PHYSICS 642 - FALL 2019
Homework Set 4

Due date: Mon., December 9, 2019

Problem 4.1

(a) Draw a quantum circuit using two qubits to construct the state

$$|\psi_N\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle$$

from $|0\rangle$ for $N = 2, 3, 4$.

(b) Draw a quantum circuit using four qubits that maps

$$|N-1\rangle \otimes |0\rangle \mapsto |N-1\rangle \otimes |\psi_N\rangle$$

where $|N-1\rangle$ is a two-qubit state, for $N = 2, 3, 4$.

Problem 4.2

(a) Let U be a 2×2 unitary matrix with $\det U = 1$. Find unitary matrices A, B, C such that

$$ABC = \mathbb{I}\,, \quad AXBXC = U$$

Hint: Use the Euler angle construction to express U in terms of rotations, $U = R_z(\alpha)R_y(\beta)R_z(\gamma)$. Also note that $XR_z(\alpha)X = R_z(-\alpha)$.

(b) Consider a two-qubit controlled phase gate: it applies $e^{i\theta}\mathbb{I}$ to the second qubit, if the first qubit is in the state $|1\rangle$, and acts trivially otherwise. Show that it is actually a single-qubit gate.

(c) Draw a quantum circuit using CNOT gates and single-qubit gates that implements controlled-U for an arbitrary 2×2 unitary matrix U.

Problem 4.3

Continuous-time database search. A quantum system consisting of n qubits has the Hamiltonian

$$H_0 = E|x_0\rangle\langle x_0|$$

where $|x_0\rangle \in \{|x\rangle, x = 0, 1, \ldots, 2^n - 1\}$ is an unknown basis state.
To find x_0, we turn on a time-independent perturbation

$$H' = E|s\rangle\langle s|$$

of the Hamiltonian, so that the total Hamiltonian becomes

$$H = H_0 + H'$$

where

$$|s\rangle = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle$$

We prepare the system in the initial state $|s\rangle$ and allow it to evolve, as governed by H, for a time T. Then we measure the state to find x_0.

(a) Solve the time-independent Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

to find the state at time T.

(b) How should T be chosen to optimize the likelihood of successfully determining x_0?

Problem 4.4

Universal quantum gates. The Hadamard and phase gates are defined, respectively, by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \ , \quad P = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$$

(a) Write out the 4×4 matrix representing the controlled phase gate (CP) in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Does it matter which qubit is the target? Explain.

(b) Consider the two-qubit unitary transformations,

$$U_i = H_i.CP.H_i \ , \quad i = 1, 2$$

where H_i is the Hadamard matrix acting on the ith qubit.

Write out the 4×4 matrices U_1 and U_2 in the standard basis, and show that they both act trivially on the states

$$|00\rangle \ , \quad \frac{1}{\sqrt{3}}(|01\rangle + |10\rangle + |11\rangle)$$

(c) It follows that U_1 and U_2 act non-trivially only in the two-dimensional space spanned by

$$\left\{\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \frac{1}{\sqrt{6}}(|01\rangle + |10\rangle - 2|11\rangle)\right\}$$

Show that, expressed in this basis, the two matrices are

$$U_1 = \frac{1}{4}\begin{pmatrix}3+i & -\sqrt{3}(1-i)\\ -\sqrt{3}(1-i) & 1+3i\end{pmatrix} \ , \quad U_2 = \frac{1}{4}\begin{pmatrix}3+i & \sqrt{3}(1-i)\\ \sqrt{3}(1-i) & 1+3i\end{pmatrix}$$

(d) Express the two matrices in the form

$$U_i = e^{i\phi_i}\left(\cos\theta_i + i\hat{n}_i\cdot\vec{\sigma}\sin\theta_i\right) \ , \quad i = 1, 2$$

and find the angles ϕ_i, θ_i and the unit vectors $\hat{n}_i$.

(e) Draw the quantum circuit for the transformation $U_2^{-1}U_1$.

Show that it performs a rotation of angle χ, where $\cos\frac{\chi}{2} = \frac{1}{4}$ in the two-dimensional space in which it acts non-trivially.

It can be deduced that H and CP form a universal gate set.