Quantum Field Theory II

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UNIT 1

Interactions

When you include interactions, the plot thickens. Until now, we have only considered free fields, described by a Hamiltonian $H = H_0$, which we were able to diagonalize (find all eigenvalues and corresponding eigenstates). But that is no physics! It is time to do some serious physics.
When you include interactions, the Hamiltonian is modified to

$$H = H_0 + H_I \tag{1.0.1}$$

where H_I describes interactions.
It is no longer possible to solve the eigenvalue problem for H, except in very few special cases (mostly in two dimensions). The only thing we can do is perturbation theory, assuming H_I is small. This does not answer deep questions, such as *"what is a proton?"*, but it provides a method for very accurate calculations (e.g., the magnetic moment of the electron is known to about 10 significant digits both theoretically and experimentally, and they agree with each other!).
Experimental results are obtained primarily through scattering. To compare with them, we need to develop scattering theory. It turns out that this type of processes (scattering) holds *all* the information of a quantum field theory.

1.1 Scattering theory

Recall in quantum mechanics,

$$H = H_0 + V \tag{1.1.1}$$

where H_0 is the kinetic energy and V the potential. If V has compact support, then at times $t \to \pm\infty$, $H = H_0$, because $V = 0$. Thus, we may define incoming and outgoing states, $|in\rangle$ (at $t \to -\infty$) and $|out\rangle$ (at $t \to +\infty$), respectively, which are eigenstates of H.
We shall attempt to do the same in quantum field theory.
Consider a state $|\text{in}\rangle$. It evolves in time as

$$e^{-iHt}|\text{in}\rangle \tag{1.1.2}$$

As $t \to -\infty$, $H \to H_0$ (no interactions), so asymptotically, our state approaches a state in the Hilbert space constructed from H_0. Call that state $|\text{in},0\rangle$. It evolves in time as

$$e^{-iH_0t}|\text{in},0\rangle \tag{1.1.3}$$

Then the statement that this is asymptotic to the state $|\text{in}\rangle$ amounts to

$$e^{-iHt}|\text{in}\rangle \longrightarrow e^{-iH_0t}|\text{in},0\rangle \ , \quad \text{as} \ \ t \to -\infty \tag{1.1.4}$$

Therefore,

$$|\text{in}\rangle = \lim_{t\to-\infty} e^{iHt}e^{-iH_0t}|\text{in},0\rangle \tag{1.1.5}$$

The operator

$$U(t) \equiv e^{iH_0 t} e^{-iHt} \tag{1.1.6}$$

(a unitary map, $U^\dagger U = \mathbb{I}$) maps a state in the Hilbert space of H_0 ($|\text{in,0}\rangle$) to a state in the Hilbert space of H ($|\text{in}\rangle$) by

$$|\text{in}\rangle = \lim_{t \to -\infty} U^\dagger(t) |\text{in,0}\rangle \tag{1.1.7}$$

Notice that if H and H_0 commute ($[H, H_0] = 0$), then we may write $U(t) = e^{-iH_I t}$, where we used (1.0.1). But this rarely happens. Thus, in general, $U(t)$ is a very complicated object.
Similarly, in the infinite future, we may map

$$|\text{out}\rangle = \lim_{t \to +\infty} U^\dagger(t) |\text{out,0}\rangle \tag{1.1.8}$$

where $|\text{out}\rangle$ ($|\text{out,0}\rangle$) is in the Hilbert space of H (H_0).
We wish to calculate the amplitude of the process

$$|\text{in}\rangle \longrightarrow |\text{out}\rangle \tag{1.1.9}$$

i.e.,

$$\langle \text{out}|\text{in}\rangle = \lim_{t_\pm \to \pm\infty} \langle \text{out,0}|U(t_+)U^\dagger(t_-)|\text{in,0}\rangle = \langle \text{out,0}|S|\text{in,0}\rangle \tag{1.1.10}$$

where

$$S \equiv \lim_{t_\pm \to \pm\infty} U(t_+)U^\dagger(t_-) \tag{1.1.11}$$

is the S-matrix.
S maps in-asymptotes to out-asymptotes (these two Hilbert spaces may, in general, be distinct, but not here). In fact, S contains *all* the information of the quantum field theory. It is an important object to study.
Let us bring it into a more convenient form. To this end, we need to establish some properties of $U(t)$ first. We have

$$\begin{aligned} \frac{dU(t)}{dt} &= iH_0 e^{iH_0 t} e^{-iHt} + e^{iH_0 t}(-iH)e^{-iHt} \\ &= -ie^{iH_0 t}(H - H_0)e^{-iHt} \\ &= -ie^{iH_0 t} H_I e^{-iHt} \\ &= -ie^{iH_0 t} H_I e^{-iH_0 t} e^{iH_0 t} e^{-iHt} \\ &= -iH_I(t)U(t) \end{aligned} \tag{1.1.12}$$

Thus $U(t)$ is a true evolution operator with (time-dependent) Hamiltonian $H_I(t)$. The latter is obtained from H_I by evolving with H_0.
The first-order ODE (1.1.12) together with the initial condition $U(0) = \mathbb{I}$ uniquely determine $U(t)$.
To solve (1.1.12), convert it into an integral equation,

$$\begin{aligned} \int_0^t \frac{dU(t')}{dt'} dt' &= -i \int_0^t dt' H_I(t') U(t') \\ U(t) &= \mathbb{I} - i \int_0^t dt' H_I(t') U(t') \end{aligned} \tag{1.1.13}$$

and then use iteration,

$$U_0(t) = \mathbb{I} \ , \quad U_n(t) = \mathbb{I} - i \int_0^t dt' H_I(t') U_{n-1}(t') \quad (n \geq 1) \tag{1.1.14}$$

We obtain

$$\begin{aligned} U_0(t) &= \mathbb{I} \\ U_1(t) &= \mathbb{I} - i\int_0^t dt' H_I(t') \\ U_2(t) &= \mathbb{I} - i\int_0^t dt' H_I(t') + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 H_I(t_1) H_I(t_2) \end{aligned} \tag{1.1.15}$$

etc..
This may be brought into a more compact form. Look at the second-order term. In it, $t \geq t_1 \geq t_2 \geq 0$. The two-dimensional integral is over a triangle in the (t_1, t_2) plane, which is half of the square $0 \leq t_1, t_2 \leq t$. The other half gives the same answer, but with t_1 and t_2 interchanged. It follows that the two-dimensional integral can be written in terms of a time-ordered product as

$$\int_0^t dt_1 \int_0^{t_1} dt_2 H_I(t_1) H_I(t_2) = \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 T[H_I(t_1)H_I(t_2)] \tag{1.1.16}$$

For the nth term, we similarly obtain

$$\int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n H_I(t_1)\cdots H_I(t_n) = \frac{1}{n!}\int_0^t dt_1 \cdots \int_0^t dt_n T[H_I(t_1)\cdots H_I(t_n)] \tag{1.1.17}$$

It follows that

$$\begin{aligned} U(t) &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} T \int_0^t dt_1 \cdots \int_0^t dt_n H_I(t_1)\cdots H_I(t_n) \\ &= T\left[e^{-i\int_0^t dt' H_I(t')}\right] \end{aligned} \tag{1.1.18}$$

which is Dyson's formula.
Next, define the generalized evolution operator

$$U(t_1, t_2) \equiv U^\dagger(t_1)U(t_2) = T\left[e^{-i\int_{t_1}^{t_2} dt' H_I(t')}\right] \tag{1.1.19}$$

It is easily seen from its definition that it has the following properties

$$U(t,t')U(t',t'') = U(t,t'') \ , \quad U^\dagger(t,t') = U(t',t) \ , \quad U^\dagger(t,t')U(t,t') = \mathbb{I} \tag{1.1.20}$$

The S-matrix can be written as

$$S = \lim_{t_\pm \to \pm\infty} U(t_+, t_-) = T\left[e^{-i\int_{-\infty}^{+\infty} dt H_I(t)}\right] \tag{1.1.21}$$

Various important properties of the S-matrix follow.

- The S-matrix is unitary,

$$S^\dagger S = \mathbb{I} \tag{1.1.22}$$

  This may look like a trivial statement (direct consequence of the unitarity of $U(t_+, t_-)$), but we need to take limits $t_\pm \to \pm\infty$, and that may spoil unitarity, unless the range of S is the entire Hilbert space. The latter is a consequence of the requirement that probability be conserved: everything that comes in should go out. There are cases where this is not true and particles may be trapped into bound states. The S-matrix has a way to address these issues. For the most part, we shall assume that no trapping occurs and the S-matrix is unitary.

- S commutes with the free Hamiltonian H_0,

$$[S, H_0] = 0 \tag{1.1.23}$$

  i.e., a scattering process conserves *unperturbed* energy.

Proof.

$$\begin{aligned} e^{i\epsilon H_0} S e^{-i\epsilon H_0} &= \lim_{t_\pm \to \pm\infty} e^{i\epsilon H_0} e^{it_- H_0} e^{-it_- H} e^{it_+ H} e^{-it_+ H_0} e^{-i\epsilon H_0} \\ &= \lim_{t_\pm \to \pm\infty} e^{iH_0(t_-+\epsilon)} e^{-iH(t_-+\epsilon)} e^{iH(t_++\epsilon)} e^{iH_0(t_--\epsilon)} \\ &= \lim_{t_\pm \to \pm\infty} U(t_-+\epsilon) U^\dagger(t_++\epsilon) \end{aligned}$$

But as $t_\pm \to \pm\infty$, adding an ϵ makes no difference, so

$$e^{i\epsilon H_0} S e^{-i\epsilon H_0} = S \tag{1.1.24}$$

Expanding in ϵ, at first order we obtain the desired result (1.1.23). □

- S is Lorentz-invariant.

Proof. The Lagrangian density (a Lorentz-invariant quantity) can be split as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{1.1.25}$$

If $\mathcal{L}_I$ contains no time derivatives, then

$$H_I = -\int d^3x \mathcal{L}_I \tag{1.1.26}$$

Therefore, using (1.1.21),

$$S = T\left[e^{-i\int_{-\infty}^{\infty} dt H_I(t)}\right] = T\left[e^{i\int d^4x \mathcal{L}_I}\right] \tag{1.1.27}$$

which is manifestly Lorentz-invariant.

The integral is over the entire space-time. It should be noted that time-ordering does not spoil Lorentz invariance. Indeed, time-ordering is frame-dependent only for spacelike separations, and fields commute at spacelike separations (causality). □

- *Corollary:* By Lorentz invariance, it follows from (1.1.23) that S commutes with the *unperturbed* four-momentum,

$$[S\ ,\ P_0^\mu] = 0 \tag{1.1.28}$$

This implies momentum conservation in scattering.

Indeed, consider an incoming (outgoing) state of free particles with momenta $p_1, p_2, \dots$ ($p_1', p_2', \dots$), i.e.,

$$|\text{in,0}\rangle = |\vec{p}_1, \vec{p}_2, \dots\rangle \quad , \quad |\text{out,0}\rangle = |\vec{p}_1', \vec{p}_2', \dots\rangle \tag{1.1.29}$$

(notation as in eqs. (1.1.7) and (1.1.8)). These are eigenstates of *unperturbed* momentum

$$P_0^\mu|\text{in,0}\rangle = (p_1^\mu + p_2^\mu + \dots)|\text{in,0}\rangle \quad , \quad P_0^\mu|\text{out,0}\rangle = (p_1'^\mu + p_2'^\mu + \dots)|\text{out,0}\rangle \tag{1.1.30}$$

It follows that

$$0 = \langle\text{out,0}|[S\ ,\ P_0^\mu]|\text{in,0}\rangle = \left(\sum p_i^\mu - \sum p_i'^\mu\right)\langle\text{out}|\text{in}\rangle$$

and so

$$\sum p_i^\mu = \sum p_i'^\mu \tag{1.1.31}$$

i.e., momentum is conserved.

1.2 Wick's theorem

Notice that in the expression for the S-matrix (1.1.27), time evolution is accomplished with the *unperturbed* Hamiltonian H_0, and not with H. Thus the fields in $\mathcal{L}_I$ are asymptotic (free) fields, and can be treated in the same way we have already discussed. Let us concentrate on the scalar field ϕ.
The field ϕ in S does *not* satisfy the full equation of motion, which is usually non-linear. Instead, it obeys the Klein-Gordon equation. To distinguish it from the true ϕ, we shall call it ϕ_I (in *interaction picture*). Since we shall be always talking about ϕ_I, we might as well drop the subscript I. Hopefully the reader will not get too confused.
Therefore, we may perform the familiar expansion

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left[a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x} \right] \tag{1.2.1}$$

and define the one-particle state

$$|\vec{p}\rangle = Ca^\dagger(\vec{k})|0\rangle \tag{1.2.2}$$

where $|0\rangle$ is the asymptotic vacuum (not the true vacuum of the whole system).
When calculating vacuum expectation values $\langle 0|\phi(x)\phi(y)\cdots|0\rangle$, it is convenient to normal-order. On the other hand, S contains time-ordered products. We need to relate the two types of orderings. To this end, define the contraction of two fields as the difference between the two types of ordering,

$$\overline{\phi(x)\phi}(y) = T(\phi(x)\phi(y)) - :\phi(x)\phi(y): \tag{1.2.3}$$

The contraction is a number (not an operator), because each time we commute a and $a^\dagger$, we get a number. To calculate it, we shall take the vacuum expectation value of both sides of (1.2.3). We obtain

$$\overline{\phi(x)\phi}(y) = \langle 0|\overline{\phi(x)\phi}(y)|0\rangle = \langle 0|T(\phi(x)\phi(y))|0\rangle - \langle 0| :\phi(x)\phi(y): |0\rangle$$

The time-ordered product is the Feynman propagator,

$$D_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \tag{1.2.4}$$

The normal-ordered product contains terms of the form $a^\dagger a^\dagger$, $a^\dagger a$ and aa, all of which have vanishing vacuum expectation values. Therefore, the contraction is the Feynman propagator,

$$\overline{\phi(x)\phi}(y) = D_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \tag{1.2.5}$$

This result can be generalized to a product of n fields. To simplify the notation, denote

$$\phi_i = \phi(x_i) \ , \quad i = 1, \dots, n \tag{1.2.6}$$

Wick's theorem states that

$$\begin{aligned} T_n \equiv T(\phi_1 \cdots \phi_n) \quad = \quad & :\phi_1 \cdots \phi_n: \\ & + \overline{\phi_1\phi_2} :\phi_3 \dots \phi_n: + \text{all other terms with 1 contraction} \\ & + \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} :\phi_5 \dots \phi_n: + \text{all other terms with 2 contractions} \\ & + \dots \end{aligned} \tag{1.2.7}$$

For $n = 1$, Wick's theorem is the trivial statement

$$T_1 = T(\phi_1) = \phi_1 =: \phi_1 :$$

For $n = 2$, it was proved above.

For $n = 3$, we need to show

$$T_3 = T(\phi_1\phi_2\phi_3) =: \phi_1\phi_2\phi_3 : + \overbracket{\phi_1\phi_2}\phi_3 + \overbracket{\phi_1\phi_3}\phi_2 + \overbracket{\phi_2\phi_3}\phi_3 \tag{1.2.8}$$

To show this, first arrange the indices $1, 2, 3$ so that times are order as $x_1^0 \geq x_2^0 \geq x_3^0$. Then

$$T_3 = T(\phi_1\phi_2\phi_3) = \phi_1\phi_2\phi_3$$

First, let us try to normal-order ϕ_3. To this end, split

$$\phi_3 = \phi_{3+} + \phi_{3-} \tag{1.2.9}$$

where ϕ_{3+} (ϕ_{3-}) contains all the creation (annihilation) operators. Then

$$\begin{aligned} T_3 = \phi_1\phi_2\phi_3 &= \phi_1\phi_2\left(\phi_{3+} + \phi_{3-}\right) \\ &= \phi_1\phi_2\phi_{3-} + \phi_{3+}\phi_1\phi_2 + \left[\phi_1, \phi_{3+}\right]\phi_2 + \phi_1\left[\phi_2, \phi_{3+}\right] \end{aligned}$$

The commutators are numbers. We have

$$\begin{aligned} [\phi_1, \phi_{3+}] &= \langle 0|\left[\phi_1, \phi_{3+}\right]|0\rangle \\ &= \langle 0|\phi_1\phi_{3+}|0\rangle \qquad (\text{since } \langle 0|\phi_{3+} = 0) \\ &= \langle 0|\phi_1\phi_3|0\rangle \qquad (\text{since } \phi_{3-}|0\rangle = 0) \\ &= \langle 0|T(\phi_1\phi_3)|0\rangle \\ &= \overbracket{\phi_1\phi_2} \end{aligned}$$

and similarly for $[\phi_2, \phi_{3+}]$. We deduce

$$T_3 = \phi_1\phi_2\phi_{3-} + \phi_{3+}\phi_1\phi_2 + \overbracket{\phi_1\phi_3}\phi_2 + \overbracket{\phi_2\phi_3}\phi_1$$

Next, we normal order $\phi_1\phi_2$,

$$\phi_1\phi_2 =: \phi_1\phi_2 : + \overbracket{\phi_1\phi_2}$$

and use

$$: \phi_1\phi_2\phi_3 : = : \phi_1\phi_2 : \ \phi_{3-} + \phi_{3+} \ : \phi_1\phi_2 :$$

together with (1.2.9) to arrive at the desired result (1.2.8).
For $n = 4$, with $x_1^0 \geq x_2^0 \geq x_3^0 \geq x_4^0$, we have

$$\begin{aligned} T_4 = \phi_1 \cdots \phi_4 &= \phi_1\phi_2\phi_3\phi_{4-} + \phi_{4+}\phi_1\phi_2\phi_3 \\ &\quad + \left[\phi_1, \phi_{4+}\right]\phi_2\phi_3 + \phi_1\left[\phi_2, \phi_{4+}\right]\phi_3 + \phi_1\phi_2\left[\phi_3, \phi_{4+}\right] \\ &= T_3\phi_{4-} + \phi_{4+}T_3 + \overbracket{\phi_1\phi_4}\phi_2\phi_3 + \overbracket{\phi_2\phi_4}\phi_1\phi_3 + \overbracket{\phi_3\phi_4}\phi_1\phi_2 \end{aligned}$$

Now use our result for T_3,

$$\begin{aligned} T_4 &= \ : \phi_1\phi_2\phi_3 : \phi_{4-} + \phi_{4+} : \phi_1\phi_2\phi_3 : \\ &\quad + \overbracket{\phi_1\phi_2}(\phi_3\phi_{4-} + \phi_{4+}\phi_3) + \cdots + \ldots \\ &\quad + \overbracket{\phi_1\phi_4}\phi_2\phi_3 + \cdots + \ldots \\ &= \ : \phi_1\phi_2\phi_3\phi_4 : + \overbracket{\phi_1\phi_2} : \phi_3\phi_4 : + \cdots + \ldots \\ &\quad + \overbracket{\phi_1\phi_4}\phi_2\phi_3 + \cdots + \ldots \end{aligned}$$

where the dots represent terms that are obtained by permutations of the indices (123).

We still need to normal order the fields in the last line. Using our result for T_2, we arrive at the desired result for T_4,

$$\begin{aligned} T_4 &= :\phi_1\phi_2\phi_3\phi_4: + \overbrace{\phi_1\phi_2} :\phi_3\phi_4: + \cdots + \ldots \\ &+ \overbrace{\phi_1\phi_4} :\phi_2\phi_3: + \cdots + \ldots \\ &+ \overbrace{\phi_1\phi_4}\overbrace{\phi_2\phi_3} + \cdots + \ldots \end{aligned}$$

The above argument can be generalized to arbitrary n by induction (assume Wick's theorem holds for all $n < N$; show it holds for N).

EXAMPLE 1: **External source**

The simplest interaction we can have is of the form

$$H_I = \int d^3x \phi(x) J(x) \tag{1.2.10}$$

where $J(x)$ is a given function (source) representing some fixed distribution of "matter" [1]
The Klein-Gordon gets modified to

$$\partial_\mu \partial^\mu \phi + m^2 \phi = -J \tag{1.2.11}$$

which may be solved exactly with the aid of the Feynman propagator. We obtain

$$\phi(x) = \int d^4y D_F(x-y) J(y) \tag{1.2.12}$$

The S-matrix is

$$S = T\left[e^{-i\int d^4x \phi(x) J(x)}\right] = \sum_{n=0}^{\infty} S^{(n)} \tag{1.2.13}$$

where

$$S^{(n)} = \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n J(x_1) \cdots J(x_n) T(\phi(x_1) \cdots \phi(x_n))$$

It can also be written as

$$S^{(n)} = \frac{(-i)^n}{n!} T[S_1 \cdots S_n] \ , \quad S_k = \int d^4x_k J(x_k) \phi(x_k) \quad (k = 1, \ldots, n)$$

Of course, all S_k are the same operator, but before we integrate over time (the x_k^0s), we need to remember to time-order. Since S_k is linear in ϕ, Wick's theorem holds for the S_ks. This allows us to turn time-ordering to normal-ordering,

$$\begin{aligned} T(S_1 \cdots S_n) &= :S_1 \cdots S_n: \\ &+ \overbrace{S_1 S_2} :S_3 \ldots S_n: + \text{permutations} \\ &+ \overbrace{S_1 S_2}\overbrace{S_3 S_4} :S_5 \ldots S_n: + \text{permutations} \\ &+ \ldots \end{aligned} \tag{1.2.14}$$

Since all S_ks are the same object, all permutations give the same result; we just need to count them. With one contraction, the number of permutations is

$$\mathcal{N}_1 = \binom{n}{2} = \frac{n!}{2!(n-2)!}$$

[1] E.g., if you think of ϕ as the electromagnetic potential, J is the current that "creates" it. Of course the electromagnetic potential (photon) has two polarizations, so there are complications, but our discussion is still valid, as it captures the essence of the field.

With two contractions, the number of permutations is the number of ways of choosing two pairs from n objects,

$$\mathcal{N}_2 = \frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \frac{n!}{2^3(n-4)!}$$

where we divided by 2 because of over-counting (identical pairs).
Generalizing to p contractions, the number is

$$\mathcal{N}_p = \frac{1}{p!}\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-(2p-2)}{2} = \frac{n!}{p!2^p(n-2p)!}$$

The sum of terms with p contractions in (1.2.14) is

$$\mathcal{N}_p \overbrace{S_1 S_2}\overbrace{S_3 S_4}\cdots\overbrace{S_{2p-1} S_{2p}} : S_{2p+1}\cdots S_n := \mathcal{N}_p\,(\overbrace{S_1 S_2})^p : S_3^{n-2p} :$$

Its contribution to $S^{(n)}$ is

$$\frac{(-i)^n}{n!}\mathcal{N}_p\,(\overbrace{S_1 S_2})^p : S_3^{n-2p} : = \frac{(-i)^n}{p!2^p(n-2p)!}\,(\overbrace{S_1 S_2})^p : S_3^{n-2p} :$$

Now let us collect all the terms with p contractions in S. Such terms exist in all $S^{(n)}$ with $n \geq 2p$. We obtain the following contribution to S:

$$\begin{aligned} A_p &= \sum_{n=2p}^{\infty} \frac{(-i)^n}{p!2^p(n-2p)!}(\overbrace{S_1 S_2})^p : S_3^{n-2p} : \\ &= \sum_{n'=0}^{\infty} \frac{(-i)^{n'+2p}}{p!2^p(n')!}(\overbrace{S_1 S_2})^p : S_3^{n'} : \\ &= \frac{(-i)^{2p}}{p!2^p}(\overbrace{S_1 S_2})^p : e^{-iS_3} : \end{aligned}$$

Summing over p, we obtain a simple expression for the S-matrix,

$$S = \sum_0^{\infty} A_p = e^{-\frac{1}{2}(\overbrace{S_1 S_2})} : e^{-iS_3} : = e^{-\alpha/2} \;\; : e^{-i\int d^4x J(x)\phi(x)} : \tag{1.2.15}$$

where the coefficient $e^{-\alpha/2}$ is a c-number, and

$$\alpha = \int d^4x d^4y J(x) D_F(x-y) J(y) \tag{1.2.16}$$

We can now study the physical properties of the system.
Evidently, the source creates particles.[2] To see this, calculate the probability amplitude that the true vacuum $|0\rangle_{\text{in}}$ will evolve to an n-particle state $|\vec{p}_1 \dots \vec{p}_n\rangle_{\text{out}}$. We have

$$_{\text{out}}\langle \vec{p}_1 \dots \vec{p}_n | 0 \rangle_{\text{in}} = \langle \vec{p}_1 \dots \vec{p}_n | S | 0 \rangle \tag{1.2.17}$$

where the states on the right-hand side are free particle states.

- $n = 0$. We have

  $$_{\text{out}}\langle 0|0\rangle_{\text{in}} = \langle 0|S|0\rangle = e^{-\alpha/2}$$

  so the probability that no particle will be created is

  $$P(0 \to 0) = e^{-\beta} \quad , \quad \beta = \Re\alpha$$

[2]This is similar to the creation of photons (*bremsstrahlung radiation*) by an accelerating charged particle.

To show that this is an honest probability, i.e., $P(0 \to 0) \leq 1$, we need to show that $\beta \geq 0$. Using (1.2.4), we obtain

$$\alpha = \int \frac{d^4k}{(2\pi)^4} \widetilde{J}(k) \frac{i}{k^2 - m^2 + i\epsilon} \widetilde{J}(-k)$$

in terms of the Fourier transform of J,

$$J(x) = \int \frac{d^4k}{(2\pi)^4} \widetilde{J}(k) e^{-ik\cdot x}$$

Since $J(x)$ is real, we have $\widetilde{J}(-k) = \widetilde{J}^*(k)$, therefore

$$\beta = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} |\widetilde{J}(k)|^2 \left[\frac{i}{k^2 - m^2 + i\epsilon} - \frac{i}{k^2 - m^2 - i\epsilon} \right]$$

Using

$$\lim_{\epsilon \to 0^+} \left[\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right] = -2\pi i \delta(x)$$

we deduce

$$\beta = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} |\widetilde{J}(k)|^2 2\pi \delta(k^2 - m^2)$$

Integrating over k_0, we obtain two identical contributions at $k_0 = \pm\omega_k$, and so

$$\beta = \int \frac{d^3k}{(2\pi)^3 2\omega_k} |\widetilde{J}(k)|^2 \geq 0 \qquad (1.2.18)$$

- $n = 1$. In this case only the first-order term in the expansion of the normal-ordered exponential in (1.2.15) contributes. We obtain

$$_{\text{out}}\langle \vec{p}_1 | 0 \rangle_{\text{in}} = \langle \vec{p}_1 | S | 0 \rangle = e^{-\alpha/2} (-i) \int d^4x J(x) \langle \vec{p}_1 | \phi(x) | 0 \rangle$$

Using $\langle \vec{p}_1 | \phi(x) | 0 \rangle = e^{ip_1 \cdot x}$, which we showed a while ago, we deduce

$$_{\text{out}}\langle \vec{p}_1 | 0 \rangle_{\text{in}} = -ie^{-\alpha/2} \widetilde{J}(p_1)$$

The probability that a single particle will be created from the vacuum is

$$P(0 \to 1) = \int \frac{d^3p_1}{(2\pi)^3 2\omega_{p_1}} \left| _{\text{out}}\langle \vec{p}_1 | 0 \rangle_{\text{in}} \right|^2 = e^{-\beta} \int \frac{d^3p_1}{(2\pi)^3 2\omega_{p_1}} |\widetilde{J}(p_1)|^2 = \beta e^{-\beta} \qquad (1.2.19)$$

For a general (given) $n \geq 1$, the nth-order term in the expansion of the normal-ordered exponential in (1.2.15) contributes. We obtain

$$\begin{aligned} _{\text{out}}\langle \vec{p}_1 \cdots \vec{p}_n | 0 \rangle_{\text{in}} &= \langle \vec{p}_1 \cdots \vec{p}_n | S | 0 \rangle \\ &= e^{-\alpha/2} \frac{(-i)^n}{n!} \langle \vec{p}_1 \cdots \vec{p}_n | \ : S_1 \cdots S_n : \ | 0 \rangle \end{aligned}$$

Only the term with n creation operators in $: S_1 \cdots S_n :$ contributes. By commuting all creation operators through to the left, it is easy to see that

$$\langle \vec{p}_1 \cdots \vec{p}_n | \ : S_1 \cdots S_n : \ | 0 \rangle = n! \widetilde{J}(p_1) \cdots \widetilde{J}(p_n)$$

The probability that n particles will be created from the vacuum is

$$\begin{aligned} P(0 \to n) &= \frac{1}{n!} e^{-\beta} \int \frac{d^3p_1}{(2\pi)^3 2\omega_{p_1}} \cdots \frac{d^3p_n}{(2\pi)^3 2\omega_{p_n}} |\langle \vec{p}_1 \cdots \vec{p}_n | S | 0 \rangle|^2 \\ &= \frac{1}{n!} e^{-\beta} \int \frac{d^3p_1}{(2\pi)^3 2\omega_{p_1}} \cdots \frac{d^3p_n}{(2\pi)^3 2\omega_{p_n}} |\widetilde{J}(p_1) \cdots \widetilde{J}(p_n)|^2 \\ &= \frac{\beta^n}{n!} e^{-\beta} \end{aligned} \qquad (1.2.20)$$

Notice that we divided by $n!$ in the expression for the probability, because these are identical particles. Thus, we have a *Poisson distribution*. Notice that

$$\sum_{n=0}^{\infty} P(0 \to n) = 1$$

Probability is conserved and the S-matrix is unitary.
The expected number of emitted quanta (photons) is

$$\langle n \rangle = \sum_{n=0}^{\infty} n P(0 \to n) = \beta \tag{1.2.21}$$

If the source is localized ($J(x) \sim \delta^4(x)$), then the Fourier transform is $\widetilde{J}(k) = 1$, so $\beta \to \infty$. The divergence comes from large k (ultraviolet (UV) divergence). Thus a source can never be perfectly localized, and no experiment can measure $\phi(x)$ precisely, in accordance with Heisenberg's uncertainty principle.

EXAMPLE 2: **Static external source**

As a special case, consider a static source. Well, it will not be completely static, because it has to be switched on at a certain time and then switched off again, after a long time. So let

$$J(x) = f(t)\rho(\vec{x}) \tag{1.2.22}$$

where $\rho(\vec{x})$ represents the static ("charge") distribution, and $f(t)$ is a smooth function of time of compact support, so that $f(t) = 0$ for $t > |T|$, say, and $f(t) = 1$ for $t < |T|$, with a smooth transition from 1 to 0 near $t = \pm T$.
We shall assume that interactions are turned on and off adiabatically. If the cutoff time T is very large, then the Fourier transform of $f(t)$,

$$\widetilde{f}(\omega) = \int dt e^{i\omega t} f(t)$$

will have narrow support (Riemann-Lebesgue Lemma). Without the smooth transition at $t = \pm T$, we have

$$\widetilde{f}(\omega) = \int_{-T}^{T} dt e^{i\omega t} = \frac{2\sin\omega T}{\omega}$$

It does not have compact support, but its width is $\Delta\omega \sim 1/T$ (i.e., $\Delta\omega\Delta t \sim 1$). As $T \to \infty$, $\widetilde{f}(\omega) \to 2\pi\delta(\omega)$, which has compact support (the point $\omega = 0$), and the frequency can be measured with infinite accuracy.
If we smoothen out the above $f(t)$, then it will have compact support of width $\sim \Delta\omega$. Let us arrange things (choose a large enough cutoff time T) so that $\widetilde{f}(\omega)$ has support within the interval $|\omega| < m$, i.e.,

$$\widetilde{f}(\omega) = 0 \ , \quad \omega \geq m \tag{1.2.23}$$

To find the probability of particle production (radiation), notice that

$$\widetilde{J}(k) = \int d^4x J(x) e^{ik\cdot x} = \widetilde{f}(\omega)\widetilde{\rho}(\vec{k})$$

In the expression for β (1.2.18), $\omega = \omega_k = \sqrt{\vec{k}^2 + m^2} \geq m$, therefore, ω is outside the support of $\widetilde{f}$, by our assumption (1.2.23), and $\widetilde{f}(\omega_k) = 0$. It follows that $\beta = 0$, and no particles can be created. This makes physical sense: no radiation is emitted by static charges.
Thus the S-matrix is pretty boring, but we may still talk about the vacuum-to-vacuum transition amplitude

$$\langle 0|S|0\rangle = e^{-\alpha/2}$$

where α is purely imaginary (we just showed that its real part $\beta = 0$). The probability is $|e^{-\alpha}| = 1$, as expected, since this is the only possible process, but what is the physical meaning of the amplitude? To answer this question, use (1.1.6) and (1.1.11) to write the amplitude as

$$e^{-\alpha/2} = \lim_{t_\pm \to \pm\infty} \langle 0|U(t_+)U^\dagger(t_-)|0\rangle = \lim_{t_\pm \to \pm\infty} \langle 0|e^{-iH(t_+-t_-)}|0\rangle$$

where we used $H_0|0\rangle = 0$.
For the true vacuum of our system, we have

$$H|0\rangle_{phys} = E_0|0\rangle_{phys}$$

where E_0 is the true ground state energy.
From the adiabatic theorem in quantum mechanics, we know that if interactions are turned on very slowly, the state $|0\rangle$ (which is the ground state in the absence of interactions) evolves into the vacuum state $|0\rangle_{phys}$ without getting excited. It follows that

$$e^{-\alpha/2} = {}_{phys}\langle 0|e^{-iH(t_+-t_-)}|0\rangle_{phys}\Big|_{t_\pm = \pm T} = e^{-2iE_0 T}$$

This is true for large cutoff time T, more precisely, as $T \to \infty$. We deduce

$$E_0 = -\frac{i}{4} \lim_{T\to\infty} \frac{\alpha}{T} \tag{1.2.24}$$

Thus the S-matrix contains information about the ground state energy of the system.
For an explicit expression, use (1.2.16) together with (1.2.4). The integrals over times x^0 and y^0 are

$$\int_{-T}^{T} dx^0 \int_{-T}^{T} dy^0 e^{-ik_0(x^0-y^0)}$$

Evidently, in the limit $T \to \infty$, this expression has support consisting of the single point $k_0 = 0$. It follows that for large T, it is approximately

$$\int_{-T}^{T} dx^0 \int_{-T}^{T} dy^0 e^{-ik_0(x^0-y^0)} \approx 2\pi C\delta(k_0)$$

The constant C is found by integrating both sides over k_0. We obtain

$$C = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_{-T}^{T} dx^0 \int_{-T}^{T} dy^0 e^{-ik_0(x^0-y^0)} = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left(\frac{2\sin k_0 T}{k_0}\right)^2 = 2T$$

It follows that for large T,

$$\alpha \approx 2iT \int d^3x d^3y \rho(\vec{x}) V(\vec{x}-\vec{y}) \rho(\vec{y})$$

and

$$E_0 = \frac{1}{2} \int d^3x d^3y \rho(\vec{x}) V(\vec{x}-\vec{y}) \rho(\vec{y}) \tag{1.2.25}$$

which is the energy of two static ("charge") distributions interacting via the potential

$$V(\vec{x}) = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 + m^2} = -\frac{e^{-mr}}{4\pi r} \tag{1.2.26}$$

This is a Yukawa potential. As $m \to 0$, it turns into the Coulomb potential.

EXAMPLE 3: **A non-linear quantum field theory**

Consider three *distinct* Klein Gordon fields ϕ_1 ($i = 1, 2, 3$) with interaction Lagrangian density

$$\mathcal{L}_I = -\lambda\phi_1(x)\phi_2(x)\phi_3(x) \tag{1.2.27}$$

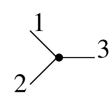
where λ is a coupling constant.
The S-matrix can be expanded in powers of λ,

$$S = T\left[e^{-i\int d^4x\lambda\phi_1\phi_2\phi_3}\right] = \sum_{n=0}^{\infty} S^{(n)} \tag{1.2.28}$$

The nth term is proportional to λ^n, and

$$S^{(0)} = 1 \ , \quad S^{(1)} = -i\lambda\int d^4x\phi_1\phi_2\phi_3 \ \text{(no need to normal} - \text{order)} \ , \quad \dots \tag{1.2.29}$$

We shall represent $S^{(1)}$ by a diagram thusly:



Each line is a field. Lines that meet at a vertex have common argument which is integrated over and multiplied by $-i\lambda$.
$S^{(1)}$ contributes to the process

$$1(p_1) \to 2(p_2) + 3(p_3) \tag{1.2.30}$$

where p_i^μ is the momentum of the particle of type i ($i = 1, 2, 3$).
The probability amplitude for this process is

$$_{out}\langle 2, p_2; 3, p_3|1, p_1\rangle_{in} = \langle 2, p_2; 3, p_3|S|1, p_1\rangle = \langle 2, p_2; 3, p_3|S^{(1)}|1, p_1\rangle + \dots \tag{1.2.31}$$

where the dots represent contributions of higher order in λ.
Using $\langle 0|\phi(x)|p\rangle = e^{-ip\cdot x}$ thrice, we obtain

$$\langle 2, p_2; 3, p_3|S^{(1)}|1, p_1\rangle = -i\lambda\int d^4x e^{-ip_1\cdot x}e^{ip_2\cdot x}e^{ip_3\cdot x} = -i\lambda(2\pi)^4\delta^4(p_1 - p_2 - p_3)$$

Therefore momentum is conserved, as expected (see discussion following eq. (1.1.28)).
For the general process $|i\rangle \to |f\rangle$, we define the scattering amplitude $\mathcal{A}_{fi}$ by

$$\langle f|S|i\rangle - \langle f|i\rangle = (2\pi)^4\delta^4(p_f - p_i)i\mathcal{A}_{fi} \tag{1.2.32}$$

where p_i^μ (p_f^μ) is the total momentum of the initial (final) state $|i\rangle$ ($|f\rangle$). Notice that we subtracted the amplitude representing no scattering, $\langle f|i\rangle$.
Earlier, for $|i\rangle = |1, p_1\rangle$, $|f\rangle = |2, p_2; 3, p_3\rangle$, we found

$$i\mathcal{A}_{fi} = -i\lambda + \mathcal{O}(\lambda^2)$$

Also, momentum conservation implies that the decay (1.2.30) is only allowed (to all orders in λ!) if

$$m_1 \geq m_2 + m_3 \tag{1.2.33}$$

Proof. By momentum conservation and using $p_i^2 = m_i^2$ $(i = 1, 2, 3)$,

$$m_1^2 = m_2^2 + m_3^2 + 2p_2 \cdot p_3$$

Also

$$p_2 \cdot p_3 = E_2 E_3 - \vec{p}_2 \cdot \vec{p}_3 \geq E_2 E_3 - |\vec{p}_2||\vec{p}_3|$$

and

$$\begin{aligned} E_2 E_3 &= \sqrt{(m_2 + \vec{p}_2^2)(m_3^2 + \vec{p}_3^2)} \\ &= \sqrt{m_2^2 m_3^2 + \vec{p}_2^2 \vec{p}_3^2 + m_2^2 \vec{p}_3^2 + m_3^2 \vec{p}_2^2} \\ &\geq \sqrt{m_2^2 m_3^2 + \vec{p}_2^2 \vec{p}_3^2 + 2m_2 m_3 |\vec{p}_2||\vec{p}_3|} \\ &= m_2 m_3 + |\vec{p}_2||\vec{p}_3| \end{aligned}$$

Therefore

$$p_2 \cdot p_3 \geq E_2 E_3 - |\vec{p}_2||\vec{p}_3| \geq m_2 m_3$$

and

$$m_1 = \sqrt{m_2^2 + m_3^2 + 2p_2 \cdot p_3} \geq \sqrt{m_2^2 + m_3^2 + 2m_2 m_3} = m_2 + m_3$$

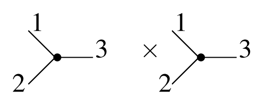
□

At $\mathcal{O}(\lambda^2)$, we have

$$S^{(2)} = \frac{(-i\lambda)^2}{2} \int d^4x d^4y T \left[\phi_1(x)\phi_2(x)\phi_3(x)\phi_1(y)\phi_2(y)\phi_3(y)\right] \tag{1.2.34}$$

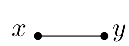
We need to re-express this in terms of normal-ordered products. This introduces contractions in the above expression. Here are the various possibilities.

- 0 contractions

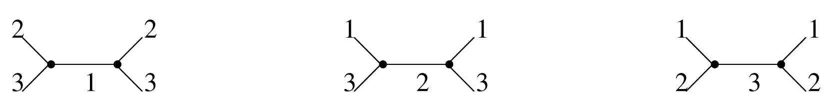


Same rules as before, but be sure to normal order! Topologically, this is a disconnected diagram. If we calculate a matrix element of S, we will get 2 δ-functions conserving momentum (of 6 particles).

- 1 contraction. Denote the contraction (propagator) $\overbrace{\phi(x)\phi(y)}$ by



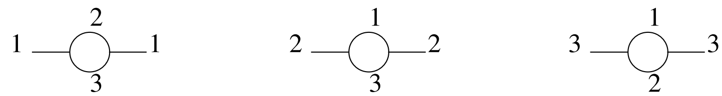
There are 3 possible contractions, each joining the two vertices,



An intermediate line labeled i denotes the contraction $\overbrace{\phi_i(x)\phi_i(y)}$ $(i = 1, 2, 3)$.

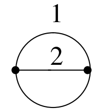
These are *not* Feynman diagrams (yet).

- 2 contractions. There are 3 possibilities.

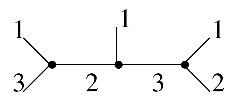


The first diagram is proportional to $\phi_2(x)\phi_2(y) \times \phi_3(x)\phi_3(y)$, etc.

- <u>3 contractions.</u> Only 1 possibility.



<u>SYMMETRY FACTORS</u>
In general, if a diagram has a *symmetry* (a permutation of its vertices that does *not* give rise to a new contraction), we need to divide by the corresponding symmetry factor.
All of the diagrams above have a symmetry: just interchange the two vertices. This explains the factor of $\frac{1}{2}$ in the expression (1.2.34) for $S^{(2)}$.
At higher orders, this symmetry need not be present. For example, consider the contribution to $S^{(3)}$,



The vertices are distinguishable, so there is no symmetry. The symmetry factor we ought to divide by is 1.
To see this in a little more detail, recall that there is a factor of $\frac{1}{3!}$ in the definition of $S^{(3)}$. On the other hand, there are 3! ways of doing the two contractions, here is one:

$$(\phi_1\phi_2\phi_3)(\phi_1\phi_2\phi_3)(\phi_1\phi_2\phi_3)$$

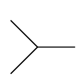
Thus, we obtain a factor of $\frac{1}{3!} \times 3! = 1$.

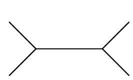
EXAMPLE 3: **A non-linear quantum field theory of a single field**

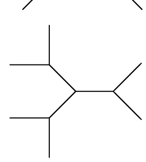
Suppose the 3 fields are identical, $\phi_1 = \phi_2 = \phi_3 = \phi$, and let the interaction Lagrangian density be

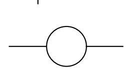
$$\mathcal{L}_I = -\frac{1}{3!}\lambda\phi^3 \tag{1.2.35}$$

where we re-defined the coupling constant λ in order to expose a factor of $\frac{1}{3!}$ for convenience.
This time, the symmetry factor is the number of permutations of vertices times the number of permutations of lines.

-  has symmetry factor 3! (number of permutations of (123)).

-  has symmetry factor 2 (vertices) $\times$ 2^2 (two pairs of external lines) $= 8$.

-  has symmetry factor 3! (vertices) $\times$ 2^3 (three pairs of external lines) $= 48$.

-  has symmetry factor 2 (vertices) $\times$ 2 (internal lines) $= 4$.

A general graph is disconnected and consists of lower-order connected graphs. Let $S^{(graph)}$ be the contribution of an individual graph to the S-matrix. It can be written as

$$S^{(graph)} = \frac{1}{n_1!n_2!\cdots}\left(S_1^{(c)}\right)^{n_1}\left(S_2^{(c)}\right)^{n_2}\cdots \tag{1.2.36}$$

where $S_i^{(c)}$ $(i = 1, 2, \dots)$ is the contribution to the S-matrix of the corresponding connected sub-graph. Notice that we had to divide by symmetry factors $n_i!$ $(i = 1, 2, \dots)$, because we have n_i identical connected sub-graphs. Be reminded that sub-graphs represent operators which do not necessarily commute with each other. However the product is normal-ordered, so the ordering of the subgraphs is immaterial. Also note that the above expression includes connected graphs, if we set $n_1 = 1$, $n_i = 0$ $(i \geq 2)$.
The entire S-matrix is the sum over all possibilities,

$$S = \sum S^{(graph)} = \sum_{n_1,n_2,\dots} \frac{1}{n_1!n_2!\cdots}\left(S_1^{(c)}\right)^{n_1}\left(S_2^{(c)}\right)^{n_2}\cdots \tag{1.2.37}$$

This looks hideous, but can, in fact, be neatly organized, because the sums over $n_1, n_2, \dots,$ are independent of each other. Thus S can be written as an infinite product,

$$S = \left(\sum_{n_1=0}^{\infty} \frac{1}{n_1!}\left(S_1^{(c)}\right)^{n_1}\right)\left(\sum_{n_2=0}^{\infty} \frac{1}{n_2!}\left(S_2^{(c)}\right)^{n_2}\right)\cdots = e^{S_1^{(c)}}e^{S_2^{(c)}}\cdots = e^{S^{(c)}} \tag{1.2.38}$$

where $S^{(c)}$ is the *connected* part of the S-matrix,

$$S^{(c)} = S_1^{(c)} + S_2^{(c)} + \dots \tag{1.2.39}$$

consisting of connected graphs only.
EXAMPLE: Let us re-visit the system with the external source (1.2.10) whose S-matrix we found exactly (eqs. (1.2.15) and (1.2.16)), albeit rather painfully.
There are only two connected graphs:

$S_1^{(c)}$ with rule: $-i\int d^4x J(x)\phi(x)$, and

$S_2^{(c)}$ one contraction.

The *connected* S-matrix is

$$S^{(c)} = S_1^{(c)} + \frac{1}{2}S_2^{(c)}$$

where I included the requisite symmetry factor.
The S-matrix is

$$S = e^{S^{(c)}} = e^{\frac{1}{2}S_2^{(c)}}e^{S_1^{(c)}}$$

The two factors can be separated because they commute with each other. The first factor is a number, in fact $S_2^{(c)} = (-i)^2\alpha = -\alpha$. The second factor is a (normal-ordered!) operator identical to the expression (1.2.15) derived earlier.

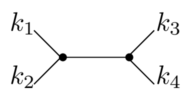
1.3 Feynman diagrams

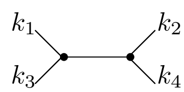
So far, we have been developing tools to calculate the S-matrix, which is an operator. Now, we shall apply these rules to actual scattering processes and find the corresponding rules due to *Feynman*.

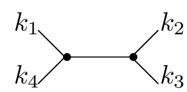
Continuing to work with the interaction Lagrangian density (1.2.35), which is physically useless, but good enough for the moment, let us consider the $2 \to 2$ interaction of particles with initial momenta k_1^μ, k_2^μ and final momenta k_3^μ, k_4^μ. The amplitude is

$$\langle k_3, k_4 | S | k_1, k_2 \rangle = \langle k_3, k_4 | S^{(2)} | k_1, k_2 \rangle + \dots \tag{1.3.1}$$

where the dots represent terms of higher order in λ. Thus, the lowest-order contribution is $\mathcal{O}(\lambda^2)$. Recall that there are various contributions, with 0, 1, 2, or 3 contractions. Since the process of interest involves 4 particles, only the part of $S^{(2)}$ with 4 external legs will contribute (the one with 1 contraction, of symmetry factor $2^3 = 8$). We need to match the momenta with these 4 external lines. There are 3 possibilities:

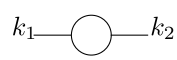






There are 8 ways to assign momenta in each of the 3 cases, so the overall symmetry factor of a diagram including momenta (i.e., a Feynman diagram) is $8 \times \frac{1}{8} = 1$. This symmetry factor may also be deduced by looking at the Feynman diagram itself: it is the number of permutations of internal lines *and* vertices (leaving external lines unchanged).

- EXAMPLE: The diagram contributing to $\langle k_1 | S | k_2 \rangle$ at $\mathcal{O}(\lambda^2)$



has symmetry factor 2 (for the internal lines). Recall that the corresponding S-matrix diagram has symmetry factor $2 \times 2 = 4$, since there are 2 ways of assigning momenta k_1 and k_2, we obtain $2 \times \frac{1}{4} = \frac{1}{2}$, which is the correct symmetry factor of the Feynman diagram.

Let us now compute the first diagram. It is

$$(-i\lambda)^2 \int d^4x d^4y \, \phi(x)\phi(y) \langle k_3, k_4 | : \phi^2(x)\phi^2(y) : | k_1, k_2 \rangle \tag{1.3.2}$$

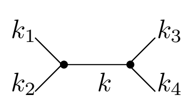
I omitted a symmetry factor anticipating that it will eventually be 1.
Using $\phi_-(x)|\vec{p}\rangle = e^{-ip\cdot x}|0\rangle$, the above expression becomes

$$(-i\lambda)^2 \int d^4x d^4y D_F(x-y) e^{ik_3\cdot x} e^{ik_4 \cdot x} e^{-ik_1\cdot y} e^{-ik_2 \cdot y} \tag{1.3.3}$$

Notice that we have a factor $e^{+ip\cdot x}$ ($e^{-ip\cdot x}$) for each outgoing (incoming) momentum.
Introducing the Fourier transform of the Feynman propagator, we obtain

$$(-i\lambda)^2 \int d^4x d^4y \frac{d^4k}{(2\pi)^4} e^{-k\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon} e^{ik_3\cdot x} e^{ik_4 \cdot x} e^{-ik_1\cdot y} e^{-ik_2 \cdot y} \tag{1.3.4}$$

which yields 2 δ-functions, $\delta^4(k_3 + k_4 - k)$ and $\delta^4(k - k_1 - k_2)$, allowing us to draw the diagram



where the intermediate line may be thought of as representing a particle of four-momentum k^μ. Momentum is conserved at each vertex, and consequently, overall,

$$k_1 + k_2 = k_3 + k_4 \tag{1.3.5}$$

as expected on *general* grounds.
The intermediate "particle" is not real. Indeed consider its decay to the two outgoing particles. For this to happen, we need $m_1 \geq m_2 + m_3$, i.e., $m \geq 2m$, which is not satisfied, unless $m = 0$. The latter is also impossible for *kinematical* reasons. We say that the intermediate particle is *"virtual"* and its existence is allowed by the Heisenberg Uncertainty Principle ($k^2 \neq m^2$ is OK in quantum mechanics).
After doing all the integrals, the diagram becomes

$$(-i\lambda)^2(2\pi)^4\delta^4(k_1 + k_2 - k_3 - k_4)\frac{i}{(k_1 + k_2)^2 - m^2 + i\epsilon} \tag{1.3.6}$$

The other 2 diagrams are similarly obtained. We deduce the scattering amplitude

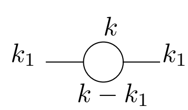
$$i\mathcal{A} = (-i\lambda)^2\left[\frac{i}{(k_1 + k_2)^2 - m^2 + i\epsilon} + \frac{i}{(k_1 - k_3)^2 - m^2 + i\epsilon}\frac{i}{(k_1 - k_4)^2 - m^2 + i\epsilon}\right] \tag{1.3.7}$$

Notice that it is a Lorentz-invariant expression, as it ought to be.
Generalizing to more complex diagrams is straightforward. One obtains the following general rules.

FEYNMAN RULES (in momentum space)

- To each external line of a diagram assign an incoming or outgoing momentum.
- To each internal line assign the most general momentum consistent with momentum conservation at each vertex.
- Include a factor $(-i\lambda)$ for each vertex.
- Include a factor $\frac{i}{k^2 - m^2 + i\epsilon}$ for each internal line of momentum k^μ.
- Integrate with $\int \frac{d^4k}{(2\pi)^4}$ for each undetermined momentum.[3]
- Divide by the appropriate symmetry factor.

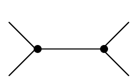
These rules yield the scattering amplitude $i\mathcal{A}$ after removing the overall factor $(2\pi)^4\delta^4(k_f - k_i)$.
EXAMPLE: Consider the propagation of a single particle of momentum k_1^μ. At $\mathcal{O}(\lambda^2)$, the diagram that contributes is



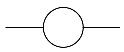
There is 1 arbitrary momentum, and the symmetry factor is $\frac{1}{2}$, so

$$i\mathcal{A} = \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}\frac{i}{(k - k_1)^2 - m^2 + i\epsilon} \tag{1.3.8}$$

We managed to derive an expression for this amplitude without much effort, but alas, it is an infinite integral. We will come back to this later.
GRAPH TOPOLOGY: The topology of a graph is related to the number of undetermined momenta, so it is important when one tries to determine the complexity of the integral(s), as well as the order in the quantum mechanical expansion of matrix elements of the S-matrix.
EXAMPLES

-  has no undetermined momenta.

[3] Notice that internal ("virtual") particles are not on the mass shell ($k^2 \neq m^2$), so we integrate over all four-momenta.

-  has one undetermined momentum.

In general, consider a connected graph with V vertices and I internal lines.

- Each vertex gives a δ-function conserving momentum. One of them is the overall factor $(2\pi)^4\delta^4(k_f - k_i)$ (for *connected graphs*), so we have $V - 1$ δ-functions constraining momenta.

- Each internal line provides an arbitrary momentum. Therefore, we have L integrals to do, where
$$L = I - (V - 1) = I - V + 1 \tag{1.3.9}$$

The following theorem, relating the integrals to the topology of the graph, holds.

Theorem. *L is the number of loops in the graph.*

Proof. The proof is by induction on L. When $L = 0$, we have a tree graph and $I = V - 1$ (because each time you add a vertex, you add an internal line, so $V - I$ =const. in a tree; if $I = 0$, then obviously $V = 1$, therefore $V - I = 1$).
To add a loop, join two external lines. This increases $I \to I + 1$ and V is unchanged. So if we start with $I - V = L - 1$, we end up with $(I + 1) - V = L = (L + 1) - 1$, which completes the inductive step. □

PHYSICAL CONSIDERATIONS. Let us go back to the $2 \to 2$ scattering process. In the center of mass frame,
$$\vec{k}_1 = -\vec{k}_2 \ , \quad \vec{k}_3 = -\vec{k}_4 \ , \quad E_1 + E_2 = E_3 + E_4$$
therefore,
$$E_1 = E_2 = E_3 = E_4 = E \ , \quad |\vec{k}_1| = |\vec{k}_2| = |\vec{k}_3| = |\vec{k}_4| = k$$
showing that we have an elastic collision. E is the *beam* energy in an accelerator. The total center-of-mass energy (a Lorentz-invariant quantity) is $E_{cm} = 2E$, because
$$E_{cm}^2 = (k_1 + k_2)^2 = (E_1 + E_2)^2 = 3E^2$$

It is also convenient to define the momentum transfers
$$\vec{q} = \vec{k}_1 - \vec{k}_3 \ , \quad \vec{q}' = \vec{k}_1 - \vec{k}_4$$

Their norms are Lorentz-invariant quantities, and found to be
$$\vec{q}^2 = (k_1 - k_3)^2 = 2k^2(1 + \cos\theta) \ , \quad \vec{q}'^2 = (k_1 - k_4)^2 = 2k^2(1 - \cos\theta)$$

where θ is the angle between the vectors $\vec{k}_1$ and $\vec{k}_3$.
The scattering amplitude (1.3.7) reads
$$\mathcal{A} = -\frac{\lambda^2}{E_{cm}^2 - m^2} + \frac{\lambda^2}{\vec{q}^2 + m^2} + \frac{\lambda^2}{\vec{q}'^2 + m^2} \tag{1.3.10}$$

The second term can be easily understood. It is the Fourier transform of a Yukawa potential (eq. (1.2.26)). This matches the result from non-relativistic quantum mechanics (Born approximation) for scattering off of a potential,
$$\langle \vec{k}_1 | V(\vec{x}) | \vec{k}_3 \rangle = \int d^3x e^{i\vec{q}\cdot\vec{x}} V(\vec{x})$$

The third term has a similar interpretation. Its presence is possible quantum mechanically because we have identical particles.
The first term has poles at $E_{cm} = \pm m$. It represents a *relativistic effect*. Note that the poles are at $E_{cm} = \pm mc^2$, so in the non-relativistic limit, $c \to \infty$, and the poles go away to infinity.

1.4 Reaction rates

To get in touch with reality and calculate quantities that an experimentalist can measure, we will need the relativistic analog of Fermi's Golden Rule. To this end, it is advantageous to put the world in a box, both in space and time (recall that we need to turn off interactions for $|t| \geq T$, where T is large; a time box is just as good and more convenient).
In one dimension, we have a basis $|k\rangle$ with $\int dk\, |k\rangle \langle k| = 1$ and $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{-ikx}$ so $\langle x|y\rangle = \int dk\, \langle x\,|k\rangle \langle k|\, y\rangle = \int \frac{dk}{2\pi} e^{-ik(x-y)} = \delta(x-y)$. We now demand $0 \leq x \leq L$ so we get discrete eigenstates $\varphi_n(x) = \frac{1}{\sqrt{L}} e^{i\frac{2\pi n}{L}x} \equiv \langle x|n\rangle$ with $\sum |n\rangle \langle n| = 1$. As $L \to \infty$, $\frac{2\pi n}{L} \to k$, and the difference in successive n's, $\frac{2\pi}{L} \to \Delta k \to dk$ and $\sum_n \to \frac{L}{2\pi} \int dk$. In three dimensions, we repeat these substitutions 3 times with $\mathbf{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$ to get $\varphi_{\mathbf{k}}(x) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{x}}$ and, as $L \to \infty$, $\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3} \int d^3k$.
To make everything concrete, we use a K-G field $\varphi(x) = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2k_0} \left(e^{-ik\cdot x} a(\mathbf{k}) - e^{ik\cdot x} a^\dagger(\mathbf{k})\right)$. $\overbrace{\varphi(x)\varphi(y)} = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2k_0} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \to \int \frac{d^3k}{(2\pi)^3 2k_0} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$; $\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \delta_{\mathbf{k},\mathbf{k}'}$ and $\langle 0\,|\varphi(x)|\,\mathbf{k}\rangle = \frac{1}{\sqrt{2Vk_0}} e^{-ik\cdot x}$. We introduce a new Feynman rule: assign to each external leg a factor of $\frac{1}{\sqrt{2EV}}$. The transition amplitude is

$$\langle f\,|S-1|\,i\rangle = iA(2\pi)^4 \delta^4(p_f - p_i) \prod_j \frac{1}{\sqrt{2E_j V}}$$

so the transition probability is

$$\begin{aligned} |\langle f\,|S-1|\,i\rangle|^2 &= |A|^2((2\pi)^4\delta^4(p_f - p_i))^2 \prod_j \frac{1}{2E_j V} \\ &= |A|^2 VT(2\pi)^4\delta^4(p_f - p_i) \prod_j \frac{1}{2E_j V} \end{aligned}$$

where the square of the delta function is

$$\begin{aligned} \left((2\pi)^4\,\delta^4(k)\right)^2 &= (2\pi)^4\,\delta^4(k) \int d^4y e^{ik\cdot y} \\ &= (2\pi)^4\,\delta^4(k) \int d^4y \\ &= (2\pi)^4\,\delta^4(k) VT \end{aligned}$$

because $\delta^4(k)$ forces $k = 0$. Thus the probability per unit time (the decay rate), as $V \to \infty$, becomes

$$\sum_{j_{out}} |A|^2 V(2\pi)^4\delta^4(p_{out} - p_{in}) \prod_j \frac{1}{2E_j V} \to \int \prod_{j\,out} \frac{d^3p_{j_{out}}}{(2\pi)^3\, 2E_{j_{out}}} |A|^2 V(2\pi)^4\delta^4(p_{out} - p_{in}) \prod_{j\,in} \frac{1}{2E_{j_{in}} V}$$

because $\sum_k \to V \int \frac{d^3k}{(2\pi)^3}$ as $VT \to \infty$. All the details of the theory are contained in $|A|^2$.

Example 1. *One incoming particle p_{in} and two outgoing particles p_1 and p_2.*

The decay rate is

$$\begin{aligned} \Gamma &= \int \frac{d^3p_1}{(2\pi)^3\, 2E_1} \frac{d^3p_2}{(2\pi)^3\, 2E_2} |A|^2 V(2\pi)^4\delta^4(p_{in} - p_1 - p_2) \frac{1}{2E_{in} V} \\ &= \frac{1}{2m} \int \frac{d^3p_1}{(2\pi)^3\, 2E_1 2E_2} |A|^2 2\pi\delta(E_{in} - E_1 - E_2) \end{aligned}$$

In the rest frame of the incoming particle $p_{in} = (m, \mathbf{0})$, $p_1 = (E_1, \mathbf{p}_1)$, and $p_2 = (E_2, \mathbf{p}_2)$. Using spherical coordinates, $d^3p_1 = p_1^2 dp_1 d\Omega$ and writing $f(\mathbf{p}_1) = m - E_1 - E_2$, we have $f'(\mathbf{p}_1) = -\frac{dE_1}{dp_1} - \frac{dE_2}{dp_1} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1 \frac{E_1+E_2}{E_1E_2} = -p_1 \frac{m}{E_1E_2}$. Then

$$\begin{aligned}\Gamma &= \frac{1}{2m} \int \frac{p_1^2 dp_1 d\Omega}{(2\pi)^3 4E_1E_2} 2\pi |A|^2 \frac{E_1E_2}{mp_{1root}} \delta\left(p_1 - p_{1root}\right) \\ &= \frac{p_1}{32\pi^2 m^2} \int d\Omega |A|^2\end{aligned}$$

In an arbitrary frame

$$\Gamma_{arb} = \frac{m}{E_{in}} \Gamma = \frac{1}{\gamma} \Gamma$$

using $E = \frac{m}{\sqrt{1-v^2}} = \gamma m$. If the incoming particle has spin 0, A is independent of Ω so $\Gamma = \frac{p_1^2}{8\pi m^2} |A|^2$.

Example 2. *One incoming particle and three outgoing particles.*

Much the same reasoning applies. We have to include three more integrals for $\frac{d^3p_3}{(2\pi)^3 2E_3}$ and four more delta functions $\delta^4(p_{in} - p_1 - p_2 - p_3)$. In the rest frame of the incoming particle

$$\Gamma = \frac{1}{2m} \int \frac{d^3p_1 d^3p_2}{(2\pi)^6 2E_1 2E_2 2E_3} 2\pi \delta(m - E_1 - E_2 - E_3) |A|^2$$

Use spherical coordinates with z-axis parallel to p_1 and let θ_{12} be the angle between p_1 and p_2; $d^3p_2 = \mathbf{p}_2^2 dp_2 d\Omega_{12}$; $dr_{12} = d(\cos\theta_{12}) d\varphi_{12}$. We will eventually use spherical coordinates to represent p_1 so we can think of the momenta as positive magnitudes.

Now specialize to the case where all outgoing particles have mass 0. Note $\mathbf{p}_i^2 = E_i^2$. The argument of the delta function $m - E_1 - E_2 - E_3 = m - p_1 - p_2 - |\mathbf{p}_1 + \mathbf{p}_2| = m - p_1 - p_2 - \sqrt{p_1^2 + p_2^2 - 2p_1p_2\cos\theta_{12}} \equiv f(\cos\theta_{12})$; $f'(\cos\theta_{12}) = \frac{-p_1p_2}{\sqrt{p_1^2+p_2^2-2p_1p_2\cos\theta_{12}}} = -\frac{E_1E_2}{E_3}$.

$$\begin{aligned}\Gamma &= \frac{1}{2m} \int \frac{d^3p_1 \mathbf{p}_2^2 dp_2 d\varphi_{12}}{(2\pi)^5 8E_1E_2E_3} \frac{E_3}{E_1E_2} |A|^2 \\ &= \frac{1}{2m} \int \frac{d^3p_1 \mathbf{p}_2^2 dp_2 d\varphi_{12}}{(2\pi)^5 8E_1^2E_2^2} |A|^2 \\ &= \frac{1}{16(2\pi)^5 m} \int \frac{\mathbf{p}_1^2 dp_1 d\Omega dp_2}{E_1^2} |A|^2 \\ &= \frac{1}{16(2\pi)^5 m} \int dE_1 d\Omega dE_2 d\varphi_{12} |A|^2\end{aligned}$$

where we have used $\mathbf{p}_i^2 = E_i^2$ and converted p_1 to spherical coordinates: $d^3p_1 = \mathbf{p}_1^2 dp_1 d\Omega$. The delta function imposes a constraint: $m = E_1 + E_2 + E_3 = E_1 + E_2 + \sqrt{E_1^2 + E_2^2 - 2E_1E_2\cos\theta_{12}}$ so $m \geq E_1 + E_2$.

Furthermore $(m - E_1 - E_2)^2 = E_1^2 + E_2^2 - 2E_1E_2\cos\theta_{12}$ so $m^2 + 2E_1E_2 - 2mE_1 - 2mE_2 = -2E_1E_2\cos\theta_{12}$ which gives $m(m - 2E_1 - 2E_2) = -2E_1E_2(1 + \cos\theta_{12})$ which implies a negative left hand side, i.e. $E_1 + E_2 \geq \frac{m}{2}$. The limits on $\cos\theta_{12}$ also mean $m(m - 2E_1 - 2E_2) \geq -4E_1E_2$ which rearranges to $\left(\frac{m}{2} - E_1\right)\left(\frac{m}{2} - E_2\right) \geq 0$. Both terms negative contradicts $m \geq E_1 + E_2$ so E_1 and E_2 are less than or equal to $\frac{m}{2}$. Thus integration is over the upper triangle in the Dalitz plot below.

In the case where all three particles have mass, the area of integration becomes a first quadrant shape (its Dalitz plot).

Example 3. *Two in, two out.*

We have the formula for the transition probability. We now calculate the cross-section. Let $\mathbf{k}_2 = 0$ (rest frame of particle 2); The volume is area times length $V = AL$; $v = \frac{L}{t}$ so $flux = \frac{\#particles}{area \cdot time} = \frac{1}{At} = \frac{v}{AL} = \frac{v}{V}$. The cross-section is

$$\begin{aligned}
\sigma &= \frac{\Gamma}{flux} \\
&= \frac{V}{v}\Gamma \\
&= \frac{V}{v}\int \frac{d^3k_3}{(2\pi)^3\,2E_3}\frac{d^3k_4}{(2\pi)^3\,2E_4}|A|^2 V(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\frac{1}{2E_1V}\frac{1}{2E_2V} \\
&= \frac{1}{vE_1E_2}\int \frac{d^3k_3}{(2\pi)^3\,2E_3}\frac{d^3k_4}{(2\pi)^3\,2E_4}|A|^2(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\frac{1}{2}\frac{1}{2}
\end{aligned}$$

The integral is clearly Lorentz invariant as is the factor in front of it since the following is Lorentz invariant

$$\begin{aligned}
\sqrt{(k_1\cdot k_2)^2 - m_1^2m_2^2} &= \sqrt{(E_1m_2)^2 - m_1^2m_2^2} \\
&= m_2\sqrt{E_1^2 - m_1^2} \\
&= m_2E_1v \\
&= E_1E_2v
\end{aligned}$$

where the left hand side has been evaluated in the rest frame of particle 2. The third equality comes from $E^2 = \gamma^2m^2$ so $E^2(1-\mathbf{v}^2) = m^2$ which gives $E^2 - m^2 = E^2\mathbf{v}^2$ or $v = \frac{|\mathbf{k}|}{E}$.
We also have, in the center of mass frame, $-\mathbf{k}_2 = \mathbf{k}_1 = \mathbf{k}$ and $-\mathbf{k}_4 = \mathbf{k}_3 = \mathbf{k}'$,

$$\begin{aligned}
\sqrt{(k_1\cdot k_2)^2 - m_1^2m_2^2} &= \sqrt{(E_1E_2 + \mathbf{k}^2)^2 - m_1^2m_2^2} \\
&= \sqrt{E_1^2E_2^2 + \mathbf{k}^4 + 2E_1E_2\mathbf{k}^2 - m_1^2m_2^2} \\
&= \sqrt{(\mathbf{k}^2+m_1^2)(\mathbf{k}^2+m_2^2) + \mathbf{k}^4 + 2E_1E_2\mathbf{k}^2 - m_1^2m_2^2} \\
&= \sqrt{2\mathbf{k}^4 + (m_1^2+m_2^2)\mathbf{k}^2 + 2E_1E_2\mathbf{k}^2} \\
&= \sqrt{(2\mathbf{k}^2 + m_1^2 + m_2^2 + 2E_1E_2)\mathbf{k}^2} \\
&= \sqrt{(E_1^2 + E_2^2 + 2E_1E_2)\mathbf{k}^2} \\
&= (E_1+E_2)\,|\mathbf{k}| \\
&= E_{CM}\,|\mathbf{k}|
\end{aligned}$$

The relative velocity of the particles is $v = \frac{(E_1+E_2)}{E_1E_2}\,|\mathbf{k}| = \frac{|\mathbf{k}|}{E_1} + \frac{|\mathbf{k}|}{E_2} = |\mathbf{v}_1 - \mathbf{v}_2|$ so

$$\begin{aligned}
\sigma &= \frac{1}{4vE_1E_2}\int \frac{d^3k_3}{(2\pi)^3\,2E_3}\frac{d^3k_4}{(2\pi)^3\,2E_4}|A|^2(2\pi)^4\delta^4(k_1+k_2-k_3-k_4) \\
&= \frac{1}{4E_{CM}\,|\mathbf{k}|}\int \frac{d^3k_3}{(2\pi)^3\,2E_3}\frac{d^3k_4}{(2\pi)^3\,2E_4}|A|^2(2\pi)^4\delta^4(k_1+k_2-k_3-k_4) \\
&= \frac{1}{4E_{CM}\,|\mathbf{k}|}\int \frac{d^3k_3}{(2\pi)^3\,2E_3 2E_4}|A|^2 2\pi\delta(E_1+E_2-E_3-E_4) \\
&= \frac{1}{16\pi^2E_{CM}\,|\mathbf{k}|}\int \frac{d\Omega}{E_3E_4}k_3^2\frac{1}{|f'(k_3)|}\,|A|^2
\end{aligned}$$

where $f(k_3) = E_1 + E_2 - E_3 - E_4 = E_{CM} - \sqrt{m_3^2 + k_3^2} - \sqrt{m_4^2 + k_3^2}$ so $f'(k_3) = \frac{k_3}{E_3} + \frac{k_3}{E_4} = \frac{E_3+E_4}{E_3E_4}k_3 = \frac{E_{CM}}{E_3E_4}k_3$. Therefore

$$\sigma = \frac{k_3}{k}\frac{1}{64\pi^2 E_{CM}^2}\int d\Omega\,|A|^2$$

where A is theory dependent.
In φ^3 theory, we have Feynman diagrams
so $iA = (-i\lambda)^2\left(\frac{i}{(k_1+k_2)^2-m^2} + \frac{i}{(k_1-k_3)^2-m^2} + \frac{i}{(k_1-k_4)^2-m^2}\right) + ...$ Define the Mandelstam variables $s = (k_1+k_2)^2$, $t = (k_1-k_3)^2$, $u = (k_1-k_4)^2$. We refer to these diagrams by the dependence of the propagators: the first diagram is the s-channel, the second is the t-channel, and the third is the u-channel. Cyclic permutations of the k_i take s to t to u to s. Also we have crossing symmetry $iA(s,t,u) = iA(u,s,t)$.

$$\begin{aligned} s+t+u &= k_1^2 + k_2^2 + 2k_1\cdot k_2 + k_1^2 + k_3^2 - 2k_1\cdot k_3 + k_1^2 + k_4^2 - 2k_1\cdot k_4 \\ &= 6m^2 + 2k_1\cdot(k_2 - k_3 - k_4) \\ &= 6m^2 - 2k_1^2 \\ &= 4m^2 \end{aligned}$$

which implies

$$iA = (-i\lambda)^2\left(\frac{i}{s-m^2} + \frac{i}{t-m^2} + \frac{i}{u-m^2}\right)$$

Poles develop as $s,t,u \to m^2$ so we expect large cross sections at those values.
Now let all particles be incoming so $k_3 \to -k_3$ and $k_4 \to -k_4$ so their energies will be negative going back in time (antiparticles). We must change the field to a complex field: two particles (b, $b^\dagger$) and two antiparticles (c, $c^\dagger$). $1+2 \to \overline{3}+\overline{4}$, $1+3 \to \overline{2}+\overline{4}$, $1+4 \to \overline{2}+\overline{3}$ (different crossing symmetries). In the center of mass frame, particles 1 - 4 have 4-momentums $(E,\mathbf{k})$, $(E,-\mathbf{k})$, $(E,\mathbf{k}')$, $(E,-\mathbf{k}')$ respectively. $1+2\to\overline{3}+\overline{4}$: $s = (2E)^2 = E_{CM}^2 \geq 4m^2 \geq 0$; $t = -(\mathbf{k}-\mathbf{k}')^2 \leq 0$; $u = -(\mathbf{k}+\mathbf{k}')^2 \leq 0$. Permute to go to another channel $1+3\to\overline{2}+\overline{4}$: $u \geq 4m^2$, $s,t\leq 0$; permute again to yet another channel $1+4\to\overline{2}+\overline{3}$: $t\geq 4m^2$, $s,u\leq 0$.
If we reverse everything, there is no effect on A. In fact $A_{|i\rangle\to|f\rangle} = A_{|f\rangle\to|i\rangle}$ is a consequence of the CPT theorem. The CPT transformation U_{CPT} is not unitary but anti-unitary because time reversal is anti-unitary. $[U_{CPT},H] = [U_{CPT},H_0] = 0$. Recall $S = \lim_{t_\pm\to\pm\infty} U(t_+)U^\dagger(t_-)$ where $U(t) = e^{iH_0t}e^{-iHt}$. Now $U_{CPT}U(t)U_{CPT}^{-1} = U(-t)$ so a change of variables shows $U_{CPT}SU_{CPT}^{-1} = \lim_{t_\pm\to\pm\infty} U(-t_+)U^\dagger(-t_-) = S^\dagger$. Under the CPT transformation $(2\pi)^4\,\delta(p_f - p_i)\,iA_{|i\rangle\to|f\rangle} = \langle f\,|S-1|\,i\rangle$

$$\begin{aligned} \langle f\,|S-1|\,i\rangle &= \langle U_{CPT}f\,|U_{CPT}(S-1)|\,i\rangle^* \\ &= \left\langle U_{CPT}f\,\middle|U_{CPT}(S-1)U_{CPT}^{-1}\middle|\,U_{CPT}i\right\rangle^* \\ &= \left\langle \overline{f}\,\middle|S^\dagger - 1\middle|\,\overline{i}\right\rangle^* \\ &= \left\langle \overline{i}\,\middle|S-1\middle|\,\overline{f}\right\rangle \end{aligned}$$

which relates to $A_{|\overline{f}\rangle\to|\overline{i}\rangle}$.

1.5 Unitarity

S is unitary. Define $iT = S-1$ so $\langle f\,|T|\,i\rangle = -i\,\langle f\,|S-1|\,i\rangle = (2\pi)^4\,\delta^4(p_f-p_i)\,A_{fi}$. Then $1 = (1+iT)^\dagger(1+iT) = 1 - iT^\dagger + iT + T^\dagger T$ so $T - T^\dagger = iT^\dagger T$.

$$\langle f\,|T|\,i\rangle - \left\langle f\,\middle|T^\dagger\middle|\,i\right\rangle = i\left\langle f\,\middle|T^\dagger T\middle|\,i\right\rangle = i\sum_n \left\langle f\,\middle|T^\dagger\middle|\,n\right\rangle\langle n\,|T|\,i\rangle$$

which translates to amplitudes

$$\left(A_{fi} - A^*_{if}\right)(2\pi)^4\,\delta^4\left(p_f - p_i\right) = i\sum_n (2\pi)^4\,\delta^4\left(p_f - p_n\right)A^*_{nf}\,(2\pi)^4\,\delta^4\left(p_n - p_i\right)A_{ni}$$

The second delta function means $p_n = p_i$ so

$$A_{fi} - A^*_{if} = i\sum_n (2\pi)^4\,\delta^4\left(p_i - p_n\right)A^*_{nf}A_{ni}$$

When $|f\rangle = |i\rangle$, we then obtain a form of the Optical Theorem

$$\mathrm{Im}\,A_{ii} = \frac{1}{2}\sum_n (2\pi)^4\,\delta^4\left(p_i - p_n\right)\left|A_{ni}\right|^2$$

Fock space gives a complete set of states $|k_1k_2...k_n\rangle = a^\dagger(k_1)a^\dagger(k_2)...a^\dagger(k_n)\,|0\rangle$, $n = 1,...\infty$ so $\sum_{n=1}^{\infty}\int\frac{d^3k_1}{(2\pi)^32E_1}\frac{d^3k_2}{(2\pi)^32E_2}\cdots\frac{d^3k_2}{(2\pi)^32E_2}\,|k_1k_2...k_n\rangle\,\langle k_1k_2...k_n|$ is the identity. If two identical particles scatter to two particles that are identical to the incoming particles, the first term of $\mathrm{Im}\,A_{ii}$ is $\frac{1}{2}\int\frac{d^3k_1}{(2\pi)^32E_1}(2\pi)^4\delta^4(p_i-k_1)$ (>- -<); the second term is $\frac{1}{2}\int\frac{d^3k_1}{(2\pi)^32E_1}\frac{d^3k_2}{(2\pi)^32E_2}(2\pi)^4\delta(p_i-k_1-k_2)$ (>= =<) and so on. This generalizes to the Optical Theorem

$$\mathrm{Im}\,A_{ii} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int\frac{d^3k_1}{(2\pi)^32E_1}\cdots\frac{d^3k_n}{(2\pi)^32E_n}\left|A_{ni}\right|^2\delta^4\left(p_i - k_1 - ... - k_n\right)$$

Example 4. *A single incoming Klein-Gordon particle of momentum p transitioning to itself.*

The only Lorentz invariant parameter is $p^2 \equiv s$ which should not be identified with the mass of the particle because intermediate particles may not be physical. Now A is a function of s: $\mathrm{Im}\,A(s) = \frac{1}{2}\int\frac{d^3k_1}{(2\pi)^32E_1}(2\pi)^4\,\delta^4\left(p-k_1\right)|A|^2+... = \frac{1}{2}2\pi\frac{\delta(E-E_1)}{2E_1}|A|^2$ which is Lorentz Invariant because the Lorentz invariant quantity $\delta\left(k_1^2-p^2\right) = \delta\left(E_1^2-\mathbf{k}_1^2-p_0^2+\mathbf{p}^2\right) = \delta\left(E_1^2-p_0^2\right) = \frac{1}{2E_1}\delta(E_1-p_0)$. Hence

$$\mathrm{Im}\,A(s) = \frac{1}{2}2\pi\frac{\delta(E-E_1)}{2E_1}|A|^2 = \pi\delta\left(k_1^2-p^2\right)|A|^2$$

to first order $(-A-)$. The higher order corrections come from diagrams $(-A=A-)$ which contributes a factor $\frac{i}{p^2-m^2+i\varepsilon}$ and $(-A=A^*-)$ which

Example 5. *One particle decays into two particles of the same species*

The decay rate calculated earlier is $\Gamma = \frac{1}{2m}\int\frac{d^3k_1}{(2\pi)^32E_1}\frac{d^3k_2}{(2\pi)^32E_2}(2\pi)^4\delta(p-k_1-k_2)\,|A|^2$. In the rest frame of the incoming particle $p = (\sqrt{s},\mathbf{0})$, $k_1 = (E',\mathbf{k})$, and $k_2 = (E',-\mathbf{k})$.

$$\Gamma = \frac{k}{32\pi^2\sqrt{s}}\int d\Omega\,|A|^2$$

which is Lorentz invariant if $\frac{k}{\sqrt{s}}$ is. $E'^2 = m^2+k^2 = \left(\frac{\sqrt{s}}{2}\right)^2$ by conservation of energy. Solve for k in terms of s and divide by $\sqrt{s}$ to get

$$\frac{2k}{2\sqrt{s}} = \sqrt{\frac{s-4m^2}{4s}}$$

which is Lorentz invariant. Then

$$m\Gamma = \frac{1}{128\pi^2}\sqrt{1-\frac{4m^2}{s}}\int d\Omega\,|A|^2$$

If $s < 4m^2$, decay is impossible (the initial energy is less than the sum of the masses of the decay products); $\Gamma = 0$ so $\mathrm{Im}\,A = 0$ and $A(s) = A^*(s^*)$.When two analytic function agree on an interval, the agree everywhere except singularities. so $\mathrm{Im}\,A = 0$ except at singularities. Graphically there is a pole at m^2 and a cut at parallel and slightly below the real axis for $s > 4m^2$. For $n = 3$, the pole is the same and there is still a single cut but for $s > 9m^2$. The physical content of the theory must approach from above.

1.6 Path Integrals

In one-dimensional non-relativistic quantum mechanics, we have position and momentum operators $\widehat{x}$ and $\widehat{p}$ with $[\widehat{x},\widehat{p}]=i$. In the Heisenberg picture, $\widehat{x}(t)=e^{iHt}\widehat{x}(0)e^{-iHt}$ and $\widehat{p}(t)=e^{iHt}\widehat{p}(0)e^{-iHt}$. The eigenstates of $\widehat{x}(t)$ ($\widehat{p}(t)$) are $|x,t\rangle = e^{i\widehat{H}t}|x,0\rangle$ ($|p,t\rangle = e^{i\widehat{H}t}|p,0\rangle$) and both collections form a complete set of states with $\langle x|x'\rangle=\delta(x-x')$, $\langle p|p'\rangle = 2\pi\delta(p-p')$, and $\langle x|p\rangle = e^{ixp}$. To compare momentum and position at different times $\langle p,t_1|x,t_2\rangle = \left\langle p,t_1\left|e^{iH(t_2-t_1)}\right|x,t_1\right\rangle \approx \left\langle p,t_1\left|1+i\widehat{H}\left(t_2-t_1\right)\right|x,t_1\right\rangle$ for small time intervals. We can regard H as a function of p and x and write

$$\begin{aligned}\langle p,t_1\left|1+iH\left(t_2-t_1\right)\right|x,t_1\rangle &= \left(1+iH(p,x)\left(t_2-t_1\right)\right)e^{-ixp}\\ &= e^{-iH(p,x)(t_2-t_1)}e^{-ixp}\end{aligned}$$

Look at the amplitude of transition from $|x_0,t_0\rangle$ to $|x',t'\rangle$

$$A=\langle x',t'|x_0,t_0\rangle \equiv \psi(x',t')$$

which is a solution to the Schrödinger equation; at $t'=t_0$, $\psi(x',t_0)=\langle x'|x_0\rangle=\delta(x'-x_0)$. At a time $t_1$between t_0 and t', insert the identity

$$\int\frac{dp'}{2\pi}\left|p',t'\right\rangle\left\langle p',t'\right|\int dx_1\left|x_1,t_1\right\rangle\left\langle x_1,t_1\right|\int\frac{dp_1}{2\pi}\left|p_1,t_1\right\rangle\left\langle p_1,t_1\right|$$

to get

$$\begin{aligned}A &= \int\frac{dp'}{2\pi}dx_1\frac{dp_1}{2\pi}\left\langle x',t'|p',t'\right\rangle\left\langle p',t'|x_1,t_1\right\rangle\left\langle x_1,t_1|p_1,t_1\right\rangle\left\langle p_1,t_1|x_0,t_0\right\rangle\\ &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}e^{ix'p}\left\langle p',t'|x_1,t_1\right\rangle e^{ix_1p_1}\left\langle p_1,t_1|x_0,t_0\right\rangle\\ &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}e^{ix'p'}e^{ix_1p_1}\left\langle p_1,t_1|x_0,t_0\right\rangle\left\langle p',t'|x_1,t_1\right\rangle\end{aligned}$$

but the remaining brackets are hard to evaluate since t_1 is not necessarily close to either t_0 or t'. We can iterate by choosing t_2 between t_1 and t' and inserting the identity

$$\int dx_2\left|x_2,t_2\right\rangle\left\langle x_2,t_2\right|\int\frac{dp_2}{2\pi}\left|p_2,t_2\right\rangle\left\langle p_2,t_2\right|$$

into the last bracket to get

$$\begin{aligned}A &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}\frac{dx_2dp_2}{2\pi}e^{ix'p'}e^{ix_1p_1}\left\langle p_1,t_1|x_0,t_0\right\rangle\left\langle p',t'|x_2,t_2\right\rangle\left\langle x_2,t_2|p_2,p_2\right\rangle\left\langle p_2,t_2|x_1,t_1\right\rangle\\ &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}\frac{dx_2dp_2}{2\pi}e^{ix'p'}e^{ix_1p_1}e^{ix_2p_2}\left\langle p_1,t_1|x_0,t_0\right\rangle\left\langle p_2,t_2|x_1,t_1\right\rangle\left\langle p',t'|x_2,t_2\right\rangle\end{aligned}$$

Now divide the interval $[t_0,t']$ into equally spaced intervals $[t_i,t_{i+1}]$ of length ε with $t_0<t_1<...$ $<t_n<t'$, then multiple iteration yields

$$\begin{aligned}A &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}...\frac{dx_ndp_n}{2\pi}e^{ix'p'}e^{ix_1p_1}..e^{ix_np_n}\left\langle p_1,t_1|x_0,t_0\right\rangle...\left\langle p_n,t_n|x_{n-1},t_{n-1}\right\rangle\left\langle p',t'|x_n,t_n\right\rangle\\ &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}...\frac{dx_ndp_n}{2\pi}e^{ix'p'}e^{ix_1p_1}..e^{ix_np_n}e^{-iH(p_1,x_0)(t_1-t_0)}e^{-ix_0p_1}...e^{-iH\left(p',x_n\right)\left(t'-t_n\right)}e^{-ix_np'}\\ &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}...\frac{dx_ndp_n}{2\pi}e^{i(x'-x_n)p'}e^{i(x_1-x_0)p_1}...e^{i(x_n-x_{n-1})p_n}e^{-i\varepsilon(H(p_1,x_0)+...+H(p_n,x_{n-1})+H(p',x_n))}\end{aligned}$$

where we have chosen n large enough to make ε small enough to apply the approximation of the first paragraph of this section. As $\varepsilon\to 0$, the choices of $x_i=x(t_i)$ ultimately give a path $x(t)$; similarly we get a path $p(t)$. We can write $H(p_i, x_{i-1}) = H(t_i) = H(t)$. $x_i - x_{i-1} = x(t_i) - x(t_{i-1}) = \varepsilon \dot{x}(t_i)$. Then

$$\begin{aligned}
A &= \int \frac{dp'}{2\pi}\frac{dx_1 dp_1}{2\pi}...\frac{dx_n dp_n}{2\pi} e^{i\varepsilon\dot{x}(t')p(t')} e^{i\varepsilon\dot{x}(t_1)p(t_1)}...e^{i\varepsilon\dot{x}(t_n)p(t_n)} e^{-i\varepsilon(H(t_1)+...+H(t_n)+H(t'))} \\
&= \int \frac{dp'}{2\pi}\frac{dx_1 dp_1}{2\pi}...\frac{dx_n dp_n}{2\pi} e^{i\varepsilon((p\dot{x}-H)(t_1)+...(p\dot{x}-H)(t_n))} \\
&\to e^{i\int dt(p\dot{x}-H)} \\
&= e^{i\int dtL} \\
&= e^{iS}
\end{aligned}$$

where $S = \int_{t_0}^{t'} dtL$ is the action. This a solution of the Schrödinger equation. Note that, in the limit $\varepsilon \to 0$, we are actually integrating over all possible paths.
If $H = \frac{p^2}{2m} + V(x)$ and ε has a small imaginary part

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dp_i}{2\pi} e^{i\varepsilon(p\dot{x}-H)} &= \int_{-\infty}^{\infty} \frac{dp_i}{2\pi} e^{i\varepsilon\left(p_i\dot{x}_i - \frac{p_i^2}{2m} - V(x)\right)} \\
&= \int_{-\infty}^{\infty} \frac{dp_i}{2\pi} e^{i\varepsilon\left(-\frac{(p_i - m\dot{x})^2}{2m} + \frac{m}{2}\dot{x}^2 - V(x)\right)} \\
&= e^{i\varepsilon\left(\frac{m}{2}\dot{x}^2 - V(x)\right)} \int_{-\infty}^{\infty} \frac{dp_i}{2\pi} e^{-i\varepsilon\frac{(p_i - m\dot{x})^2}{2m}} \\
&= e^{i\varepsilon L(x,\dot{x})} \sqrt{\frac{m}{2\pi i\varepsilon}}
\end{aligned}$$

If we define $[dx] = \lim_{\varepsilon\to 0} \prod \frac{dx_i}{\sqrt{2\pi i\varepsilon/m}}$, we have

$$\langle x', t' | x_0, t_0 \rangle = \int [dx]\, e^{i\int dtL(x,\dot{x})}$$

To look at the classical limit, divide i by $\hbar$ and let $\hbar \to 0$ so $\frac{i}{\hbar}\int dtL \to \infty$. If we look at S and $S + \delta S$, the difference will be very large (complete destructive interference) unless $\delta S = 0$ which implies the classical equations of motion.
Green function
To look at the Green function $\langle x', t' | x(t_1)x(t_2) | x_0, t_0 \rangle$ with $t_0 < t_2 < t_1 < t'$, insert the identity $\int dx_1 |x_1, t_1\rangle \langle x_1, t_1| \int dx_2 |x_2, t_2\rangle \langle x_2, t_2|$ to get

$$\begin{aligned}
\langle x', t' | x(t_1)x(t_2) | x_0, t_0 \rangle &= \int dx_1 dx_2 \langle x', t' | x(t_1) | x_1, t_1 \rangle \langle x_1, t_1 | x_2, t_2 \rangle \langle x_2, t_2 | x(t_2) | x_0, t_0 \rangle \\
&= \int dx_1 dx_2 x_1 x_2 \langle x', t' | x_1, t_1 \rangle \langle x_1, t_1 | x_2, t_2 \rangle \langle x_2, t_2 | x_0, t_0 \rangle \\
&= \int dx_1 dx_2 x_1 x_2 \int_{x_1,t_1\to x',t'} [dx]\, e^{iS} \int_{x_2,t_2\to x_1,t_1} [dx]\, e^{iS} \int_{x_0,t_0\to x_2,t_2} [dx]\, e^{iS} \\
&= \int_{x_0,t_0\to x',t'} dx_1 dx_2 x(t_1) x(t_2) e^{iS}
\end{aligned}$$

Note that $x_1(t_1)$ and $x_2(t_2)$ are operators on the left hand side but are just numbers in the result; numbers commute but operators may not. This is not a contradiction since we started with a time ordered product. We generalize to

$$\langle x', t' | T(x(t_1)...x(t_2)) | x_0, t_0 \rangle = \int [dx]\, x(t_1)...x(t_n) e^{iS}$$

(operators on the left, numbers on the right). This is all non-relativistic quantum mechanics, but it is easily brought into QFT. (x, t) is a 4-vector, φ is an operator and $|\varphi(x)\rangle$ denotes the eigenstate of φ.

$$\langle\varphi(x')|T(\varphi(x_1)...\varphi(x_n))|\varphi(x_0)\rangle = \int [d\varphi]\,\varphi(x_1)...\varphi(x_n)e^{iS}$$

is a non-perturbative way of calculating any Green function.

Example 6. *To compare to scattering results we have already obtained, let $t_0 \to -\infty$ and $t' \to \infty$. In any theory $\mathbf{1} = |\Omega\rangle\langle\Omega| + \int dE\,|E\rangle\langle E|$ where Ω is the vacuum.*

Let φ be a Klein-Gordon field. $H|0\rangle = 0$ and $H|k\rangle = \sqrt{k^2+m^2}\,|k\rangle$. The spectrum of H has one discrete energy (the vacuum) and all others are continuous starting at m.

$$\begin{aligned}|\varphi(x_0)\rangle &= \langle\Omega|\varphi(x_0)|\Omega\rangle + \int dE\,\langle E|\varphi(x_0)|E\rangle \\ &= e^{iE_0t_0}|\varphi(\mathbf{x}_0,0)\rangle + \int dEg(E)e^{iEt_0}\langle E|\varphi(\mathbf{x}_0,0)|E\rangle\end{aligned}$$

because $|\varphi(\mathbf{x}_0,t_0)\rangle = e^{iHt_0}|\varphi(\mathbf{x}_0,0)\rangle$ and Ω and E are eigenvalues of H. The factor $g(E)$ is the degeneracy at E. We have expressed $|\varphi(x_0)\rangle$ as a function $f(t_0)$; it may be impossible to calculate, but we only need its value as $t_0 \to -\infty$. $f(t_0)$ is the Fourier transform of the function $g(E)\langle E|\varphi(\mathbf{x}_0,0)|E\rangle$; the Riemann-Lesbegue lemma shows that the Fourier transform of a function approaches 0 as $t \to \pm\infty$. If E is large, the phase differences are large and, as $t_0 \to \pm\infty$, there is full destructive interference.

$$\begin{aligned}|\varphi(x_0,t_0)\rangle &\to e^{-iE_0t'}|\varphi(\mathbf{x},0)\rangle \\ \langle\varphi(x',t')| &\to e^{-iE_0t'}|\varphi(\mathbf{x}',0)\,|\Omega\rangle\langle\Omega| \\ \int [d\varphi]\,\varphi(x_1)...\varphi(x_n)e^{iS} &\to e^{iE_0(t_0-t')}\langle\varphi(\mathbf{x}',0)|\Omega\rangle\langle\Omega|\varphi(\mathbf{x}_0,0)\rangle\,\varphi(\mathbf{x}',0)\,\langle\Omega|T(\varphi(x_1)...\varphi(x_n))|\Omega\rangle\end{aligned}$$

The last bracket is definitely a physical object, the propagator for $n = 2$ or the scattering amplitude. Thus

$$\langle\Omega|T(\varphi(x_1)...\varphi(x_n))|\Omega\rangle = \frac{\int [d\varphi]\,\varphi(x_1)...\varphi(x_n)e^{iS}}{\int [d\varphi]\,e^{iS}}$$

In terms of diagrams, the components of $\int [d\varphi]\,e^{iS}$ have no external legs (the diagram is a bubble) because it is a vacuum to vacuum transition. Looking at $\int [d\varphi]\,\varphi(x_1)\varphi(x_2)e^{iS}$, the simplest graph is just a line segment, the K-G propagator. We have the connected graphs
The disconnected graphs contribute which can be written as (1+bubbles)(all graphs with no bubbles). When divided by (1+bubbles), we have the propagator. Bubbles represent vacuum to vacuum transitions, but physical processes involve excitations of the vacuum. Therefore bubbles do not contribute to physical reality although they presumably make up the cosmological constant.
Let $G^{(n)}(x_1,...,x_n) \equiv \langle\Omega|T(\varphi(x_1)...\varphi(x_n))|\Omega\rangle = \frac{\int[d\varphi]\varphi(x_1)...\varphi(x_n)e^{iS}}{\int[d\varphi]e^{iS}}$ (graphs without bubbles).
Add an external source J so $S \to S + \int J\varphi$. $\int J\varphi$ is shorthand for $\int d^4xJ(x)\varphi(x)$. Define

$$Z[J] = \frac{\int [d\varphi]\,e^{i(S+\int J\varphi)}}{\int [d\varphi]\,e^{iS}}$$

Note

$$e^{i\int J\varphi} = 1 + i\int J\varphi + \frac{i^2}{2!}\left(\int J\varphi\right)^2 + \frac{i^3}{3!}\left(\int J\varphi\right)^3 + ...$$

so

$$Z[J] = 1 + \frac{i\int [d\varphi] \int J\varphi e^{iS}}{\int [d\varphi]\, e^{iS}} + \frac{\frac{i^2}{2!}\int [d\varphi]\, e^{iS} \int J\varphi \int J\varphi}{\int [d\varphi]\, e^{iS}} + ...$$

Note that

$$\frac{\delta Z}{\delta J(x)}|_{J=0} = \frac{i\int [d\varphi]\, \varphi e^{iS}}{\int [d\varphi]\, e^{iS}} = G^{(1)}(x)$$

Similarly

$$\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}|_{J=0} = \frac{i\int [d\varphi]\, \varphi(x_1)\varphi(x_2) e^{iS}}{\int [d\varphi]\, e^{iS}} = G^{(2)}(x_1, x_2)$$

etc.