PHYSICS 642 - FALL 2019
Homework Set 3

Due date: Wed., November 13, 2019

Problem 3.1

Alice and Bob share many identical copies of the two-qubit state

$$|\psi\rangle = \sqrt{1-2x}|00\rangle + \sqrt{x}|01\rangle + \sqrt{x}|10\rangle$$

where $0 < x < \frac{1}{2}$. They conduct many trials in which each measures his/her qubit in the basis $\{|0\rangle, |1\rangle\}$, and they learn that, if Alice's outcome is $|1\rangle$, then Bob's is always $|0\rangle$, and if Bob's outcome is $|1\rangle$, then Alice's is always $|0\rangle$.

Alice and Bob conduct further experiments in which Alice measures in the basis $\{|0\rangle, |1\rangle\}$, whereas Bob measures in the orthonormal basis $\{|\phi_0\rangle, |\phi_1\rangle\}$. They discover that, if Alice's outcome is $|0\rangle$, then Bob's outcome is always $|\phi_0\rangle$ and never $|\phi_1\rangle$. Similarly, if Bob measures in the basis $\{|0\rangle, |1\rangle\}$ and Alice measures in the basis $\{|\phi_0\rangle, |\phi_1\rangle\}$, then if Bob's outcome is $|0\rangle$, Alice's outcome is always $|\phi_0\rangle$ and never $|\phi_1\rangle$.

(a) Calculate $|\phi_0\rangle$ and $|\phi_1\rangle$.

(b) If Alice and Bob both measure in the basis $\{|\phi_0\rangle, |\phi_1\rangle\}$, what is the probability $P(x)$ that both obtain the outcome $|\phi_1\rangle$?

(c) Find the maximum value of the probability $P(x)$.

(d) Consider this argument: *Suppose that Alice and Bob both measure in the basis $\{|\phi_0\rangle, |\phi_1\rangle\}$, and that they both obtain the outcome $|\phi_1\rangle$. If Alice had measured in the basis $\{|0\rangle, |1\rangle\}$, instead, we can be certain that her outcome would have been $|1\rangle$, since experiment has shown that if Alice had obtained $|0\rangle$, then Bob could not have obtained $|\phi_1\rangle$. Similarly, if Bob had measured in the basis $\{|0\rangle, |1\rangle\}$, then he certainly would have obtained the outcome $|1\rangle$. We conclude that, if Alice and Bob both measured in the basis $\{|0\rangle, |1\rangle\}$, both would have obtained the outcome $|1\rangle$. But this is a contradiction, for experiment has shown that it is not possible for both Alice and Bob to obtain the outcome $|1\rangle$, if they both measure in the basis $\{|0\rangle, |1\rangle\}$. We are therefore forced to conclude that, if Alice and Bob both measure in the basis $\{|\phi_0\rangle, |\phi_1\rangle\}$, it is impossible for both to obtain the outcome $|\phi_1\rangle$.*

The above argument leads to the erroneous conclusion that $P(x) = 0$. What is wrong with this argument?

Problem 3.2

Consider two particles, A (that belongs to Alice), and B (that belongs to Bob), whose trajectories are described by coordinates q_A and q_B, and their momenta are p_A and p_B, respectively. We have $[q_A, p_A] = [q_B, p_B] = i\hbar$. Consider the entangled state

$$|Q, P\rangle = \frac{1}{\sqrt{2\pi}} \int dq e^{iPq/\hbar} |q\rangle_A |q + Q\rangle_B$$

where $|q\rangle_A$ ($|q\rangle_B$) is an eigenstate of the position operator $\boldsymbol{q}_A$ ($\boldsymbol{q}_B$) with eigenvalue q.

(a) Show that $|Q, P\rangle$ is a common eigenstate of the relative position operator, $\boldsymbol{Q} = \boldsymbol{q}_B - \boldsymbol{q}_A$, and the total momentum, $\boldsymbol{P} = \boldsymbol{p}_A + \boldsymbol{p}_B$.

(b) Verify that the states $|Q, P\rangle$ form an orthonormal set by showing that

$$\langle Q', P'|Q, P\rangle = \delta(Q' - Q)\delta(P' - P)$$

(c) Calculate the coefficients $A(Q, P; q_A, q_B)$ in the expansion of a position eigenstate,

$$|q_A\rangle_A |q_B\rangle_B = \int dQdP\, A(Q, P; q_A, q_B)|Q, P\rangle$$

(d) Alice and Bob share the entangled state $|Q, P\rangle$ of their two particles A and B. Charlie sends a third particle C with position operator $\boldsymbol{q}_C$ and momentum operator $\boldsymbol{p}_C$ to Alice in the state $|\psi\rangle_C$. Alice wants to teleport it to Bob.

Design a protocol that they can execute to achieve teleportation. What should Alice measure? What *classical* information should she send to Bob? What should Bob do when he receives this information so that particle B will be prepared in the state $|\psi\rangle_B$?

Problem 3.3

Consider a *noisy* entangled pair shared by Alice and Bob with density matrix

$$\rho_\lambda = (1 - \lambda)|\psi^-\rangle\langle\psi^-| + \frac{\lambda}{4}\mathbb{I}$$

where $0 < \lambda < 1$, and $|\psi^-\rangle$ is the singlet state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

(a) If Alice measures the spin of her qubit along the $\hat{n}$-axis and Bob measures his qubit along the $\hat{m}$-axis, what is the probability that they will both get spin up?

(b) Find the fidelity F that can be attained if the state ρ_λ is used to teleport a qubit from Alice to Bob.

Problem 3.4

The no-cloning theorem shows that we cannot build a unitary machine that will make a perfect copy of an unknown quantum state. But what if we are willing to settle for an *imperfect* copy? What fidelity might we achieve?

(a) Consider a machine that acts on three qubits according to

$$\begin{aligned}
|000\rangle_{ABC} &\longrightarrow \sqrt{\frac{2}{3}}|00\rangle_{AB}|0\rangle_C + \frac{1}{\sqrt{3}}|\psi^+\rangle_{AB}|1\rangle_C \\
|100\rangle_{ABC} &\longrightarrow \sqrt{\frac{2}{3}}|11\rangle_{AB}|1\rangle_C + \frac{1}{\sqrt{3}}|\psi^+\rangle_{AB}|0\rangle_C
\end{aligned}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

Design a machine that can do this, i.e., find a unitary matrix that implements the above transformation.

(b) For an input $|\Psi_{\rm in}\rangle_{ABC} = |\psi\rangle_A|00\rangle_{BC}$, where $|\psi\rangle_A = a|0\rangle_A + b|1\rangle_A$, let $|\Psi_{\rm out}\rangle_{ABC}$ be the output. Show that the output contains two identical, but *imperfect*, copies of $|\psi\rangle$, by showing that $\rho_A = \rho_B$, where

$$\rho_A = {\rm Tr}_{BC}|\Psi_{\rm out}\rangle\langle\Psi_{\rm out}|\ , \ \ \rho_B = {\rm Tr}_{AC}|\Psi_{\rm out}\rangle\langle\Psi_{\rm out}|$$

are the respective states of the qubits A and B, each observed in isolation.

(c) Calculate the fidelity of the copy,

$$F = {}_A\langle\psi|\rho_A|\psi\rangle_A$$