PHYSICS 611 - QUANTUM FIELD THEORY

Two Fundamental Theorems

George Siopsis - Fall 2002

The Axioms for scalar fields

A relativistic quantum field theory (RQFT) is defined on a Hilbert space $\mathcal{H}$. The states in the theory correspond to *rays* in $\mathcal{H}$. We also need to introduce a set of *fields* $\phi^a(x^\mu)$, $(a = 1, \dots, N)$, where $x^\mu \in \mathbb{R}^4$. The fields are operator-valued distributions in the sense that, if $f(x^\mu)$ is a given smooth function on $\mathbb{R}^4$ of finite support (smearing ϕ^a out), then

$$\phi^a_f = \int d^4x f(x^\mu)\phi^a(x^\mu)$$

is a linear operator in $\mathcal{H}$ ($\phi_f : \mathcal{H} \to \mathcal{H}$).

The theory is based on the following five axioms due to Wightman.

AXIOM 1 (RELATIVISTIC TRANSFORMATION LAW) *Each Lorentz transformation* $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ *(with* $\det\Lambda = 1$, $\Lambda^0{}_0 \geq 1$*) has a* unitary *representation* $U(\Lambda)$ *acting on* $\mathcal{H}$ *such that*

$$U(\Lambda)\phi^a(x^\mu)U^\dagger(\Lambda) = \phi^a(\Lambda^\mu{}_\nu x^\nu)$$

Each translation is similarly represented by $U(a) = e^{ia^\mu P_\mu}$*, i.e.,*

$$U(a)\phi^a(x^\mu)U^\dagger(a) = \phi^a(x+a)$$

where P_μ *has eigenvalues satisfying*

$$P^\mu P_\mu = m^2 \geq 0 \quad , \quad P_0 \geq 0$$

AXIOM 2 (THE VACUUM) *There exists a unique state (ray in* $\mathcal{H}$*), called the vacuum* $|0\rangle$*, which is invariant under Lorentz transformations* $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ *and spacetime translations* $x^\mu \to x^\mu + a^\mu$*.*

It follows that the vacuum is annihilated by the cooresponding unitary operators,

$$U(\Lambda)|0\rangle = U(a)|0\rangle = 0$$

Also, any correlator,

$$G(x_1, \dots, x_n) = \langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle$$

where I omitted indices on ϕ for simplicity, is a function of differences $x_i - x_j$ only.

Axiom 3 (Completeness) *The set of all polynomials in (smeared) fields* ϕ^a *acting on the vacuum is* dense *in* $\mathcal{H}$.

Thus, we may create all states by acting repeatedly with ϕ^a on the vacuum.

Axiom 4 (Causality) *Fields separated by a spacelike distance commute with each other:*

$$[\phi^a(x)\ ,\ \phi^b(y)] = 0 \quad , \quad (x-y)^2 < 0$$

All axioms so far are based on basic physical principles. The next (and final) axiom is there for technical reasons.

Axiom 5 (Polynomial Boundedness) *The matrix elements of the fourier transform of a field,* $\widetilde{\phi}^a(p)$, *are bounded:*

$$|\langle A|\widetilde{\phi}^a(p)|B\rangle| \ \leq \ P_n(p)$$

where $P_n(p)$ *is a polynomial in the momentum* p^μ *of degree* n *(unspecified).*

This axiom ensures that the field ϕ^a is no more signular a distribution than the nth derivative of a δ-function, for some $n < \infty$.

Complexification

A useful tool in the proof of any theorem based on the above axioms is the analytic continuation of the real coordinates x^μ to complex coordinates. This allows us to make use of the powerful machinery of complex analysis.

This leads us to consider complex Lorentz transformations $\Lambda^\mu_{\ \nu} \in \mathbb{C}$ obeying

$$\Lambda^T \eta \Lambda = \eta \quad , \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with $\det \Lambda = 1$. For real $\Lambda^\mu_{\ \nu}$, this group consists of two disconnected components, with $\Lambda^0_{\ 0} \geq 1$ and $\Lambda^0_{\ 0} \leq -1$, respectively. For example, the transformation

$$U_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

cannot be connected to the identity I by a continuous path. This is not true for complex $\Lambda^\mu_{\ \nu}$. Indeed, consider the boost in the x-direction,

$$U(\zeta) = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can take ζ along a continuous path from $\zeta_0 = 0$ to $\zeta_1 = i\pi$, e.g., by setting $\zeta = i\theta$ ($\theta \in [0, \pi]$). Clearly, $U(\zeta_0) = I$ and $U(\zeta_1) = U_1$. Thus, $U(\zeta)$ connects I to U_1 via a continuous path in the complex plane.

In conclusion: the complexified Lorentz group has a single connected component.

Correlation (n-point) functions

$$G_n(x_1, \dots, x_n) = \langle 0|\phi(x_1) \cdots \phi(x_n)|0\rangle$$

may also be continued analytically to $x_1, \dots, x_n \in \mathbb{C}$. The domain of analyticity is the region defined by

$$\Im(x_i^0 - x_{i+1}^0) < 0$$

i.e., consecutive time differences have negative imaginary parts. We will show this for the two-point function ($n = 2$). The argument may be straightforwardly extended to higher-order functions.

Using Axiom 1, we may write

$$G_2(x_1, x_2) = \langle 0|\phi(0)e^{-i(x_1-x_2)^\mu P_\mu}\phi(0)|0\rangle = \sum_k \; e^{-i(x_1-x_2)\cdot P_k} \; |\langle 0|\phi(0)|k\rangle|^2$$

where $P_{k0} = \sqrt{m^2 + \vec{P}_k^2}$. (We used translation invariance of the vacuum and inserted a sum over momentum eigenstates. The sum is symbolic; it is an integral over continuous eigenvalues of the momentum - alternatively, place the system in a box to discretize the spectrum.) This sum (integral) converges if $\langle 0|\phi(0)|\vec{k}\rangle$ behaves (as a function of $\vec{k}$), which is ensured by polynomial boundedness (Axiom 5), and if $(x_1 - x_2)^0$ has a negative imaginary part (since $k_0 \geq 0$), providing an exponential cutoff for large momenta $|\vec{k}|$.

We may not continue these functions analytically beyond $\Im(x_i^0 - x_{i+1}^0) = 0$, because they in general possess cuts there. These cuts are due to the non-commutativity of fields separated by timelike distances.

The CPT Theorem

First consider a single hermitian scalar field ϕ so that charge conjugation is meaningless. Then we shall prove a PT Theorem. Recall that P (parity) and T (time reversal) are not connected to the identity through a continuous path of Lorentz transformations, since $\det P = \det T = -1$. The same is true for PT, even though $\det(PT) = 1$. This is because $(PT)^0_{\ 0} = -1$.

In the complexified Lorentz group, the above statement is still true for P and T, so there exist QFTs in which either P or T are not symmetries. However, PT is continuously connected to the identity. Therefore, it will be a symmetry of any theory if we can show that the operator implementing PT, i.e., the operator U_{PT} with the property

$$U_{PT} \; \phi(x^\mu) \; U_{PT}^{-1} = \phi(-x^\mu)$$

is *anti*-unitary (recall that U_T cannot be unitary; it has to be *anti*-unitary). We shall prove the following theorem.

THEOREM 1 (PT THEOREM) *U_{PT} is* anti-*unitary, i.e.,*

$$\langle U_{PT}\Psi_1|U_{PT}\Psi_2\rangle = \langle\Psi_2|\Psi_1\rangle = \langle\Psi_1|\Psi_2\rangle^*$$

PROOF: By the completeness axiom, it suffices to consider inner products of the form

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle$$

on which the statement of anti-unitarity (to be proved) reads

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle = \langle 0|\phi(-x_1)\cdots\phi(-x_n)|0\rangle^* = \langle 0|\phi(-x_n)\cdots\phi(-x_1)|0\rangle$$

We shall prove this statement in three steps.
<u>STEP 1:</u> Using the causality axiom, we may commute the fields freely to conclude

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle = \langle 0|\phi(x_n)\cdots\phi(x_1)|0\rangle$$

for spacelike distances, $(x_i - x_j)^2 < 0$. The domain of validity of the above statement extends into the complex plane by analytic continuation (complex $x_1, \dots, x_n$).

- **CAUTION:** The above statement is most certainly not true for timelike distances, because the commutators of fields do not vanish. Timelike distances lie on the boundary of the domain of analyticity. To see how the above statement is modified as we approach the boundary, consider the two-point function, which is a function of $z^2 = (x_1 - x_2)^2$ only, due to translational invariance of the vacuum (AXIOM 2) and Lorentz invariance. When we approach the *positive* real axis in the z^2-plane, say, at $z^\mu = (t, \vec{x})$, where $z^2 = t^2 - \vec{x}^2 > 0$, we may only do so as

  $$G_2(t,\vec{x}) = \lim_{\epsilon\to 0^+} G_2(t - i\epsilon, \vec{x}) \quad , \quad G_2(-t,-\vec{x}) = \lim_{\epsilon\to 0^+} G_2(-t - i\epsilon, -\vec{x})$$

  i.e., below and above the positive real axis cut, respectively. It follows that

  $$G_2(t,\vec{x}) = G_2^*(-t,-\vec{x})$$

  (and <u>not</u> $G_2(t,\vec{x}) = G_2(-t,-\vec{x})$ as one would naïvely have extrapolated; that would lead to the erroneous conclusion that fields at timelike distances commute).

  EXAMPLE: Verify the above statements for the two-point function of a free field of mass m,

  $$G_2(t,\vec{x}) = \frac{1}{4\pi^2\sqrt{-t^2+\vec{x}^2}} \int_m^\infty \frac{d\omega\ \omega}{\sqrt{\omega^2 - m^2}}\, e^{-\omega\sqrt{-t^2+\vec{x}^2}}$$

<u>STEP 2:</u> In the domain of analyticity of the n-point function, the PT transformation

$$\Lambda_{PT} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

is a complex Lorentz transformation that may reached continuously from the identity. We may therefore use AXIOM 1 to conclude

$$\langle 0|\phi(x_n)\cdots\phi(x_1)|0\rangle = \langle 0|\phi(\Lambda_{PT}x_n)\cdots\phi(\Lambda_{PT}x_1)|0\rangle = \langle 0|\phi(-x_n)\cdots\phi(-x_1)|0\rangle$$

STEP 3: Combining the results of STEP 1 and STEP 2, we conclude

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle = \langle 0|\phi(-x_n)\cdots\phi(-x_1)|0\rangle$$

in the domain of analyticity of the n-point function. To complete the proof of the PT theorem, we need to extend this statement to the boundary of the region of analyticity (which includes $x_i \in \mathbb{R}$). To this end, use translational invariance to write

$$G_n(x_1,\ldots,x_n) = \langle 0|\phi(x_1-x_n)\phi(x_2-x_n)\cdots\phi(0)|0\rangle$$

(shifting all coordinates by x_n), and

$$G_n(-x_n,\ldots,-x_1) = \langle 0|\phi(x_1-x_n)\cdots\phi(x_1-x_2)\phi(0)|0\rangle$$

(shifting all coordinates by $-x_1$). Introducing variables

$$z_1 = x_1 - x_2 \quad , \quad z_2 = x_2 - x_3 \quad , \quad \ldots$$

if we write

$$G_n(x_1,\ldots,x_n) = f(z_1,\ldots,z_{n-1})$$

then it is easy to see that

$$G_n(-x_n,\ldots,-x_1) = f(z_{n-1},\ldots,z_1)$$

so the statement $G_n(x_1,\ldots,x_n) = G_n(-x_n,\ldots,-x_1)$ translates to

$$f(z_1,\ldots,z_{n-1}) = f(z_{n-1},\ldots,z_1)$$

If this is true in the region of analyticity of f, it is certainly true on the boundary. This concludes the proof of the PT theorem.

This can be easily generalized to a set of real fields ϕ^a ($a = 1,\ldots,N$), as well as to *complex* fields. In the latter case, the following theorem holds

THEOREM 2 (CPT THEOREM) *The CPT operator, defined by its action on a complex field ϕ,*

$$U_{CPT}\,\phi(x^\mu)\,U_{CPT}^{-1} = \phi^*(-x^\mu)$$

is anti-unitary.

PROOF: An exercise.

The Spin-Statistics Theorem

Another important consequence of the Axioms of RQFT is the spin-statistics connection. We shall demonstrate this connection for a complex scalar field ϕ.

Theorem 3 *If ϕ* anti-*commutes with ϕ^* at spacelike separations,*

$$\{\phi(x)\ ,\ \phi^*(y)\} = 0 \quad , \quad (x-y)^2 < 0$$

then the Hilbert space $\mathcal{H}$ is trivial.

Proof: Introduce the two-point functions

$$G_2(z) = \langle 0|\phi(z)\phi^*(0)|0\rangle \quad , \quad \widehat{G}_2(z) = \langle 0|\phi^*(z)\phi(0)|0\rangle$$

Our assumption implies

$$G_2(z) + \widehat{G}_2(-z) = 0 \quad , \quad z^2 < 0$$

The above may be extended to the region of analyticity of G_2, which is the region $\Im z^0 < 0$.

Also, by *complex* Lorentz invariance,

$$\widehat{G}_2(-z) = \widehat{G}_2(-\Lambda_{PT}\, z) = \widehat{G}_2(z)$$

in the region of analyticity. Therefore,

$$G_2(z) + \widehat{G}_2(z) = 0$$

This can now be extended to the boundary of the region of analyticity, in particular it is valid in the limit $z \longrightarrow 0$. Since $G_2(0) = \widehat{G}_2(0)$, we obtain

$$G_2(0) = \langle 0|\phi(0)\phi^*(0)|0\rangle = 0$$

which is the statement that $|\phi\rangle = \phi(0)|0\rangle$ has zero norm ($\langle\phi|\phi\rangle = 0$). Therefore,

$$|\phi\rangle = \phi(0)|0\rangle = 0$$

By translational invariance, this generalizes to

$$\phi(x^\mu)|0\rangle = 0$$

for all points x^μ. Since the entire Hilbert space $\mathcal{H}$ is constructed by acting with ϕ on the vacuum, it follows that $\mathcal{H}$ is trivial.

It follows that ϕ ought to *commute* with ϕ^* at spacelike distances, so the Causality Axiom (Axiom 4) should be complimented by

$$[\phi(x)\ ,\ \phi^*(y)] = 0 \quad , \quad (x-y)^2 < 0$$

Next we turn to the commutation rules for ϕ.

Theorem 4 *If in the Causality Axiom (*Axiom 4*), we replace commutators with* anti-*commutators,*

$$\{\phi(x)\ ,\ \phi(y)\} = 0 \quad , \quad (x-y)^2 < 0$$

then the Hilbert space $\mathcal{H}$ is trivial.

Proof: Consider the state

$$|z\rangle = \phi(z)\phi(0)|0\rangle \quad , \quad z^2 < 0$$

We have

$$\langle z|z\rangle = \langle 0|\phi^*(0)\phi^*(z)\phi(z)\phi(0)|0\rangle$$

Commuting $\phi(0)$ through, we pick up a minus sign,

$$\langle z|z\rangle = -\langle 0|\phi^*(0)\phi(0)\phi^*(z)\phi(z)|0\rangle$$

At very large separations ($z \to \infty$), we have (*cluster decomposition*)

$$\langle 0|\phi^*(0)\phi(0)\phi^*(z)\phi(z)|0\rangle \to \Big(\langle 0|\phi^*(0)\phi(0)|0\rangle\Big)^2$$

This can be seen by inserting a complete set of momentum eigenstates to write

$$\begin{aligned}\langle 0|\phi^*(0)\phi(0)\phi^*(z)\phi(z)|0\rangle &= \langle 0|\phi^*(0)\phi(0)\ e^{iz^\mu P_\mu}\ \phi^*(0)\phi(0)|0\rangle \\ &= \textstyle\sum_k\ e^{iz\cdot P_k}\ \langle 0|\phi^*(0)\phi(0)|k\rangle\ \langle k|\phi^*(0)\phi(0)|0\rangle\end{aligned}$$

In the limit $z \to \infty$, only the vacuum contributes to the sum, so

$$\sum_k e^{iz\cdot P_k}\ \langle 0|\phi^*(0)\phi(0)|k\rangle\ \langle k|\phi^*(0)\phi(0)|0\rangle \to \Big(\langle 0|\phi^*(0)\phi(0)|0\rangle\Big)^2$$

as advertized. It follows that for sufficiently large z,

$$\langle 0|\phi^*(0)\phi(0)\phi^*(z)\phi(z)|0\rangle \geq 0$$

and so

$$\langle z|z\rangle = -\langle 0|\phi^*(0)\phi(0)\phi^*(z)\phi(z)|0\rangle \leq 0$$

Since $\langle z|z\rangle \geq 0$, we conclude $\langle z|z\rangle = 0$ and n the limit $z \to \infty$, we deduce

$$\Big(\langle 0|\phi^*(0)\phi(0)|0\rangle\Big)^2 = 0$$

therefore $\langle \phi|\phi\rangle = 0$, which implies

$$|\phi\rangle = \phi(0)|0\rangle = 0$$

leading to a trivial Hilbert space, as before.