PHYSICS 642 - FALL 2019

Homework Set 1

Due date: Fri., September 27, 2019

Problem 1.1

(a) Show with an explicit calculation that $\hat{n}\cdot\vec{\sigma}$ has eigenvalues ± 1 and that the projectors onto the corresponding eigenstates are $P_{\pm} = \frac{1}{2}(\mathbb{I} \pm \hat{n}\cdot\vec{\sigma})$.

(b) If the system is in the state $|0\rangle$, what is the probability of obtaining the result $+1$ for a measurement of $\hat{n}\cdot\vec{\sigma}$, and what is the state of the system after the measurement, if the outcome is $+1$?

Problem 1.2

Consider a composite system consisting of systems A and B.

(a) If the composite system is in the state

$$|\Psi\rangle = |\psi_1\rangle_A \otimes |\psi_2\rangle_B$$

calculate the density matrices of the two systems, ρ_A and ρ_B, and show that they are both pure.

(b) If the composite system is in the state

$$|\Psi\rangle = \mathcal{N}\left[|\psi_1\rangle_A \otimes |\psi_2\rangle_B + |\psi_2\rangle_A \otimes |\psi_1\rangle_B\right]$$

with $\langle\psi_1|\psi_2\rangle \neq 0$, calculate the normalization constant $\mathcal{N}$, and find the density matrices ρ_A and ρ_B.

Show that $\mathrm{Tr}\rho_A = \mathrm{Tr}\rho_B = 1$.

Problem 1.3

Consider a qubit in the state $|\psi\rangle$. It is rotated by an angle θ around the $\hat{n}$ axis by acting on it with the operator

$$R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}$$

The qubit is mapped onto the vector $\vec{P}$ on the Bloch sphere, where $\vec{P}^2 = 1$.

(a) What is the density matrix ρ in terms of $\vec{P}$?

(b) Let ρ' be the density matrix after the qubit is rotated by an angle θ around the $\hat{n}$ axis. Show that ρ' corresponds to a vector $\vec{P}'$ on the Bloch sphere, which is obtained by rotating $\vec{P}$ by an angle θ around the $\hat{n}$ axis.

Problem 1.4

Find the Schmidt decomposition of the two-qubit states

$$\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]\ ,\ \frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]\ ,\ \frac{1}{\sqrt{3}}\left[|00\rangle + |01\rangle + |10\rangle\right]$$

Problem 1.5

Gleason's Theorem assigns probabilities $p(E_\lambda)$ to projectors $E_\lambda = |\lambda\rangle\langle\lambda|$. The assignment for each E_λ is independent of the other assignments.

A *hidden-variable theory* asserts that probabilities are derived from a more fundamental deterministic description. If all hidden variables are precisely known, then each probability $p(E_\lambda)$ should either be 0 or 1. The probabilities of quantum measurements are obtained by averaging over unknown values of the hidden variables. It turns out that the hidden-variable theory violates the above assumption of *Gleason's Theorem*.

To see this, consider the nine two-qubit observables

$$\begin{array}{ccc} X \otimes \mathbb{I} & \mathbb{I} \otimes X & X \otimes X \\ \mathbb{I} \otimes Y & Y \otimes \mathbb{I} & Y \otimes Y \\ X \otimes Y & Y \otimes X & Z \otimes Z \end{array}$$

(a) Show that the observables in each row and in each column are mutually commuting, and find their common eigenstates. Thus, we obtain six different complete sets of one-dimensional projectors for two qubits.

(b) Each of the observables has eigenvalues ± 1. A *hidden-variable theory* would assign a definite value, either $+1$ or -1 to each observable. This deterministic assignment has to be consistent; e.g., the product of operators in the first row is the identity, which must be assigned $+1$, therefore either all observables in the first row are assigned $+1$, or two are assigned -1 and one $+1$. This introduces 6 constraints on the assignments. Show that there is no way to satisfy all 6 constraints simultaneously.

Problem 1.6

Charlie prepares system A in one of two *non-orthogonal* states, $|\psi_1\rangle_A$ or $|\psi_2\rangle_A$, and challenges Alice to uncover information about which state he prepared without in any way disturbing the state. To meet the challenge, Alice prepares an ancillary system B in the state $|\phi\rangle_B$, and applies a unitary on the composite system acting as

$$U\ :\ \begin{array}{lcl} |\psi_1\rangle_A \otimes |\phi\rangle_B & \to & |\psi_1\rangle_A \otimes |\phi'\rangle_B \\ |\psi_2\rangle_A \otimes |\phi\rangle_B & \to & |\psi_2\rangle_A \otimes |\phi''\rangle_B \end{array}$$

which does indeed leave the state of system A undisturbed.

(a) Will Alice succeed in meeting the challenge?

[Hint: Use the unitarity of U to relate the states $|\phi'\rangle_B$ and $|\phi''\rangle_B$]

(b) Can Alice succeed if the states $|\psi_1\rangle_A$ and $|\psi_2\rangle_A$ are, instead, orthogonal?