PHYSICS 642 - FALL 2019
Homework Set 2

Due date: Mon., October 21, 2019

Problem 2.1

Consider a game in which Alice prepares a state ρ_1 with probability p_1, or ρ_2 with probability p_2 ($p_1 + p_2 = 1$). To guess which state Alice prepared, Bob performs a POVM with outcomes corresponding to two non-negative Hermitian operators F_i ($i = 1, 2$, $F_1 + F_2 = \mathbb{I}$). If Bob's outcome is F_1, he guesses that Alice's state was ρ_1, and if it is F_2, he guesses ρ_2.

(a) Show that the probability Bob guesses wrong is

$$p_{\text{error}} = p_1 \mathrm{Tr}(\rho_1 F_2) + p_2 \mathrm{Tr}(\rho_2 F_1)$$

(b) Let $|i\rangle$ be an eigenstate of the Hermitian operator $p_2\rho_2 - p_1\rho_1$ with corresponding eigenvalue λ_i. Show that

$$p_{\text{error}} = p_1 + \sum_i \lambda_i \langle i|F_1|i\rangle$$

(c) Find F_1 that minimizes p_{error} and show that the minimum value is

$$p_{\text{error,min}} = p_1 + \sum_{i, \lambda_i < 0} \lambda_i$$

where the sum is over the negative eigenvalues of $p_2\rho_2 - p_1\rho_1$.

(d) As an example, let $\rho_i = |\psi_i\rangle\langle\psi_i|$, $p_i = \frac{1}{2}$ ($i = 1, 2$), where

$$|\psi_1\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} , \quad |\psi_2\rangle = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} , \quad 0 < \theta < \frac{\pi}{4}$$

Find Bob's optimal two-outcome measurement $\{F_1, F_2\}$, and compute the minimum error probability $p_{\text{error,min}}$.

Problem 2.2

Alice is equipped to prepare either one of the two states

$$|\psi_1\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} , \quad |\psi_2\rangle = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} , \quad 0 < \theta < \frac{\pi}{4}$$

Alice sends either one of these states to Bob randomly (i.e., with probability $p_1 = p_2 = \frac{1}{2}$), and Bob makes a measurement to determine what she sent.

(a) Bob cannot identify Alice's qubit with certainty, so he performs a POVM with three possible outcomes: NOT-1, NOT-2, or IDK. If he obtains NOT-1, then he knows he received $|\psi_2\rangle$. If he obtains NOT-2, then he knows he received $|\psi_1\rangle$. If the result is IDK, then his measurement is inconclusive. The corresponding operators are

$$F_{\mathrm{NOT-1}} = A(\mathbb{I} - |\psi_1\rangle\langle\psi_1|) , \quad F_{\mathrm{NOT-2}} = A(\mathbb{I} - |\psi_2\rangle\langle\psi_2|) , \quad F_{\mathrm{IDK}} = \mathbb{I} - F_{\mathrm{NOT-1}} - F_{\mathrm{NOT-2}}$$

where $A > 0$.

Find A that minimizes the probability of the outcome IDK. What is the minimal IDK probability?

[HINT: If A is too large, then F_{IDK} will have negative eigenvalues and will not be part of a POVM.]

(b) Eve also wants to know what Alice is sending to Bob. Hoping that Alice and Bob won't notice, she intercepts each qubit that Alice sends by performing an orthogonal measurement that projects onto the states $|0\rangle$ and $|1\rangle$. If she obtains the outcome $|0\rangle$, she sends the state $|\psi_1\rangle$ to Bob, and if she obtains the outcome $|1\rangle$, she sends $|\psi_2\rangle$ to Bob. Therefore, each time Bob's POVM has a conclusive outcome, Eve knows with certainty what that outcome is. Unfortunately for Eve, sometimes Bob obtains a "conclusive" outcome that differs from what Alice sent. What is the probability of such an error?

Problem 2.3

A harmonic oscillator is damped because it emits a photon with energy $\hbar\omega$ when making a transition from the energy level $E_n = n\hbar\omega$ to $E_{n-1} = (n-1)\hbar\omega$ (we have subtracted the ground-state energy $\frac{1}{2}\hbar\omega$, for simplicity). The probability that a photon is emitted in a small time interval Δt is $\Gamma\Delta t$, where Γ is the emission rate. Suppose that initially, the coupled system of the harmonic oscillator and the electromagnetic field is in the state

$$|\Psi(0) = |\psi\rangle_{\text{h.o.}}|0\rangle_{\text{EM}}$$

where $|n\rangle_{\text{EM}}$ is the state of the electromagnetic field with n photons.
At time Δt, the system evolves into the state

$$|\Psi(\Delta t)\rangle = \sqrt{\Gamma\Delta t}\boldsymbol{a}|\psi\rangle_{\text{h.o.}}|1\rangle_{\text{EM}} + \left(\mathbb{I} - \frac{1}{2}\Gamma\Delta t\boldsymbol{a}^\dagger\boldsymbol{a}\right)|\psi\rangle_{\text{h.o.}}|0\rangle_{\text{EM}}$$

where $\boldsymbol{a}$ acts on $|\psi\rangle_{\text{h.o.}}$ reducing the excitation level by one unit.

(a) Check that the evolution is unitary by showing that

$$\langle\Psi(\Delta t)|\Psi(\Delta t)\rangle = 1 + \mathcal{O}\left((\Delta t)^2\right)$$

(b) Show that the density matrix of the harmonic oscillator

$$\boldsymbol{\rho} = \text{Tr}_{\text{EM}}|\Psi\rangle\langle\Psi|$$

evolves as

$$\boldsymbol{\rho}(\Delta t) = \Gamma\Delta t\boldsymbol{a}\boldsymbol{\rho}(0)\boldsymbol{a}^\dagger + \left(\mathbb{I} - \frac{1}{2}\Gamma\Delta t\boldsymbol{a}^\dagger\boldsymbol{a}\right)\boldsymbol{\rho}\left(\mathbb{I} - \frac{1}{2}\Gamma\Delta t\boldsymbol{a}^\dagger\boldsymbol{a}\right)$$

Explain why this is true even if the initial state of the harmonic oscillator is not pure.

(c) Suppose that the initial state of the oscillator is a coherent state,

$$|\psi\rangle = |\alpha\rangle \equiv e^{-|\alpha|^2/2} e^{\alpha \boldsymbol{a}^\dagger} |0\rangle$$

where α is a complex number. Calculate the average position $\langle x \rangle$ and uncertainty Δx in terms of α to show that this is a minimum uncertainty wave packet centered at $x_0 = \langle x \rangle$. What is its energy?

Show that

$$\left(\mathbb{I} - \frac{1}{2}\Gamma \Delta t \boldsymbol{a}^\dagger \boldsymbol{a}\right) |\alpha\rangle = e^{-\Gamma \Delta t |\alpha|^2/2} |\alpha e^{-\Gamma \Delta t/2}\rangle + \mathcal{O}\left((\Delta t)^2\right)$$

and therefore,

$$\boldsymbol{\rho}(\Delta t) = |\alpha e^{-\Gamma \Delta t/2}\rangle\langle \alpha e^{-\Gamma \Delta t/2}|$$

if $\rho(0) = |\alpha\rangle\langle\alpha|$.

Explain why this implies

$$\rho(t) = |\alpha e^{-\Gamma t/2}\rangle\langle \alpha e^{-\Gamma t/2}|$$

for finite time t, and that the energy decays like $e^{-\Gamma t}$ (i.e., Γ is the damping rate of the harmonic oscillator).

(d) Now suppose that the initial state of the harmonic oscillator is the *cat* state

$$|\psi\rangle = N\left[|\alpha\rangle + |-\alpha\rangle\right]$$

Calculate the normalization constant N.

If α is large, then show that this state consists of two well-separated minimum uncertainty wave packets.

Show that at time t, the operator $|\alpha\rangle\langle -\alpha|$ (that contributes to $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$) becomes

$$e^{i\phi} e^{-2\Gamma t |\alpha|^2} |\alpha e^{-\Gamma t/2}\rangle\langle -\alpha e^{-\Gamma t/2}|$$

and find the phase ϕ. Thus, this term in ρ decays exponentially at a rate

$$\Gamma_{\text{decohere}} = 2\Gamma|\alpha|^2$$

Estimate the damping time $1/\Gamma$ and decoherence rate Γ_{decohere} for a harmonic oscillator with mass $m = 1$ g, frequency $\omega = 1\ \text{s}^{-1}$, quality factor $Q \equiv \omega/\Gamma = 10^9$ (unrealistically good), and wavepackets initially separated by 1 cm.

Problem 2.4

(a) Consider a depolarizing qubit that is subjected to *Pauli errors* at a rate $\tilde{\Gamma}$, where X, Y, Z are all equally likely. The depolarization can be described by a master equation with Lindblad operators $\sqrt{\frac{\tilde{\Gamma}}{3}}X, \sqrt{\frac{\tilde{\Gamma}}{3}}Y, \sqrt{\frac{\tilde{\Gamma}}{3}}Z$. Show that this master equation has the form

$$\frac{d\boldsymbol{\rho}}{dt} = -i[\boldsymbol{H}, \boldsymbol{\rho}] - \Gamma\left(\boldsymbol{\rho} - \frac{1}{2}\mathbb{I}\right)$$

where we set $\hbar = 1$, and find Γ in terms of $\tilde{\Gamma}$.

(b) Up to an irrelevant term proportional to the identity, the most general Hamiltonian is

$$\boldsymbol{H} = \frac{\omega}{2}\hat{n} \cdot \vec{\sigma}$$

where $\hat{n}$ is a unit vector. Use this form of $\boldsymbol{H}$ and the Bloch parametrization

$$\boldsymbol{\rho} = \frac{1}{2}(\mathbb{I} + \vec{P} \cdot \vec{\sigma})$$

to show that the master equation can be rewritten as

$$\frac{d\vec{P}}{dt} = \omega\hat{n} \times \vec{P} - \Gamma\vec{P}$$

Thus the polarization precesses uniformly with frequency ω about the axis along $\hat{n}$ as it contracts with lifetime $\tau = \frac{1}{\Gamma}$.

(c) Alice and Bob play a game in which Alice decides to turn on one of the two Hamiltonians

$$\boldsymbol{H} = \frac{\omega}{2}Z \;, \quad \boldsymbol{H}' = 0$$

with equal probabilities, and Bob is to guess which Hamiltonian Alice chose. Bob has a supply of qubits, and he can observe whether the qubits “precess” in order to distinguish $\boldsymbol{H}$ from $\boldsymbol{H}'$. However, according to the master equation, his qubits are also subject to depolarization at the rate Γ. Suppose that Bob prepares his qubits at $t = 0$ with polarization $\vec{P}(0) = \hat{x}$. At time $t > 0$, find the polarization $\vec{P}(t)$ if the Hamiltonian is $\boldsymbol{H}$, and $\vec{P}'(t)$ is the Hamiltonian is $\boldsymbol{H}'$.

(d) What is Bob’s optimal measurement for distinguishing the two polarizations $\vec{P}(t)$ and $\vec{P}'(t)$? What is the optimal probability of error?

(e) Find the time that Bob needs to wait for before measuring in order for the probability of error to be minimum. Does your answer make sense in the limits $\Gamma \gg \omega$ and $\Gamma \ll \omega$?