2\pi i\tau}$, $|\tilde{z}| \approx |\tilde{z}|e^{2\pi\tau_2}$: unit circle $\approx$ circle of radius $e^{2\pi\tau_2}$ with twist τ_1.

Quantum Mechanics

Start with a state $|n\rangle$, evolve for "time" $2\pi\tau$, $|n\rangle \to |n(\tau)\rangle = e^{iH(2\pi\tau)}|n\rangle$. For the left-movers, $H = L_0 - \frac{D}{24}$.
Transition amplitude: $A_n = \langle n|n(\tau)\rangle = \langle n|e^{i(L_0-D/24)(2\pi\tau)}|n\rangle$ Define $Z(\tau) = \sum_n A_n(\tau) = \sum_n \langle n|e^{i(L_0-D/24)(2\pi\tau)}|n\rangle$. If n is an eigenfunction of the "Hamiltonian", $(L_0 - D/24)|n\rangle = E_n|n\rangle$ then $Z(\tau) = \sum e^{2\pi\tau i E_n} = \text{Tr}\,(e^{2\pi\tau i(L_0-D/24)})$.
We need to include the right-movers. Define

$$Z(\tau) = \text{Tr}\,(e^{2\pi i\tau(L_0-D/24)}e^{-2\pi i\bar{\tau}(\tilde{L}_0-D/24)}).$$

If $q = e^{2\pi i\tau}$, then

$$Z(\tau) = \text{Tr}\,(q^{(L_0-D/24)}\bar{q}^{(\tilde{L}_0-D/24)})$$

Let us calculate it. Recall ...

$$L_0 + \tilde{L}_0 = \frac{\alpha'}{2}p^2 + N + \tilde{N}, \quad L_0 - \tilde{L}_0 = N - \tilde{N}$$

Therefore

$$Z(\tau) = (q\bar{q})^{-D/24}\text{Tr}\,(e^{-2\pi\tau_2(\alpha'/2p^2)}q^N\bar{q}^{\tilde{N}})$$

where $N = \sum_n \alpha^\mu_{-n}\alpha_{\mu n}$, $\tilde{N} = \sum_n \tilde{\alpha}^\mu_{-n}\tilde{a}_{\mu n}$. For each n and μ, $\alpha^\mu_{-n}\alpha_{\mu n}$ has eigenvalues $nN_{\mu n} \in \mathbb{N}_0$, where $N_{\mu n}$ is the occupation number.
Therefore

$$\text{Tr}\,q^N\bar{q}^{\tilde{N}} = \prod_{\mu n}\sum_{N,\tilde{N}}(q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}}) = \prod_{\mu,n}\left(\sum_{N=0}^{\infty} q^{nN_{\mu n}}\right)\left(\sum_{\tilde{N}=0}^{\infty}\bar{q}^{n\tilde{N}_{\mu n}}\right)$$

Each sum is a geometric series, therefore

$$\text{Tr}\,q^N\bar{q}^{\tilde{N}} = \prod_{n=1}^{\infty}(1-q^n)^{-2D}.$$

To calculate $\text{Tr}\,e^{-2\pi\tau_2(\alpha'/2p^2)}$, make the space finite and Euclidean. Then p^μ has discrete eigenvalues (n/L), where L is the size of the box.

$$\sum_n f(n/L) = L\sum_{\frac{1}{L}} f(2\pi n/L) = L\int \frac{dp_x}{2\pi}f(p_x).$$

Repeat for other dimensions and we get

$$\text{Tr}\;\to L^D\int\frac{d^Dp}{(2\pi)^D}f(p)$$

The t-component $E = p_0\tilde{\psi}ip_0$ to make the integral finite, so

$$\text{Tr}\,e^{-2\pi\tau_2(\alpha'p^2/2)} \;=\; iL^D\int\frac{d^Dp}{(2\pi)^D}e^{-2\pi\tau_2(\alpha'p^2/2)}$$

$$= \quad iL^D \left(\int \frac{dp}{2\pi} e^{-2\pi\tau_2(\alpha' p^2/2)} \right)^2$$
$$= \quad iL^D \left(2\pi\sqrt{\alpha'\tau_2} \right)^{-D}$$

Putting everything together,

$$Z(\tau) = iL^D \left(2\pi\sqrt{\alpha'\tau_2}|q^{1/24} \prod_{n=1}^{\infty}(1-q^n)|^2 \right)^{-D}.$$

We now need to check the modular invariance. $\tau \to \tau + 1 \Rightarrow q \to q$, obvious invariance! $\tau \to -1/\tau$ is not at all obvious! In order to show the invariance we may use the powerful machinery developed by Jacobi centuries ago, known as the Jacobi-Theta functions, Θ.

Θ-functions

These are functions with nice modular properties.
Definition:

$$\vartheta(\nu,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau + 2\pi i n\nu} = \sum_{n=-\infty}^{\infty} q^{n^2/2} z^n, \;\; q = e^{2\pi i\tau}, \; z^{2\pi i\nu}$$

The Jacobi-Theta functions satisfy the heat equation

$$\frac{i}{\pi}\frac{\partial^2\vartheta}{\partial\nu^2} + 4\frac{\partial\vartheta}{\partial\tau} = 0.$$

Periodicity properties:

$$\begin{aligned} \vartheta(\nu+1,\tau) &= \vartheta(\nu,\tau) \\ \vartheta(\nu+\tau,\tau) &= e^{-\pi i\tau - 2\pi i\nu}\vartheta(\nu,\tau) \end{aligned} \tag{5.2.1}$$

These **uniquely** define ϑ up to a multiplicative constant.
Check: $\nu \to \nu+1 \Rightarrow z \to z \therefore \vartheta(\nu+1,\tau) = \vartheta(\nu,\tau)$. $\nu \to \nu+\tau \Rightarrow z \to qz$

$$\begin{aligned} \vartheta(\nu+\tau,\tau) &= \sum q^{n^2/2} z^n q^n = q^{-1/4}\sum q^{(n+1)^2/2} z^n \\ &= q^{-1/4} z^{-1} \sum q^{(n+1)^2/2} z^{n+1} \\ &= q^{-1/4} z^{-1}\vartheta(\nu,\tau) \\ &= e^{-\pi i\tau - 2\pi i\nu}\vartheta(\nu,\tau) \end{aligned}$$

Equations (5.2.1) admit a different form of solution (which must be the same by uniqueness).

$$\vartheta(\nu,\tau) = \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2})$$

Check: $\nu \to \nu + 1$ is trivial to check $\nu \to \nu + \tau$: $z \to qz$ therefore

$$\prod \to \prod \times \frac{1 + z^{-1}q^{-1/2}}{1 + zq^{1/2}} = (\prod) \times z^{-1}q^{-1/2}$$

as before. The normalization constants also match. Check the limit $q \to 0$. ϑ is usually called ϑ_3. A general ϑ function is given by

$$\vartheta \left[\begin{array}{c} a \\ b \end{array} \right] (\nu, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+a)^2 \tau + 2\pi i (n+a)(\nu + b)} = e^{\pi i a^2 \tau + 2\pi i a(\nu + b)} \vartheta_3(\nu + a\tau + b, \tau)$$

where

$$\vartheta_3(\nu, \tau) = \vartheta \left[\begin{array}{c} 0 \\ 0 \end{array} \right] (\nu, \tau).$$

Interesting ϑ:

$$\begin{aligned} \vartheta_1 &= -\vartheta \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (\nu, \tau) = i \sum_{n=-\infty}^{\infty} (-)^n q^{(n-1/2)^2/2} z^{n-1/2} \\ &= 2e^{i\pi\tau/4} \sin \pi\nu \prod_{m=1}^{\infty} (1 - q^m)(1 - zq^m)(1 - z^{-1}q^m) \end{aligned}$$

The product forms are useful to find zeros of ϑ_3.

$$z = -q^{\pm(m-1/2)} \Rightarrow e^{2\pi i \nu} = e^{\pi i \tau (2m-1) + \pi i}$$

Therefore

$$\begin{aligned} \nu &= \frac{1}{2}(\tau + 1), \;\; \nu + 1, \;\; \nu + 2, ... \\ &= n_1 + n_2\tau, \;\; \nu + \tau, \;\; \nu + 2\tau, ... \end{aligned}$$

Modular Transformations

$T: \; \tau \to \tau + 1$:

$$\begin{aligned} \vartheta_3(\nu, \tau) &= \sum e^{\pi i n^2 \tau + \pi i n^2 + 2\pi i n \nu} \\ &= \sum e^{\pi i n^2 \tau + 2\pi i n(\nu + 1/2) + \pi i n^2 - \pi i n} \\ &= e^{\pi i n^2 \tau + 2\pi i n(\nu + 1/2)}, \quad e^{\pi i n(n-1)} = e^{2\pi i k} = 1 \\ &= \vartheta_3(\nu + 1/2, \tau) \end{aligned}$$

$S: \; \tau \to -\frac{1}{\tau}$: This transformation is more difficult. First, convert the sum to an integral:

$$\vartheta_3(\nu, \tau) = \sum q^{n^2/2} z^n = \int_{-\infty}^{\infty} dx q^{x^2/2} z^x \sum_n \delta(x - n)$$

and

$$\sum_n \delta(x-n) = \sum_{m=-\infty}^{\infty} e^{2\pi ixm}.$$

Proof: If $x \in \mathbb{Z}$, then clearly $\sum \to \infty$. If $x \notin \mathbb{Z}$, then

$$\begin{aligned}\sum_{m=-\infty}^{\infty} e^{2\pi ixm} &= \frac{1}{1-e^{2\pi ix}} + \frac{1}{1-e^{-2\pi ix}} - 1 \\ &= 2Re(1-e^{2\pi ix})^{-1} - 1 = 2Re\frac{e^{-\pi ix}}{-2i\sin(\pi x)} - 1 = 0\end{aligned}$$

Also,

$$\int_{-1/2}^{1/2} \sum_{m=-\infty}^{\infty} e^{2\pi ixm} = \sum_{m\neq 0} \frac{1}{2\pi im} e^{2\pi ix}\Big|_{-1/2}^{1/2} + 1$$

If $m \neq 0$, then $e^{2\pi imx}\big|_{-1/2}^{1/2} \sim \sin\pi m = 0$, therefore $\int_{-1/2}^{1/2} = 1$. By periodicity, $\sum e^{2\pi ixm} = \sum_n \delta(x-n)$. Therefore

$$\begin{aligned}\vartheta_3(\nu,\tau) &= \sum_m \int_{-\infty}^{\infty} dx q^{x^2/2} z^x e^{2\pi ixm} \\ &= \sum_m \int_{-\infty}^{\infty} dx e^{\pi i\tau x^2} e^{2\pi i\nu x} e^{2\pi imx} \\ &= \sum_m \int_{-\infty}^{\infty} dx e^{\pi i\tau(x+(\nu+m)/\tau)^2 - \pi i(\nu+m)^2/\tau} \\ &= \frac{1}{\sqrt{-i\tau}} e^{-i\pi\nu^2/\tau} \sum_m e^{-\pi im^2/\tau - 2\pi i\nu m/\tau} \\ &= \frac{1}{\sqrt{-i\tau}} e^{-\pi i\nu^2/\tau} \vartheta_3(\nu/\tau| -1/\tau)\end{aligned}$$

Therefore

$$\vartheta_3\left(\frac{\nu}{\tau}| -\frac{1}{\tau}\right) = \sqrt{-i\tau} e^{\pi i\nu^2/\tau} \vartheta_3(\nu|\tau).$$

Similarly for ϑ_1, we obtain

$$\begin{aligned}\vartheta_1(\nu|\tau+1) &= e^{\pi i/4}\vartheta_1(\nu|\tau) \\ \vartheta_1\left(\frac{\nu}{\tau}| -\frac{1}{\tau}\right) &= -\sqrt{-i\tau} e^{\pi i\nu^2/\tau} \vartheta_1(\nu|\tau)\end{aligned}$$

ϑ_1 can be related to the partition function $Z(\tau)$ as follows. Recall

$$\vartheta_1(\nu|\tau) = 2e^{\pi i\tau/4} \sin\pi\nu \prod_{m=1}^{\infty} (1-q^m)(1-zq^m)(1-z^{-1}q^m).$$

To get $Z(\tau)$, we can not just set $z=1$, because then $\sin\pi\nu=0$, so $\vartheta_1=0$. We need to differentiate with respect to ν first.

$$\partial_\nu\vartheta_1(\nu|\tau)=2\pi e^{\pi i\tau/4}\left(\cos\pi\nu\prod(...)+\sin\pi\nu\partial_\nu\prod(...)\right)$$

Now set $\nu=0$ and the second term vanishes. Therefore

$$\partial_\nu\vartheta_1(\nu|\tau)=2\pi e^{\pi i\tau/4}\left[\prod_{m=1}^{\infty}(1-q^m)\right]^3=2\pi\left[q^{1/24}\prod_{m=1}^{\infty}(1-q^m)\right]^3.$$

Notice the appearance of $1/24$ in the exponent! It is needed for the modular invariance! Therefore

$$\eta(\tau)=q^{1/24}\prod_{m=1}^{\infty}(1-q^m)=\left(\frac{\partial_\nu\vartheta_1(\nu|\tau)}{2\pi}\right)^3$$

where $\eta(\tau)$ is the Dedekind η-function.

Modular Properties of the Dedekind function

$\tau\to\tau+1$

$$\eta(\tau+1)=e^{\pi 1/12}\eta(\tau)$$

$$\begin{aligned}\frac{1}{\tau}\partial_\nu\vartheta_1\left(0|-\frac{1}{\tau}\right) &= -\sqrt{-i\tau}\partial_\nu\vartheta_1(0|\tau)\\ \partial_\nu\vartheta_1\left(0|-\frac{1}{\tau}\right) &= (-i\tau)^{3/2}\partial_\nu\vartheta_1(0|\tau)\end{aligned}$$

$$\eta\left(-\frac{1}{\tau}\right)=\sqrt{-i\tau}\eta(\tau)$$

So we see that ϑ and η are not invariant under modular transformations. We really only care about the partition function, which depends on η. Let us check how the partition function transforms under the modular transformations from knowing how η transforms. The partition function is

$$Z(\tau)=iL^D\left(\frac{1}{2\pi\sqrt{\alpha'\tau_2}}|\eta(\tau)|^{-2}\right)^D$$

Under $\tau\to\tau+1$, τ_2 does not change, nor does $|\eta(\tau)|$. Under $\tau\to-\frac{1}{\tau}$, $\tau_2\to\frac{\tau_2}{|\tau|^2}$, so $\frac{1}{\sqrt{\tau_2}}\to\frac{|\tau|}{\sqrt{\tau_2}}$ and $|\eta(\tau)|^{-2}\to\frac{1}{|\tau|}|\eta(\tau)|^{-2}$
Therefore $\frac{1}{\sqrt{\tau_2}}|\eta(\tau)|^{-2}$ is modular invariant and so is $Z(\tau)$.

5.3 The bc system

Recall $L_0 = \sum n : b_{-n}c_n : -1$. This is in the "sphere" picture. To go back to the cylinder picture, use

$$L_0 \to L_0 + \frac{c}{12}$$

where $c = 13$. Therefore

$$L_0 = \sum n : b_{-n}c_n : + \frac{1}{12}$$

which is the generator of the t-translations for left-movers.

$$Z_{bc} = q^{1/2}\bar{q}^{1/2}\mathrm{Tr}\ q^{L_0}\bar{q}^{\tilde{L}_0}$$

To calculate $\mathrm{Tr}\ q^{L_0}$, start with $b_{-1}c_1 + c_{-1}b_1$. $b_{-1}c_1$ which has eigenvectors

$$|0\rangle = |\psi\rangle, \quad |1\rangle = b_{-1}|\psi\rangle$$

with respective eigenvalues $0, 1$.
We can show $(b_{-1}c_1)^2$ is a projection operator by using the bc algebra,

$$\{b_{-m}, c_n\} = \delta_{0,m+n}, \ \ \{b_m, b_n\} = \{c_m, c_n\} = 0.$$

$$(b_{-1}c_1)^2 = b_{-1}c_1b_{-1}c_1 = b_{-1}(1 - b_{-1}c_1)c_1 = b_{-1}c_1.$$

Therefore, the eigenvalues $0, 1$ are the only eigenvalues. Now

$$\mathrm{Tr}\ |\psi\rangle\langle\psi| = \langle\chi|\psi\rangle = \langle\psi|c_0|\psi\rangle$$

is the only sensible definition. Define $\langle 0| = \langle\psi|$, $\langle 1| = \langle\psi|c_1$.

$$\mathrm{Tr}\ |1\rangle\langle 1| = \langle 1|1\rangle = \langle\psi|c_1c_1b_{-1}|\psi\rangle = -1.$$

The eigenvectors of $c_{-1}b_1$ are $|\psi\rangle$ and $c_{-1}|\psi\rangle$ so overall, the $b_{-1}c_1 + c_{-1}b_1$ eigenvalues of $|\psi\rangle,\ b_{-1}|\psi\rangle,\ c_{-1}|\psi\rangle,\ c_{-1}b_{-1}|\psi\rangle$ are $0, 1, 1, 2$ respectively. Therefore

$$\mathrm{Tr}\ q^{b_{-1}c_1 + c_{-1}b_1} = q^0 - q^1 - q^1 + q^2 = (1-q)^2.$$

Similarly, we obtain

$$\mathrm{Tr}\ q^{n(b_{-n}c_n + c_{-n}b_n)} = (1-q^n)^2.$$

Therefore

$$\mathrm{Tr}\ q^{L_0} = \prod (1-q^n)^2,$$

and

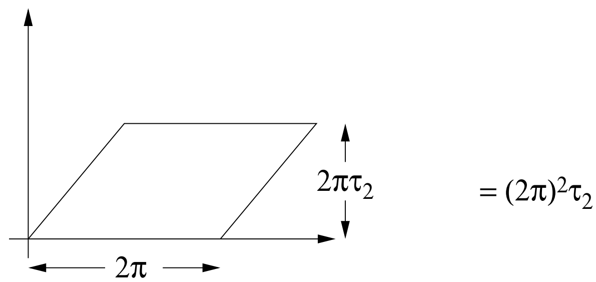
$$Z_{bc}(\tau) = (q\bar{q})^{1/2}\prod_{n=1}^{\infty}|1-q^n|^4 = |\eta(\tau)|^4.$$

5.4 Vacuum Energy

Combine $Z(\tau)$ for X^μ with $Z_{bc}(\tau)$ to get the total partition function.

$$Z_{total}(\tau) = iL^D \left(\frac{1}{2\pi\sqrt{\alpha'\tau_2}} |eta(\tau)|^{-1} \right)^D |\eta(\tau)|^4 = iL^D (2\pi\sqrt{\alpha'\tau_2})^{-D} |\eta(\tau)|^{-2(D-2)}.$$

So $D \to D-2$ which is good since only the transverse modes should contribute. This is not modular invariant. However, note that we have a conformal transformation left: rigid translations (similar to the sphere where we had $SL(2,\mathbb{C})$ invariance). There is a one to one correspondence between points on the torus and translations. Therefore the number of translations is proportional to the volume of the torus.



We need to *average* over translations, so divide by the volume of the group $((2\pi)^2\tau_2)$. We also have a reflection symmetry, so we need to multiply by $1/2$. If we have vertex operators, we need to fix *one* of the positions (c.f. a sphere where we fixed three positions). Finally the one-loop vacuum string amplitude is

$$Z_{physical} = iL^D \int_{F_0} \frac{d\tau d\bar{\tau}}{2(2\pi)^2\tau_2} (2\pi\sqrt{\alpha'\tau_2})^{-D} |\eta(\tau)|^{-2(D-2)}.$$

This is the cosmological constant. Why? C.f. with the point particle.

$$E = \sqrt{\vec{p}^2 + m^2}.$$

On a circle of length ℓ the temperature is $T \sim 1/\ell$. The partition function is given by

$$\begin{aligned} Z(\ell) &= L^D \int \frac{d^D p}{(2\pi)^D} e^{-\chi^\ell}, \quad \chi = \frac{1}{2}(-E^2 + \vec{p}^2 + m^2) = \frac{1}{2}(p^2 + m^2) \\ &= iL^D \, (2\pi\ell)^{-D/2} \, e^{-m^2\ell/2}. \end{aligned} \tag{5.4.1}$$

What information can we gain by evaluating the vacuum amplitude for the string? The vacuum amplitude defined earlier is given by,

$$Z = \int_0^\infty \frac{dl}{2\pi} Z(l).$$

where

$$Z(l) = L^D \int \frac{d^D p}{(2\pi)^D} e^{-\chi l} = iL^D \left(\frac{1}{\sqrt{2\pi l}}\right)^D e^{-m^2 l/2}$$

There is an ultraviolet divergence when $l = 0$. Cut off the integration region at ϵ and later let $\epsilon \to 0$. Apply a Wick rotation ($p_0 \to ip_0$) and integrate over p_0.

$$\begin{aligned} Z(l) &= i\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \exp[-\frac{l}{2}(p_0^2 + \vec{p}^2 + m^2)] \\ &\sim \frac{1}{\sqrt{l}} \exp[-\frac{l}{2}(\vec{p}^2 + m^2)] \end{aligned}$$

Now do the integral over l:

$$\begin{aligned} Z &= \int_{\epsilon}^{\infty} \frac{dl}{l}\frac{1}{\sqrt{l}} \exp[-\frac{l}{2}(\vec{p}^2 + m^2)] \\ &\sim \sqrt{\vec{p}^2 + m^2} \int_{\epsilon}^{\infty} \frac{dx}{x^{3/2}}\, \mathrm{e}^{-\mathrm{x}/2} \\ &\sim \sqrt{\vec{p}^2 + m^2} * \text{constant} \end{aligned}$$

$$\begin{aligned} Z(l) &\sim L^D \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} \sqrt{\vec{p}^2 + m^2} \\ &\sim L^D \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} E(\vec{p}) \end{aligned}$$

This is equivalent to integrating over all possible modes of the string. We can define the energy density as Λ.

$$\Lambda = \frac{Z}{L^D} \sim \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} E(\vec{p}) \qquad \text{“Cosmological Constant”}$$

This implies the cosmological constant is the sum of the vacuum energies from each of the states of the string.
How does the Cosmological constant play a role in one loop correction of the vacuum?

$$\Lambda \sim \int_{F_0} \frac{d\tau\, d\bar{\tau}}{\tau_2} (2\pi\sqrt{\alpha'\tau_2})^{-D} |\eta(\tau)|^{-2(D-2)}$$

Notice that there is no ultraviolet divergence, because τ is not allowed to approach zero ($|\tau| \geq 1$ in F_0). One the other hand, if $\tau_2 \to \infty$ (very long torus)

$$|q| = |e^{2\pi i\tau}| = e^{-2\pi\tau_2} \to 0$$

so we may expand

$$|\eta(\tau)|^{-2} = |q|^{-1/12}|1 - q + ...|^{-2} = e^{\pi\tau_2/6}(1 + 4Req + ...).$$

The dominant contribution

$$\int^{\infty}\frac{d\tau_2}{\tau_2}(\sqrt{4\pi^2\alpha'\tau_2})^{-26}e^{4\pi\tau_2}$$

c.f. with $(2\pi\tau_2)^{-D/2}\exp(-m^2\ell/2)$. Since $\ell = 2\pi\alpha'\tau_2$ the square of the mass becomes $m^2 = -4/\alpha'$ and we have a tachyon! This diverges due to $m^2 < 0$. Other terms in the sum are due to other modes with $m^2 \geq 0$, so they all converge.

5.5 Thermodynamics

In this section we look at various thermodynamic properties of the partition function.

$$\tau_1 = 0 \qquad \tau = i\tau_2 \qquad q = e^{-2\pi i\tau_2}$$

In the limits above what can we find out about the partition function?

$$\begin{aligned}
Z(\tau_2) &= |q|^{-D/12}\mathrm{Tr}\left(q^{L_0}\bar{q}^{\tilde{L}_0}\right) \\
&= \mathrm{e}^{-\pi\mathrm{D}\tau_2/6}\mathrm{Tr}\left(\mathrm{e}^{-2\pi\tau_2\mathrm{H}}\right) : \qquad \mathrm{H} = \mathrm{L}_0 + \tilde{\mathrm{L}}_0 \\
&= \mathrm{e}^{-\pi\mathrm{D}\tau_2/6}\sum_{\mathrm{n}}\mathrm{e}^{-2\pi\tau_2\mathrm{E_n}} : \qquad \frac{1}{\mathrm{T}} = 2\pi\tau_2
\end{aligned}$$

High-temperature limit: $\tau_2 \to 0$. Then $q \to 1$ and $Z(\tau_2)$ is dominated by high-weight states. By modular invariance, $\tau_2 \to -1/\tau_2$, it is related to $Z(1/\tau_2)$.
Small-temperature limit: $\tau_2 \to \infty \Rightarrow q \to q' = \mathrm{e}^{-2\pi/\tau_2} \to 0$

$$Z(\tau_2) = \sum_n \mathrm{e}^{-\mathrm{E_n}/\mathrm{T}} \simeq \mathrm{e}^{-\mathrm{E_0}/\mathrm{T}} + \sum_{\mathrm{n}\neq 0}\mathrm{e}^{-\mathrm{E_n}/\mathrm{T}} + \ldots$$

Only the $E_0 = 0$ will survive in the sum.

$$Z\left(\frac{1}{\tau_2}\right) = \mathrm{e}^{-\frac{\pi\mathrm{D}}{6\tau_2}}$$

$$\tau_2 \to 0 \ : \qquad Z(\tau_2) = Z\left(\frac{1}{\tau_2}\right) = \mathrm{e}^{-\frac{\pi\mathrm{D}}{6\tau_2}}$$

Entropy

$$Z = \sum_E \rho(E)\mathrm{e}^{-\mathrm{E}/\mathrm{T}} \qquad \text{saddle point approximation}$$

There exists a stationary exponent when

$$d(\ln\rho - E/T) = 0.$$

Define the entropy as $S = \ln\rho$. Therefore $dE = TdS$.

$$Z = \sum \mathrm{e}^{\mathrm{S}-\mathrm{E}/\mathrm{T}} \qquad \mathrm{d}(\mathrm{S} - \mathrm{E}/\mathrm{T}) = 0 \Rightarrow \mathrm{dS} = \frac{\mathrm{dE}}{\mathrm{T}}$$

Free energy

$$F = -T \ln Z = -T\left(S - \frac{E}{T}\right) = E - TS$$

The entropy can be found from the Free energy, $S = -\frac{\partial F}{\partial T}$. Comparing with $Z(\tau_2)$ we see there is extra factor of $e^{-D/12T}$. Modular invariance implies $Ze^{-D/12T}$ is invariant. Therefore

$$\begin{aligned} Z(\tau)\mathrm{e}^{-\mathrm{D}/12\mathrm{T}} &= Z\left(\frac{(2\pi)^2}{T}\right)\mathrm{e}^{-\pi^2\mathrm{DT}/12} \\ Z(\tau) &= \mathrm{e}^{\mathrm{D}/12\mathrm{T}}\mathrm{e}^{-\pi^2\mathrm{DT}/12}\mathrm{Z}\left(\frac{(2\pi)^2}{\mathrm{T}}\right) \end{aligned}$$

$Z(1/T)$ is a slowly varying function in the saddle point approximation. When we go from statistical mechanics to thermodynamics $Z\left(\frac{1}{T}\right)$ is ignored.

$$\ln Z(T) = \frac{D}{12T} - \frac{\pi^2 DT}{12}$$

$$F = -\frac{D}{12} + \frac{\pi^2 DT^2}{12} = \frac{D}{12}(\pi^2T^2 - 1), \quad S = -\frac{\partial F}{\partial T} = \frac{D\pi^2 T}{6},$$

$$E = F + TS = \frac{D}{12}(\pi^2T^2 + 1) \qquad S \leq \pi E.$$

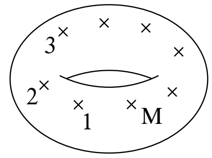
At high temperature the energy is proportional to the square of the temperature ($E \propto T^2$). This represents the "Casimir energy in 2-D", which in four-dimensions is given by $E \propto T^4$. We may express S in terms of E

$$\begin{aligned} \frac{D}{12}\pi^2T^2 &= E - \frac{D}{12}, \\ S^2 &= \left(\frac{D\pi^2}{6}\right)^2 T^2 = \frac{D^2}{26}\pi^4\frac{12}{\pi^2 D}\left(E - \frac{D}{12}\right) = 4\frac{D\pi^2}{12}\left(E - \frac{D}{12}\right), \\ S &= 2\pi\sqrt{\frac{D}{12}\left(E - \frac{D}{12}\right)} \qquad \text{“Cardy”} \end{aligned}$$

or $S \leq \pi E$ which is the same as the black hole entropy, where Bekenstein's equation is $S \leq S_B \sim E$. If the equation is generalized to 4-D, it gives the Friedman-Robertson-Walker universe equation for the Hubble constant.

5.6 Amplitudes on a torus

For M tachyons, $V_i =: e^{ik_i\cdot X}(z_i)$ together with the right movers, the amplitude can be writen as



$$A(z_1, z_2, ..., z_M) = \text{Tr}\, V_1 \, ... \, V_M q^{(L_0 - D/24)} \times (c.c.)$$

where

$$X_L = x + \frac{\alpha'}{2} pz + i\sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{n} \alpha_n e^{-inz}, \quad [x,p] = i$$

At the moment, let us omit the complex conjugate. We can insert it when needed. There are two parts to the amplitude: the p-integral and the N-modes.

$$\int \frac{d^D p}{(2\pi)^D} \langle p | e^{-\pi\tau_2 \alpha' p^2} e^{ik_1 \cdot x/2} e^{i\alpha' k_1 \cdot pz/2} \, \cdots \, e^{ik_m \cdot x/2} e^{i\alpha' k_M \cdot pz_M/2} | p \rangle .$$

using $X_L = x + \frac{\alpha'}{2} pz + i\sqrt{\frac{\alpha'}{2}} \sum_{n>0} \frac{1}{n} \alpha_n e^{-inz}$. To evaluate the integral, commute all $e^{ik_1 \cdot x/2}$ through. When they hit $|p\rangle$, they change $|p\rangle \to |p + k_1 + \ldots + k_M\rangle$. Then $\langle p | p + k_1 + \ldots + k_M \rangle = \delta^D (k_1 + \ldots + k_M)$ (conservation of momentum).
As we push the factors through, the amplitude becomes

$$\int \frac{d^D p}{(2\pi)^D} e^{i\pi\tau p^2} e^{i(k_1 z_1 + \ldots + k_M z_M) \cdot p} \delta^D (k_1 + \ldots + k_M)$$

This is a Gaussian integral. We need to shift the momentum $p \to p - Im(k_1 z_1 + \ldots + k_M z_M)/\pi\tau_2$. The amplitude becomes

$$A \sim (2\pi\sqrt{\alpha' \tau_2})^{-D/2} \delta^D (k_1 + \ldots + k_M) e^{-\alpha'/\pi\tau_2 [\text{Im}(k_1 z_1 + \ldots + k_M z_M)]^2} \prod_{i<j} e^{-\alpha'/2 k_i \cdot k_j (z_i - z_j)} \times c.c.$$

Using conservation of momentum, we may also write

$$(k_1 z_1 + \ldots + k_M z_M)^2 = -\sum_{i<j} k_i \cdot k_j (z_1 - z_j)^2 .$$

Next, do the oscillators. We can do each oscillator separately, so fix μ, n and do $\alpha_{-n^\mu}, \; \alpha_n^\mu$. The trace is

$$T_n^\mu = \text{Tr} \prod_{i=1}^{M} \exp\left(\frac{1}{n} k_{i\mu} \alpha_{-n}^\mu\right) \exp\left(-\frac{1}{n} k_{i\mu} \alpha_n^\mu e^{-niz}\right) q^{\alpha_{-n}^\mu \alpha_{n\mu}}$$

where there is no sum on μ!

To evaluate the trace we need to insert the identity $I = \sum_E |E\rangle\langle E|$ and then use $\mathrm{Tr}\, A = \sum_E \langle E|A|E\rangle$. For a complete set of states, choose

$$|\kappa\rangle = e^{\frac{1}{n}\kappa\cdot\alpha_{-n}}|0\rangle,$$

where $\kappa \in \mathbb{C}$. The identity operator is

$$I = \frac{1}{n\pi}\int d^2\kappa e^{-|\kappa|^2/n}|\kappa\rangle\langle\kappa|.$$

To show this, consider an eigenstate of the number operator $N = \sum \alpha_{-n}\alpha_n$ (dropped the μ to avoid confusion).

$$|\ell\rangle = \frac{1}{\sqrt{\ell!}}\frac{(\alpha_{-n})^\ell}{\sqrt{n}}|0\rangle, \quad \langle\ell|\ell\rangle = 1.$$

Then

$$\begin{aligned} \langle\ell|\kappa\rangle &= \frac{1}{\ell!}\langle\ell|(\frac{\kappa}{n})^\ell\alpha_{-n}^\ell|0\rangle, \\ &= \left(\frac{\kappa}{n}\right)^\ell\frac{1}{\sqrt{\ell!}}, \end{aligned}$$

$$\langle\ell|I|\ell'\rangle = \frac{1}{n\pi}\frac{1}{\sqrt{(\ell!)^2}}\int d^2\kappa\, e^{-|\kappa|^2/n}\left(\frac{\bar{\kappa}}{n}\right)^\ell\left(\frac{\kappa}{n}\right)^{\ell'}$$

For $\ell \neq \ell'$, $\langle\ell|I|\ell'\rangle = 0$ which can be shown by using polar coordinates. For $\ell = \ell'$

$$\langle\ell|I|\ell\rangle = \frac{1}{n\pi\alpha'}\int d^2ke^{-|\kappa|^2/n}\left(\frac{|\kappa|}{n}\right)^{2\ell} = 1,$$

which implies

$$<\ell|I|\ell'>=\delta_{\ell\ell'}.$$

Also $q^N|\kappa\rangle = |\kappa q^n\rangle$.
PROOF:

$$\begin{aligned} q^N\alpha_{-n}q^{-N} &= \alpha_{-n} + \ln q[N,\alpha_{-n}] + \frac{1}{2}(\ln q)^2[N,[N,[\alpha_{-n}]] \\ &= \alpha_{-n} + n\ln q\alpha_{-n} + \frac{n^2}{2}(\ln q)^2\alpha_{-n} + ... \\ &= e^{n\ln q}\alpha_{-n} = q^n\alpha_{-n} \\ q^N(\alpha_{-n})^\ell q^{-N}) &= q^{n\ell}(\alpha_{-n})^\ell \\ q^N e^{\frac{1}{n}\kappa\alpha_{-n}}q^{-N} &= e^{\frac{1}{n}q^n\kappa\alpha_{-n}}, \end{aligned}$$

where we used the Glauber identity ($e^Ae^B = e^Be^Ae^{[A,B]}$). Therefore

$$q^N|\kappa\rangle = |\kappa q^N\rangle.$$

The trace becomes

$$T_n^\mu = \frac{1}{n\pi}\int d^2\kappa e^{-|\kappa|^2/n}\langle\kappa|\prod_{i=1}^m e^{\frac{1}{n}k_i\cdot\alpha_{-n}e^{niz_i}}e^{-\frac{1}{n}k_i\cdot\alpha_n e^{-niz}}e^{\frac{1}{n}q^n\kappa\alpha_{-n}}|0\rangle.$$

As we push $\exp\left(-\frac{1}{n}k_1\alpha_n e^{-niz}\right)$ through, the commutators contribute to a new factor given by

$$\exp\left(-\frac{1}{n}k_1 q^n\kappa e^{-niz_1}\right).$$

After moving the exponentials over to act on the states, the commutators contribute an overall factor of

$$\prod_{i<j}\exp\left(-\frac{1}{n}k_i\cdot k_j e^{ni(z_i-z_j)}\right),\quad \prod_i \exp\left(-\frac{1}{n}k_i q^n\kappa e^{-niz_i}\right)$$

Finally push $\exp\left(\frac{1}{n}\bar{x}\alpha_n\right)$ through. We get a factor $\exp\left(\frac{1}{n}k_i\bar{\kappa}e^{niz_i}\right)$ from each vertex and $\exp\left(\frac{1}{n}q^n|\kappa|^2\right)$ from $|q^n\kappa\rangle$. Therefore

$$T_n^\mu = \frac{1}{n\pi}\int d^2\kappa e^{-(1-q^n)|\kappa|^2/n}\prod_{i<j}\exp\left(-\frac{1}{n}k_i\cdot k_j e^{ni(z_i-z_j)}\right)\prod_i \exp\left(-\frac{1}{n}k_i q^n\kappa e^{-niz_i}\right)$$

This is a Gaussian integral: $\frac{1}{\pi}\int d^2\kappa e^{-c|\kappa|^2}e^{a\kappa+b\bar{\kappa}} = \frac{1}{c}e^{-ab/c}$.
Therefore

$$T_n^\mu \sim \frac{1}{1-q^n}\prod_{i<j}e^{-\frac{1}{n}k_i\cdot k_j e^{-ni(z_i-z_j)}}\exp\left\{-\frac{(\sum k_i e^{niz_i})(\sum k_i e^{-niz_i})q^n}{n(1-q^n)}\right\}.$$

Use

$$\sum_{i<j}k_i\cdot k_j = -\frac{1}{2}\sum k_i 2 = -M.$$

$$\prod_{n,\mu}T_n^\mu = \prod_{m=1}^\infty (1-q^m)^{-D}\prod_{i<j}e^{-k_i\cdot k_j}\frac{e^{mi(z_i-z_j}+q^m e^{-mi(z_i-z_j)}-2q^m}{m(1-q^m)}$$

where we see the whole second product is a new contribution. If we use the identity

$$\sum_{m=1}^\infty \frac{1}{m}\frac{x^m}{1-y^m} = -\sum_{n=0}^\infty \ln(1-xy^n)$$

we find

$$\begin{aligned}\prod_{n,\mu}T_n^\mu &= \prod_{m=1}^\infty (1-q^m)^{-D}\prod_{i<j}e^{-k_i\cdot k_j\sum_{n=0}^\infty \ln(1-q^n e^{i(z_i-z_j)})+\ln(1-q^{n+1}e^{-i(z_i-z_j)})-2\ln(1-q)} \\ &= \prod_{m=1}^\infty (1-q^m)^{-D}\prod_{i<j}(1-qe^{i(z_i-z_j)})^{\alpha' k_i\cdot k_j}\prod_{n=1}^\infty\left[\frac{(1-q^n e^{i(z_i-z_j)})(1-q^n e^{-i(z_i-z_j)})}{(1-q^n)^2}\right]^{\alpha' k_i\cdot k_j}.\end{aligned}$$

The first factor $\prod_{i<j}(1-qe^{i(z_i-z_j)})^{\alpha' k_i\cdot k_j}$ is combines with $\prod_{i<j} e^{-\alpha/2(z_i-z_j)k_i\cdot k_j}$ to give

$$\prod_{i<j}\sin(z_i-z_j)^{\alpha' k_i\cdot k_j/2}.$$

Then

$$\sin(z_i-z_j)\prod_{n=1}^{\infty}\left[\frac{(1-q^n e^{i(z_i-z_j)})(1-q^n e^{-i(z_i-z_j)})}{(1-q^n)^2}\right]^{\alpha' k_i\cdot k_j}\sim\frac{\vartheta_1\left(\frac{z_i-z_j}{2\pi}|\tau\right)}{\partial_\nu\vartheta_1(0|\tau)},$$

where

$$\vartheta_1(\nu|\tau)=2e^{\pi i\tau/4}\sin\pi\nu\prod(1-q^m)(1-zq^m)(1-z^{-1}q^m).$$

As we recall, ϑ_1 has nice modular properties, given by

$$\begin{aligned}\vartheta_1(\nu|\tau+1) &= e^{i\pi/4}\vartheta_1(\nu|\tau),\\ \vartheta_1\left(\frac{\nu}{\tau}|-\frac{1}{\tau}\right) &= -i\sqrt{-i\tau}e^{\pi i\nu^2/\tau}\vartheta_1(\nu|\tau).\end{aligned}$$

Let us compare the amplitude to the sphere amplitude. The M-point amplitude for the sphere was

$$A=\langle V_1 ... V_M\rangle\sim\prod_{i<j}|z_i-z_j|^{\alpha' k_k\cdot k_j}.$$

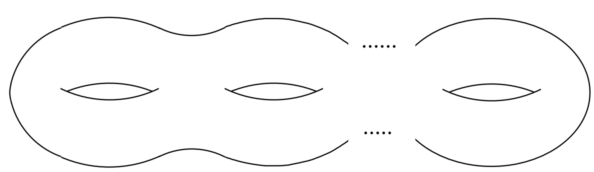
For small $|z_i-z_j|$, we may make the approximation

$$\vartheta_1\left(\frac{z_i-z_j}{2\pi}|\tau\right)\sim\sin\pi\nu\sim z_i-z_j.$$

So, the torus is similar to the sphere at short distances.

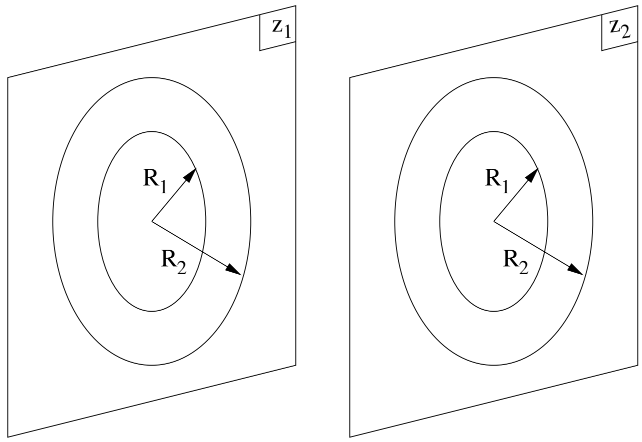
5.7 Higher Genus Surfaces

All two-dimensional surfaces may be classified by their genus. These surfaces are classified the number of handles they posess. There is no classification for surfaces of dimension greater than two. The genus of a sphere and torus are zero and one respectively.

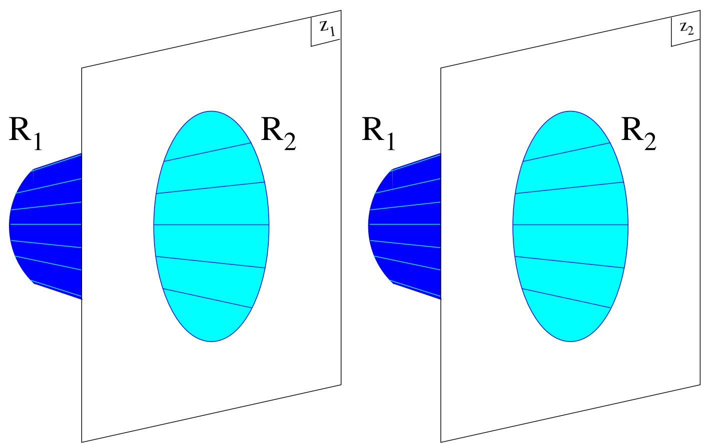


Plumbing Fixture

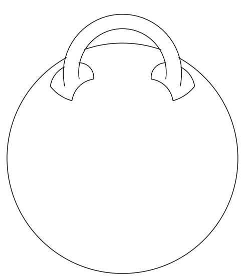
This is a method of creating higher genus surfaces from lower genus surfaces.



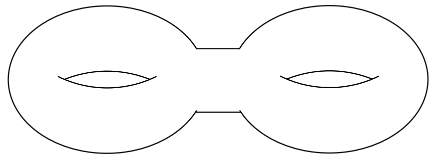
Coordinate patches z_1 and z_2 may belong to different or same surface. Cut a disk of radius R_1 around the origin of each. Then glue annular regions $R_1 < |z_i| < R_2$. $(i = 1, 2)$ by identifying



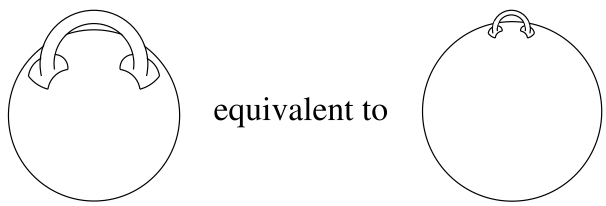
Example 1: z_1 and z_2 on the same plane. Add a handle to create a torus.

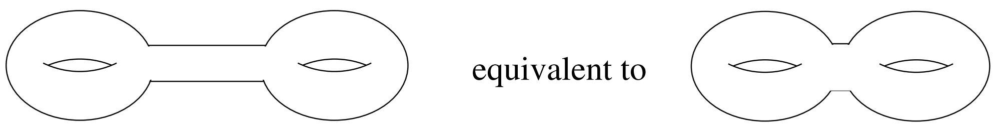


Example 2: z_1 and z_2 on two tori.



Plumbing is a reversible process. One can cut handles and then patch disks to create a lower genus surface. Let us look at an interesting case, $q \to 0$. Then for fixed R_1, $R_2 \to 0$, so an annulus maps to a semi-infinite cylinder, an object with **physical** meaning!
Examples:





This is the boundary of the moduli space. Recall that violations to the BRST symmetry may also come from the boundary of the moduli space, so pinched handles are important.

Amplitudes

We will start with a trivial example: two spheres connected by a cylinder.

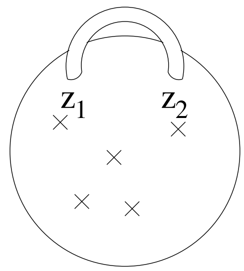
$$\begin{aligned} A_1 &= \langle 0|V_1 ... V_M|E\rangle = \langle 0|V_1 ... V_M V_E|0\rangle \\ A_2 &= \langle E|V_{M+1} ... V_N|0\rangle = \langle 0|V_E^* V_{M_1} ... V_N|0\rangle \end{aligned}$$

where

$$|E\rangle = V_E|0\rangle \quad \text{and} \quad \mathrm{I} = \sum_{\mathrm{E}} |\mathrm{E}\rangle\langle\mathrm{E}|$$

$$A = \sum_E A_1 A_2 = \langle 0|V_1 ... V_M V_{M+1} V_N|0\rangle = \langle 0|V_1 ... V_N|0\rangle$$

This is just a N-point amplitude on a sphere, obviously since $S^2 \cup S^2 = S^2$
Next, let us move to an example which is nontrivial, but known.



If $q = 1$, then the identification: $z_1 z_2 = 1$ which is the two patches on the sphere around zero and infinity respectively. We might as well choose $z_1 = 0,\ z_2 = \infty$. Then the amplitude, A, becomes

$$A = \langle E|V_1 ... V_M|E\rangle = \langle 0|V_E^* V_1 ... V_M V_E|0\rangle$$

and the trace of the amplitude becomes

$$\mathrm{Tr}\ A = \sum_E A = \mathrm{Tr}\ V_1 ... V_M.$$

Now let us have $z_1 z_2 = q$, need $z_2 \to z_2 q$, a conformal transformation under which

$$V_E \to \left(\frac{\partial z_2'}{\partial z_2}\right)^h \left(\frac{\partial \bar{z}_2'}{\partial \bar{z}_2}\right)^{\tilde{h}} V_E = q^h \bar{q}^{\tilde{h}} V_E.$$

$\{E\}$ is the set of h's (weights), where

$$L_0|E\rangle = L|E\rangle, \quad \tilde{L}_0|E\rangle = \tilde{L}|E\rangle.$$

Therefore

$$\sum_E A = \mathrm{Tr}\ \left(V_1 ... V_M q^{L_0} \bar{q}^{\tilde{L}_0}\right), \text{ for } q \neq 1.$$

Generalize ...

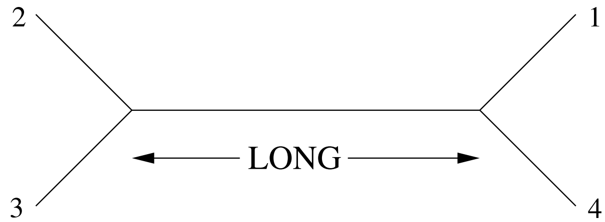
$$A = \sum_E \langle V_1 ... V_M|E\rangle\langle E|V_{M+1} ... V_N|0\rangle q^h \bar{q}^{\tilde{h}}$$

to obtain an amplitude of a higher genus surface.

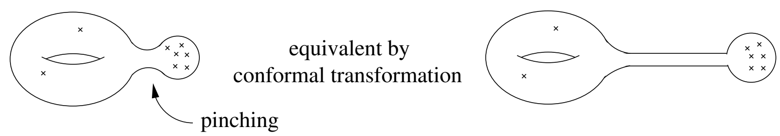
Divergences

We have already seen that there are no ultra-violet divergences that plague quantum field theories (short distance effects- breakdown of theory, i.e., new physics at short distances). String theory is the ultimate theory, no matter how short the distance.
Example: The torus
Puzzle: as vertices come together on a torus, we get a singularity. Is this a short distance effect? No! It is like the case of a sphere (topology plays no

role). Recall the singularity in $\langle V_1(\infty)V_2(1)V_3(z)V_4(0)\rangle$ as $z \to 1$. We obtain poles, which after a conformal transformation seem to come from a particle propagating for a long time.

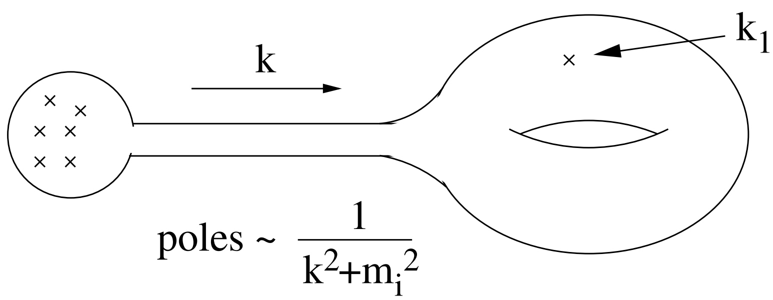


That is a long distance effect, i.e., an "infrared divergence" which is expected. $z \to 1$ means the distance becomes small on the worldsheet (bogus?). A long intermediate state is a **spacetime** concept (REAL!). Similarly, on a torus



Special Case I

All vertices but one come together.



Conservation of momentum implies $k = k_1$, but $k_1^2 = -m^2$, so $1/(k^2+m^2) = 1/0$, a singularity (infrared).
We get rid of it in quantum field theory as follows. Let $k^2 \neq -m^2$. Then

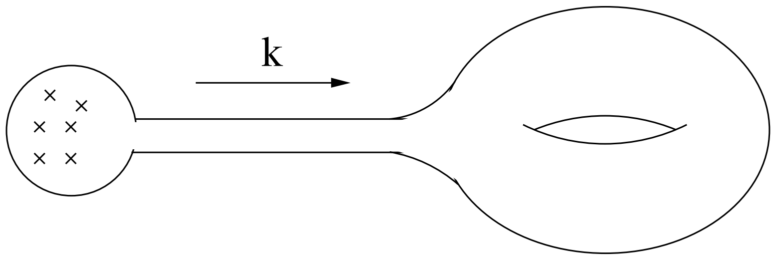
$$\begin{aligned} &= \delta + \frac{\delta^2}{k^2+m^2} + \frac{\delta^3}{(k^2+m^2)^2} + \dots \\ &= \frac{\delta}{1-\frac{\delta}{k^2+m^2}} = \frac{\delta(k^2+m^2)}{k^2+m^2-\delta} \end{aligned}$$

Notice that the pole has been shifted by δ, so the mass is corrected by quantum effects: $\delta m^2 = -\delta$. This means that we should have started with a particle of mass $m^2 - \delta$ and **not** m^2. It was a poor perturbative expansion (singular

perturbation theory). It is the **same** in string theory. We got $1/0$ because we asked the **wrong** question.
Note: Massless particles receive no quantum corrections as required by gauge invariance.

Special Case II

All vertices come together



So the pole from the massless modes goes as $1/k^2 = 1/0$! (Bad)
Again, we are asking the wrong question. To calculate the contribution of the massless modes, after we insert $\sum_E |E\rangle\langle E|$, we pick $E = 0$. That is an insertion of $\partial X^\mu \bar{\partial} X_\mu$. If we regulate the integral, e.g., by cutting the length of the connecting cylinder by L_{max}, we obtain

$$Amp \sim C \int d^2 z \partial X^\mu \bar{\partial} X_\mu$$

where C is an infinite constant.
Now this can also come from a perturbation in the action $\int d^2 z \partial X^\mu \bar{\partial} X_\mu$ which tells us that the metric in **spacetime**, instead of being flat, $G_{\mu\nu} = \eta_{\mu\nu}$ (Lorentz) should be $G_{\mu\nu} = (1 - C)\eta_{\mu\nu}$. The flat background is **not** a good zeroth order approximation because of the gravitational effects of the string.