Quantum Field Theory II

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Contents

UNIT 1

Interactions

1.1 The interaction picture

Previously we have only considered systems that did not interact. They were described by a Hamiltonian $H = H_0$. We would now like to consider interactions and our Hamiltonian will take the from $H = H_0 + H_{int}$. The physics of what happens when particles interact can be very complicated. Many times we will not be able to not be able to describe the State space that belongs to H. We can however describe the State space belonging to H_0.

1.2 Scattering theory and the S-matrix

Consider a scattered particle. At a time in the infinite past (infinite past for a particle in a scattering experiment can be a few seconds ago, we just mean that it was long enough ago that the particle did not feel H_{int}) it was in a state $|in\rangle_0$, this state belongs to the state space of H_0. At some time in the infinite future it is in a state $|out\rangle_0$ that also belongs to the state space of H_0. Between these two states the particle occupies states that belong to the state space of H, which is complicated.
Let $|\Psi\rangle \in \mathbb{H}_0$, then we let $|in\rangle \in \mathbb{H}$ so that

$$\lim_{t\to-\infty} e^{-iHt}|in\rangle = e^{-iH_0t}|\Psi\rangle$$
$$|in\rangle = \lim_{t\to-\infty} e^{iHt}e^{-iH_0t}|\Psi\rangle$$

We define

$$U(t) = e^{-iHt}e^{iH_0t} \tag{1.2.1}$$

Since $[H, H_0] = 0$ we can write that $U(t) = e^{-iH_{int}t}$. We can similarly talk about the particle after it has been scattered

$$|out\rangle = \lim_{t\to+\infty} U^\dagger|\chi\rangle$$

Now we can talk about transistion probablilities. We relate the states that we know to the states that we don't know via S, known as the S-Matrix

$$\langle out|in\rangle = \langle\chi|S|\Psi\rangle = \lim_{t_\pm\to\pm\infty}\langle\chi|U(t_+)U^\dagger(t_-)|\Psi\rangle \tag{1.2.2}$$

1.2.1 Time Ordering

We would now like to talk about the time evolution of H_{int}.

$$\begin{gathered}\frac{dU(t)}{dt} = iH_0e^{iH_0t}e^{-iHt} + e^{iH_0t}(-iH)e^{-iHt}\\ = -ie^{iH_0t}(H-H_0)e^{-iHt} = -ie^{iH_0t}(H_{int})e^{-iHt}\\ = -ie^{iH_0t}(H_{int})e^{-iH_0t}e^{iH_0t}e^{-iHt}\\ = -iH(t)U(t)\end{gathered}$$

H_{int} is time dependent in a very complicated way. We have found a first order equation that we can solve by iteration. We know that initially that $U(0) = \mathbb{I}$

$$\begin{gathered}\int_0^t \frac{dU(t')}{dt'}dt' = -i\int_0^t dt' H_{int}(t')U(t')\\ U(t) = \mathbb{I} - i\int_0^t dt' H_{int}(t')U(t')\\ \Rightarrow U_0 = \mathbb{I}\\ U_1 = \mathbb{I} - i\int_0^t dt' H_{int}(t')\mathbb{I}\\ U_2 = \mathbb{I} - i\int_0^t dt' H_{int}(t')\mathbb{I} + (i)^2\int_0^t dt' H_{int}(t')\int_0^{t'} dt'' H_{int}(t'')\end{gathered}$$

With the last term we now must consider time ordered products. We can rewrite that term as

$$(i)^2\int_0^t dt' H_{int}(t')\int_0^{t'} dt'' H_{int}(t'') = \frac{(i)^2}{2}\int_0^t dt' T(H_{int}(t')\int_0^{t'} dt'' H_{int}(t''))$$

Using time ordered products we can write the n-th term as

$$U_n = U_{n-1} + \frac{(-i)^n}{n!}\int_0^t dt_1\dots dt_n T(\prod_1^n H_{int_m}) = U_{n-1} + \frac{(-i)^n}{n!}T(\int_0^t dt' H_{int})^n$$

Which we immediately recognize as

$$U(t) = T(e^{-i\int_0^{t'} dt' H_{int}}) \tag{1.2.3}$$

1.2.2 Properties of the S-Matrix

1.1

$$S^\dagger S = \mathbb{I} \tag{1.2.4}$$

We will only consider cases such that S itself is a Unitary operator. In full generality just Because S is the product of 2 unitary operators doesn't mean that it is itself a unitary operator. An example of when unitarity of the S matrix breaks down is in bound states. We will not be considering bound states.

1.2

$$[S, H_0] = 0 \tag{1.2.5}$$

This means that S preserves the Energy Levels of H_0. **Proof.**

$$e^{i\epsilon H_0} S e^{-i\epsilon H_0} = \lim_{t_\pm \to \pm\infty} e^{i\epsilon H_0} e^{it_- H_0} e^{-it_- H} e^{it_+ H} e^{-it_+ H_0} e^{-i\epsilon H_0}$$
$$= \lim_{t_\pm \to \pm\infty} e^{iH_0(t_- + \epsilon} e^{-iH(t_- + \epsilon)} e^{iH(t_+ + \epsilon)} e^{iH_0(t_- - \epsilon)}$$
$$= \lim_{t_\pm \to \pm\infty} U(t_- + \epsilon) U^\dagger(t_+ + \epsilon)$$

But as $t_\pm \to \pm\infty$ adding an ϵ doesn't matter so $e^{i\epsilon H_0} S e^{-i\epsilon H_0} = S$

we see that S doesn't evolve $\therefore [S, H_0]] = 0$

■

1.3 S is Lorentz Invariant. To show that S is lorentz invariant we need to show that each component of S is Lorentz invariant.

$$S = \lim T(e^{-i\int_0^{t'_-} dt' H_{int}}) T(e^{i\int_0^{t'_+} dt' H_{int}}) = \lim T(e^{i\int_{t'_-}^0 dt' H_{int}}) T(e^{i\int_0^{t'_+} dt' H_{int}})$$
$$= \lim T(e^{i\int_{t'_-}^{t'_+} dt' H_{int}}) = T(e^{i\int_{-\infty}^{\infty} dt' H_{int}})$$

You can only combine the two integrals due to time ordering. For S to be Lorentz Invariant we will need to place constraints.

- The potential responsible for the interaction can not depend on velocity. Then we have that

$$H_{int} = \int d^3x \mathcal{H}_{int} = -\int d^3x \mathcal{L}_{int}$$

  By definition $\mathcal{L}$ is a lorentz invariant quantity.
- Time Ordering: We know that T is a Lorentz Invariant for timelike distances, it has no choice but to be. But what about Space like distances? It isn't, but Causality fixes this problem for us. Remember $[A(x), B(y)] = 0$, therefore the fact that Time ordering isn't lorentz invariant for space like distances is irrelevant.

1.4 Corollary (2+3) $[S, P_0^\mu] = 0$ To see this we will need to consider a scattering event. Consider the case of two particles in the state $|\vec{P}_1, \vec{P}_2\rangle$ being scattered. The process is irrelevant. The final product of this is $|\vec{P'}_1, \vec{P'}_2, \vec{P'}_3, \vec{P'}_4\rangle$. We define the action of P_0^μ as usual

$$P_0^\mu |\vec{P}_1, \vec{P}_2\rangle = (P_1^\mu + P_2^\mu)|\vec{P}_1, \vec{P}_2\rangle$$
$$P_0^\mu |\vec{P'}_1, \vec{P'}_2, \vec{P'}_3, \vec{P'}_4\rangle = (P_1^\mu + P_2^\mu + P_3^\mu + P_4^\mu)|\vec{P'}_1, \vec{P'}_2, \vec{P'}_3, \vec{P'}_4\rangle$$

then we have

$$
\begin{aligned}
& \langle \vec{P}_1, \vec{P}_2 | [S, P_0^\mu] | \vec{P'}_1, \vec{P'}_2, \vec{P'}_3, \vec{P'}_4 \rangle \\
= \; & (P_1^\mu + P_2^\mu + P_3^\mu + P_4^\mu) \langle \vec{P}_1, \vec{P}_2 | S | \vec{P'}_1, \vec{P'}_2, \vec{P'}_3, \vec{P'}_4 \rangle \\
- \; & (P_1^\mu + P_2^\mu) \langle \vec{P}_1, \vec{P}_2 | S | \vec{P'}_1, \vec{P'}_2, \vec{P'}_3, \vec{P'}_4 \rangle \\
= \; & \{ (P_1^\mu + P_2^\mu + P_3^\mu + P_4^\mu) - (P_1^\mu + P_2^\mu) \} \langle \vec{P}_1, \vec{P}_2 | S | \vec{P'}_1, \vec{P'}_2, \vec{P'}_3, \vec{P'}_4 \rangle
\end{aligned}
$$

Momentum is always conserved $\therefore [S, P_0^\mu] = 0$.

1.3 Wick's theorem

Earlier when we first wrote our Hamiltonian for a scalar field, we found that it corresponded to infinite energy densities. This was fixed with normal ordering. We would now like to find a way to relate normal ordering with time ordering so that our theory is both causaul and free of the problems we encountered earlier. For two fields the relation ship is simple.

$$T(\phi(x)\phi(y)) - :\phi(x)\phi(y): = \overbrace{\phi(x)\phi(y)}$$

Where the $:$ surrounding something tells you that it is normal ordered. Contraction gives us a number to see this

$$\langle 0 | \overbrace{\phi(x)\phi(y)} | 0 \rangle = \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle - \langle 0 | :\phi(x)\phi(y): | 0 \rangle$$

The second term in this expersion simply gives zero, since the normal ordered product annhilates the ground state. The first term was defined in QFT I and is the Feynman propogator for the Klein Gordon Field. Can we find a way to generalize this to n fields?. We will drop the arguements of the fields, they are now implied to correspond with the index of the field. Also we will take

$$T(\phi_1 \phi_2 \ldots \phi_n) = \phi_1 \phi_2 \ldots \phi_n$$

and we will say that the decomposition of the field into modes is the $\phi = \phi_+ + \phi_-$. We have done 2 fields so lets do 3.

$$
\begin{aligned}
\phi_1 \phi_2 \phi_2 = \phi_1 \phi_2 (\phi_{3-} + \phi_{3+}) &= \phi_1 \phi_2 \phi_{3-} + \phi_{3+} \phi_1 \phi_2 \\
&+ [\phi_1, \phi_{3+}] \phi_2 + \phi_1 [\phi_2, \phi_{3-}]
\end{aligned}
$$

We now need to find these commutators

$$
\begin{aligned}
\langle 0 | [\phi_1, \phi_{3+}] | 0 \rangle &= \langle 0 | \phi_1 \phi_{3+} - \phi_{3+} \phi_1 | 0 \rangle \\
\langle 0 | \phi_1 \phi_{3+} - \phi_{3+} \phi_1 \| 0 \rangle - \phi_{3-}) | 0 \rangle - \langle 0 | \phi_{3+} \phi_1 | 0 \rangle &= \langle 0 | \phi_1 \phi_3 | 0 \rangle
\end{aligned}
$$

But this was our definition of of a contraction. Carrying this out for each field, for 3 fields we can write

$$\phi_1 \phi_2 \phi_3 =: \phi_1 \phi_2 \phi_3 : + \overbrace{\phi_1 \phi_2} \phi_3 + \overbrace{\phi_1 \phi_3} \phi_2 + \overbrace{\phi_2 \phi_3} \phi_3$$

From here it is proof by induction to show that in general for n fields you will have

$$\phi_1\phi_2\ldots\phi_n =: \phi_1\phi_2\ldots\phi_n : + \overbrace{\phi_1\phi_2} \\ \times : \phi_3\ldots\phi_n : + permutations + \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} : \phi_5\ldots\phi_n : + permutations + \ldots$$

This is Wick's thereom.

1.3.1 Example: $\phi(x)J(x)$

The simplest interaction we can have is of the form

$$H_{int} = \int d^3x\phi(x)J(x) \tag{1.3.1}$$

If we consider a series expansion of S we can write the n'th term as

$$S^{(n)} = \frac{(-i)^n}{n!}T(-i\int d^4x\phi(x)J(x))$$

Lets let $A = -i\int d^4x\phi(x)J(x)$, we can use Wick's theorem. Instead of having n-different objects we have one object n times, so we can do something.

$$T(A^n) =: A^n : + \begin{pmatrix} n \\ 2 \end{pmatrix} \overbrace{AA} : A^{n-2} : + \tfrac{1}{2}\begin{pmatrix} n \\ 2 \end{pmatrix}\begin{pmatrix} n-2 \\ 2 \end{pmatrix}\overbrace{AA}\overbrace{AA} : A^{n-4} : \\ + \cdots + \tfrac{1}{p!}\begin{pmatrix} n \\ 2 \end{pmatrix}\begin{pmatrix} n-2 \\ 2 \end{pmatrix}\ldots\begin{pmatrix} n-p \\ 2 \end{pmatrix}\overbrace{AA}\ldots\overbrace{AA} : A^{n-p} : \tag{1.3.2}$$

we can collect all the terms contractions p

$$\begin{aligned} A_p &= \sum_{n=2p}^{\infty}\frac{(-i)^n}{n!}\frac{n!}{p!2^p(n-2p)!}(\overbrace{AA})^p : A^{n-2p} : \\ &= \sum_{n'}^{\infty}(-i)^{n'+2p}\frac{1}{p!2^p(n')!}(\overbrace{AA})^p : A^{n'} : \\ &= \frac{(-i)^{2p}}{p!2^p}(\overbrace{AA})^p : e^{-iA} : \end{aligned}$$

This gives us a very simple expression for S

$$S = \sum_0^{\infty} A_p = e^{\frac{-1}{2}(\overbrace{AA})} : e^{-iA} : \tag{1.3.3}$$

Nothing to Nothing

Since we have S we can now calculate transition amplitudes. Nothing to Nothing is simplest and we shall start there.

$$\langle 0|S|0\rangle = e^{\frac{-1}{2}(\overbrace{AA})}$$

Nothing to Something

As long as you have a source you can have a particle appear from nothing.

$$\begin{aligned} \langle \vec{K}|S|0\rangle &= e^{\frac{-1}{2}(\overline{AA})}\langle \vec{K} : e^{-iA} : |0\rangle \\ \langle \vec{K} : e^{-iA} : |0\rangle &= \langle \vec{K}|1 - iA + \frac{(-i)^2}{2!} : A^2 : + \ldots |0\rangle \end{aligned}$$

Only one A is going to survive this inner product since the others raise the vacuum too much.

$$\begin{aligned} \langle \vec{K}|A|0\rangle &= -i\langle \vec{K}| \int d^4x J(x)\phi(x)|0\rangle \\ &= -i\tilde{J}(\vec{K}) \\ \therefore \langle \vec{K}|S|0\rangle &= -ie^{\frac{-1}{2}(\overline{AA})}\tilde{J}(\vec{K}) \end{aligned}$$

Nothing to n-particles

The generalization to n-particles is quite easy

$$\langle \vec{K}_1\vec{K}_2 \ldots \vec{K}_n|S|0\rangle = -ie^{\frac{-1}{2}(\overline{AA})}\frac{(-i)^n}{n!}J(\tilde{x}_1)\ldots J(\tilde{x}_n)$$

Now that we have the general expresion for the transition amplitude we can calculate the probability of transition from 0 to n

$$P(0 \to n) = \prod_n \int \frac{d^4k_n}{(2\pi)^3 2k_n^0}\frac{1}{n!}|\langle 0|S|0\rangle|^2$$

We now need to calculate what the Contraction of 2 A's is.

$$\begin{aligned} \overline{AA} &= \int d^4x J(x)\phi(x) \int d^4y J(y)\phi(y) \\ &= \int d^4x d^4y J(x)J(y)D_F(x-y) \\ &= \int \frac{d^4k}{(2\pi)^4}\tilde{J}(k)\tilde{J}(-k)\frac{i}{k^2 - m^2 + i\epsilon} = c \end{aligned}$$

Since we want to square our exponential we would like to find $c + c^*$

$$c + c^* = \int \frac{d^4k}{(2\pi)^4}|\tilde{J}(k)|^2 \left(\frac{i}{k^2 - m^2 + i\epsilon} - \frac{i}{k^2 - m^2 - i\epsilon}\right)$$

we will use the fact that $\lim_{\epsilon \to 0^+} \left(\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon}\right) = 2\pi\delta(x)$

$$\int \frac{d^4k}{(2\pi)^3}|\tilde{J}(k)|^2\delta(k_0^2 - k^2 - m^2) = \int \frac{d^3k}{(2\pi)^3 2|k_0|}|\tilde{J}(k)|^2$$

We end up with a probability that is of the form $\frac{1}{n!}e^{-\beta}\beta^n$. The sum over all n of a poisson distribution is 1, which is what we should see. There is a 100% chance that all transitions can occur.

Comment

If we localize J, $J(x) \to \delta(x)$, then $\int \frac{d^3k}{(2\pi)^3 2|k_0|}$ diverges and you have what is known as Ultraviolet Divergence.

1.3.2 Example:f(t)J($\vec{x}$)$\phi(x)$

We will now consider a static source J($\vec{x}$) with a turning off and on function f(t) so that for some time ≫1 f(t) is zero, but less than that and it is constant. We will call this cut off time T. In the limit that T goes to infinite $f(\omega)$ becomes $2\pi\delta(\omega)$, the Riemann-Lebesgue lemma tells us then that all terms with order A greater than or equal to 1 are zero. These means that a static source cannot radiate or absorb. So that tells us that the probability for nothing to nothing is 1.

nothing to nothing

In this case the contraction of our 2 A's will give us a number, α, which is purely imaginary. Remember that the action of nature's Hamiltonian on the physical ground state is

$$H_{nat}|0_{phys}\rangle = E_0|0_{phys}\rangle$$

Then using the S-matrix we can say that

$$\begin{aligned}
\langle 0|S|0\rangle &= \lim\langle 0|e^{-iH(t_+-t_-)}|0\rangle \\
&= \langle 0_{phys}|e^{-iH_{nat}(2T)}|0_{phys}\rangle = e^{-iE_0(2T)} \\
E_0 &= \lim_{T\to\infty}\frac{i}{2T}\ln\langle 0|S|0\rangle = \lim_{T\to\infty}\frac{-i\alpha}{2T}
\end{aligned}$$

Now lets calculate α

$$\alpha = \frac{1}{2}\int d^4x_1 \int d^4x_2 J(x_1)D_F(x_1-x_2)J(x_2)$$

Where

$$D_F(x_1-x_2) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik(x_1-x_2)}$$

plugging this in and doing the integrals over x gives

$$= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}|f(\tilde{k}_0)\tilde{\rho}(\vec{k})|^2$$

We can safely let ϵ go to zero. It can be shown that

$$\lim_{T\to\infty}\frac{1}{T}|\tilde{f}(k_0)|^2 = 2\pi\delta(k_0)$$

doing the integral over k_0 fixes $k_0 = 0$ and we have

$$\begin{aligned} E_0 &= \int \frac{d^3k}{(22\pi)^3}\frac{i}{\vec{k}^2 - m^2 + i\epsilon}|\tilde{\rho}(\vec{k})|^2 \\ &= \int \frac{d^3k}{2(2\pi)^3}\int d^3x_1 \int d^3x_2 \frac{i}{\vec{k}^2 - m^2 + i\epsilon}\rho(\vec{x}_1)\rho(\vec{x}_2) \\ &= \frac{1}{2}\int d^3x_1 \int d^3x_2 \rho(\vec{x}_1)\rho(\vec{x}_2)V(\vec{x}_1 - \vec{x}_2) \end{aligned}$$

Where the potential V is known as the Yukawa potential. Doing the integral over k yields

$$V = -\frac{1}{4\pi|\vec{x}|}e^{-m|\vec{x}|} \tag{1.3.4}$$

1.3.3 Wick Diagrams

Doing all of this by hand gets to be rather tedious. There are however diagrams that will help us calculate things. Consider a Klein Gordon Lagrangian with interaction term

$$\mathcal{L}_{int} = -\lambda\phi_1\phi_2\phi_3$$

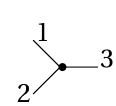
We can write our S in the usual way

$$S = T(e^{-i\int d^4x\lambda\phi_1\phi_2\phi_3})$$

We can power expand this so that

$$S = S^{(0)} + S^{(1)} + \cdots + S^{(n)}$$

Where the label simply tells you which order of λ the term is. We can use diagrams to represent each of these terms. For example the first order term would look like

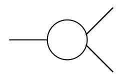


Where the large dot tells us to integrate the things that meet there over x, in this case do the integral

$$-i\lambda\int d^4x\phi_1\phi_2\phi_3$$

Example

Lets look at 1 particle going to 2 particles



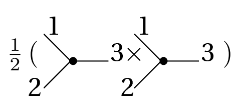
This is just a cartoon and doesn't mean anything. For this Process

$$\begin{aligned}
\langle k_2 k_3 | S | k_1 \rangle &= \langle k_2 k_3 | S^{(0)} | k_1 \rangle + \langle k_2 k_3 | S^{(1)} | k_1 \rangle + \dots \\
&= -i\lambda \langle k_2 k_3 | \int d^4 x \phi_1 \phi_2 \phi_3 | k_1 \rangle + \dots \\
&= -i\lambda \int d^4 x \langle k_2 | \phi_2 | 0 \rangle \langle k_3 | \phi_3 | 0 \rangle \langle 0 | \phi_1 | k_1 \rangle + \dots \\
&= -i\lambda \int d^4 x e^{-ik_1 x} e^{ik_2 x} e^{ik_2 x} + \dots = i\lambda (2\pi)^4 \delta^4 (k_1 - k_2 - k_3) + \dots
\end{aligned}$$

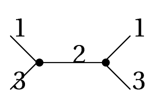
Immediately we see that the S-matrix formalism requires conservation of Momentum. We define the Amplitude of a transition as

$$\langle f | S | i \rangle - \langle f | i \rangle = (2\pi)^4 \delta^4 (p_f - p_i) i\mathcal{A} \tag{1.3.5}$$

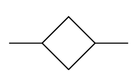
Where we have subtracted off the amplitude for nothing to happen, and factored out the conservation of momentum term since it will belong to every term. For $S^{(2)}$ we have



As is this diagram does not correspond to 1 particle going to 3. This would be process describing 6 particles with 2 conservation laws. We are interested in 3 particles with one conservation law. There are 2 ways that we can combine these equations that make sense. one is



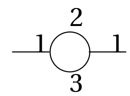
Or any other permutation of the legs. While the other is



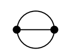
We will consider the first diagram. If you have 2 vertices connected by a line this tells you that a contraction must occur, in the case of our picture the associated integral would be

$$\int d^4x d^4y \phi(y)_2 \phi(x)_2 : \phi_1(x)\phi_3(x)\phi_1(y)\phi_3(y) :$$

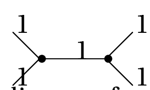
If we have 2 contractions



This represents a 2nd order correction, and is a quantum correction to the classical solution. We can have 3 Contractions



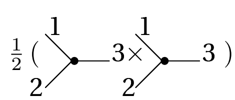
Where the legs should be labeled 1,2,3 from top to bottom or any permutation there of. When dealing with these diagrams one needs to consider the symmetery of the diagram. If one considers the ϕ^3-theory all the lines will correspond to the same field.



You can flip the left side of the diagram for a factor of 2, the contraction for a factor of 2 and the right side for yet another factor of 2, giving a symmetry factor of $2 \times 2 \times 2 = 8$

Connected versus Disconnected

The first second order diagram I showed you



is an example of a disconnected diagram. Notice that this is the product of 2 connected diagrams. In general the disconnected diagrams of S will just be the product of the connected diagrams of S

$$S^{(m)}_{(disconected)} = \frac{1}{n_1!}\frac{1}{n_2!}\frac{1}{n_3!}\ldots(S^{(1)}_{(connected)})^{n_1}(S^{(2)}_{(conn)})^{n_2}(S^{(3)}_{(conn)})^{n_2}\ldots$$

where we have divided by the factors of n! to take care of the different combinations of the diagrams that are equivalent due to symmetry. The total S is going to be

$$\begin{aligned} S &= \sum_m S^{(m)}_{(disconected)} = \sum_m \prod_n^m \frac{1}{n!}(S^{(n)}_{(connected)})^n \\ &= \prod_n e^{S^{(n)}_{(connected)}} = e^{\sum_n S^{(n)}_{(connected)}} \end{aligned}$$

Therefore the connected diagrams are the only important ones.
An assigned momentum may be fixed by the conservation of momentum at each vertex or it may be undetermined. Let I be the number of internal lines and let V be the number of vertices. There are I unknowns and V equations but there are only $V-1$ independent ones because of the overall conservation of momentum. Thus there are $I-V+1$ undetermined momenta to integrate over. Let L be the number of loops in the diagram.

Theorem 1 $L = I - V + 1$, *i.e. the number of loops L in the diagram is exactly the number of integrals.*

Proof. The proof is by induction on L. When $L = 0$, $I = V - 1$ (a tree) so the momenta are completely determined. The induction assumption is that $L = I-V+1$. Given a diagram with I' internal lines, V' vertices, and $L' = L+1$ loops, remove one edge of the loop. The number of vertices is unchanged, the number of edges is reduced by one and the number of loops is reduced by one so the induction assumption applies. Hence $L'-1 = (I'-1)-V'+1 = I'-V'$ so $L' = I' - V' + 1$.

Example 2 *A collision of two particles which produces two particles.*

■

We can take the incoming particles to have momentum $k_1 = (E_1, \mathbf{k}_1)$, $k_2 = (E_2, \mathbf{k}_2)$; in the center of mass frame, $E_1 = E_2 \equiv E$ and $\mathbf{k}_2 = -\mathbf{k}_1$. The scattered particles will have momenta $k_3 = (E, \mathbf{k}_3)$ and $k_4 = (E, -\mathbf{k}_3)$ by conservation of 4-momentum. Then

$$(k_1 + k_2)^2 = 4E^2$$
$$(k_1 - k_3)^2 = -(\mathbf{k}_1 - \mathbf{k}_2)^2 \equiv -\mathbf{q}^2 \text{ (}\mathbf{q}\text{ is the momentum transfer)}$$
$$(k_1 - k_4)^2 = -(\mathbf{k}_1 - \mathbf{k}_4)^2 = -(\mathbf{k}_1 + \mathbf{k}_3)^2 \equiv -(\mathbf{q}')^2$$

Then

$$\begin{aligned} iA &= (-i\lambda)^2 \left(\frac{i}{(k_1+k_2)^2 - m^2} + \frac{i}{(k_1-k_3)^2 - m^2} + \frac{i}{(k_1-k_4)^2 - m^2} \right) \\ &= (-i\lambda)^2 \left(\frac{i}{4E^2 - m^2} - \frac{i}{\mathbf{q}^2 + m^2} - \frac{i}{(\mathbf{q}')^2 + m^2} \right) \end{aligned}$$

In non-relativistic quantum mechanics, the last two terms are Born approximation of the scattering amplitude, $\int e^{-i\mathbf{q}\cdot\mathbf{x}} V(x)$, for the Yukawa potential, $V(x) = \frac{\lambda^2}{4\pi} \frac{e^{-m|x|}}{|x|}$ [C-T, p. 958]. The first term is relativistic; in fact it has a pole at $m = 2E$ which goes away as $c \to \infty$, i.e. in the non-relativistic limit.
To relate to experiment, we will need a relativistic analog of Fermi's Golden Rule [C-T, p. 1299]. In one dimension, we have a basis $|k\rangle$ with $\int dk\, |k\rangle \langle k| = 1$ and $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{-ikx}$ so $\langle x|y\rangle = \int dk\, \langle x\,|k\rangle \langle k|\, y\rangle = \int \frac{dk}{2\pi} e^{-ik(x-y)} = \delta(x-y)$. We now demand $0 \le x \le L$ so we get discrete eigenstates $\varphi_n(x) = \frac{1}{\sqrt{L}} e^{i\frac{2\pi n}{L}x} \equiv \langle x|n\rangle$ with $\sum |n\rangle \langle n| = 1$. As $L \to \infty$, $\frac{2\pi n}{L} \to k$, and the difference in successive n's, $\frac{2\pi}{L} \to \Delta k \to dk$ and $\sum_n \to \frac{L}{2\pi} \int dk$. In three dimensions, we repeat these substitutions 3 times with $\mathbf{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$ to get $\varphi_{\mathbf{k}}(x) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{x}}$ and, as $L \to \infty$, $\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3} \int d^3k$.
To make everything concrete, we use a K-G field $\varphi(x) = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2k_0} \left(e^{-ik\cdot x} a(\mathbf{k}) - e^{ik\cdot x} a^\dagger(\mathbf{k}) \right)$.
$\overbrace{\varphi(x)\varphi(y)} = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2k_0} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \to \int \frac{d^3k}{(2\pi)^3 2k_0} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$; $\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \delta_{\mathbf{k},\mathbf{k}'}$

and $\langle 0 |\varphi(x)| \mathbf{k}\rangle = \frac{1}{\sqrt{2Vk_0}}e^{-ik\cdot x}$. We introduce a new Feynman rule: assign to each external leg a factor of $\frac{1}{\sqrt{2EV}}$. The transition amplitude is

$$\langle f |S-1| i\rangle = iA(2\pi)^4\delta^4(p_f-p_i)\prod_j \frac{1}{\sqrt{2E_jV}}$$

so the transition probability is

$$\begin{aligned} |\langle f |S-1| i\rangle|^2 &= |A|^2((2\pi)^4\delta^4(p_f-p_i))^2\prod_j \frac{1}{2E_jV} \\ &= |A|^2VT(2\pi)^4\delta^4(p_f-p_i)\prod_j \frac{1}{2E_jV} \end{aligned}$$

where the square of the delta function is

$$\begin{aligned} \left((2\pi)^4\,\delta^4(k)\right)^2 &= (2\pi)^4\,\delta^4(k)\int d^4y e^{ik\cdot y} \\ &= (2\pi)^4\,\delta^4(k)\int d^4y \\ &= (2\pi)^4\,\delta^4(k)VT \end{aligned}$$

because $\delta^4(k)$ forces $k=0$. Thus the probability per unit time (the decay rate), as $V\to\infty$, becomes

$$\sum_{j_{out}}|A|^2V(2\pi)^4\delta^4(p_{out}-p_{in})\prod_j\frac{1}{2E_jV}\to\int\prod_{j\ out}\frac{d^3p_{j_{out}}}{(2\pi)^3\,2E_{j_{out}}}|A|^2V(2\pi)^4\delta^4(p_{out}-p_{in})\prod_{j\ in}\frac{1}{2E_{j_{in}}V}$$

because $\sum_k \to V\int\frac{d^3k}{(2\pi)^3}$ as $VT\to\infty$. All the details of the theory are contained in $|A|^2$.

Example 3 *One incoming particle p_{in} and two outgoing particles p_1 and p_2.*

The decay rate is

$$\begin{aligned} \Gamma &= \int\frac{d^3p_1}{(2\pi)^3\,2E_1}\frac{d^3p_2}{(2\pi)^3\,2E_2}|A|^2V(2\pi)^4\delta^4(p_{in}-p_1-p_2)\frac{1}{2E_{in}V} \\ &= \frac{1}{2m}\int\frac{d^3p_1}{(2\pi)^3\,2E_1 2E_2}|A|^2 2\pi\delta(E_{in}-E_1-E_2) \end{aligned}$$

In the rest frame of the incoming particle $p_{in}=(m,\mathbf{0})$, $p_1=(E_1,\mathbf{p}_1)$, and $p_2=(E_2,\mathbf{p}_2)$. Using spherical coordinates, $d^3p_1=p_1^2dp_1d\Omega$ and writing $f(\mathbf{p}_1)=m-E_1-E_2$, we have $f'(\mathbf{p}_1)=-\frac{dE_1}{dp_1}-\frac{dE_2}{dp_1}=-\frac{p_1}{E_1}-\frac{p_1}{E_2}=-p_1\frac{E_1+E_2}{E_1E_2}=-p_1\frac{m}{E_1E_2}$. Then

$$\begin{aligned} \Gamma &= \frac{1}{2m}\int\frac{p_1^2dp_1d\Omega}{(2\pi)^3 4E_1E_2}2\pi|A|^2\frac{E_1E_2}{mp_{1root}}\delta\left(p_1-p_{1root}\right) \\ &= \frac{p_1}{32\pi^2m^2}\int d\Omega|A|^2 \end{aligned}$$

In an arbitrary frame

$$\Gamma_{arb}=\frac{m}{E_{in}}\Gamma=\frac{1}{\gamma}\Gamma$$

using $E=\frac{m}{\sqrt{1-v^2}}=\gamma m$. If the incoming particle has spin 0, A is independent of Ω so $\Gamma=\frac{p_1^2}{8\pi m^2}|A|^2$.

Example 4 *One incoming particle and three outgoing particles.*

Much the same reasoning applies. We have to include three more integrals for $\frac{d^3p_3}{(2\pi)^3 2E_3}$ and four more delta functions $\delta^4(p_{in} - p_1 - p_2 - p_3)$. In the rest frame of the incoming particle

$$\Gamma = \frac{1}{2m}\int \frac{d^3p_1 d^3p_2}{(2\pi)^6 2E_1 2E_2 2E_3} 2\pi\delta(m - E_1 - E_2 - E_3)|A|^2$$

Use spherical coordinates with z-axis parallel to p_1 and let θ_{12} be the angle between p_1 and p_2; $d^3p_2 = \mathbf{p}_2^2 dp_2 d\Omega_{12}$; $dr_{12} = d(\cos\theta_{12})d\varphi_{12}$. We will eventually use spherical coordinates to represent p_1 so we can think of the momenta as positive magnitudes.

Now specialize to the case where all outgoing particles have mass 0. Note $\mathbf{p}_i^2 = E_i^2$. The argument of the delta function $m - E_1 - E_2 - E_3 = m - p_1 - p_2 - |\mathbf{p}_1 + \mathbf{p}_2| = m - p_1 - p_2 - \sqrt{p_1^2 + p_2^2 - 2p_1p_2\cos\theta_{12}} \equiv f(\cos\theta_{12})$; $f'(\cos\theta_{12}) = \frac{-p_1p_2}{\sqrt{p_1^2+p_2^2-2p_1p_2\cos\theta_{12}}} = -\frac{E_1E_2}{E_3}$.

$$\begin{aligned}
\Gamma &= \frac{1}{2m}\int \frac{d^3p_1 \mathbf{p}_2^2 dp_2 d\varphi_{12}}{(2\pi)^5 8E_1E_2E_3}\frac{E_3}{E_1E_2}|A|^2 \\
&= \frac{1}{2m}\int \frac{d^3p_1 \mathbf{p}_2^2 dp_2 d\varphi_{12}}{(2\pi)^5 8E_1^2E_2^2}|A|^2 \\
&= \frac{1}{16(2\pi)^5 m}\int \frac{\mathbf{p}_1^2 dp_1 d\Omega dp_2}{E_1^2}|A|^2 \\
&= \frac{1}{16(2\pi)^5 m}\int dE_1 d\Omega dE_2 d\varphi_{12}|A|^2
\end{aligned}$$

where we have used $\mathbf{p}_i^2 = E_i^2$ and converted p_1 to spherical coordinates: $d^3p_1 = \mathbf{p}_1^2 dp_1 d\Omega$. The delta function imposes a constraint: $m = E_1 + E_2 + E_3 = E_1 + E_2 + \sqrt{E_1^2 + E_2^2 - 2E_1E_2\cos\theta_{12}}$ so $m \geq E_1 + E_2$.

Furthermore $(m - E_1 - E_2)^2 = E_1^2 + E_2^2 - 2E_1E_2\cos\theta_{12}$ so $m^2 + 2E_1E_2 - 2mE_1 - 2mE_2 = -2E_1E_2\cos\theta_{12}$ which gives $m(m - 2E_1 - 2E_2) = -2E_1E_2(1 + \cos\theta_{12})$ which implies a negative left hand side, i.e. $E_1 + E_2 \geq \frac{m}{2}$. The limits on $\cos\theta_{12}$ also mean $m(m - 2E_1 - 2E_2) \geq -4E_1E_2$ which rearranges to $\left(\frac{m}{2} - E_1\right)\left(\frac{m}{2} - E_2\right) \geq 0$. Both terms negative contradicts $m \geq E_1 + E_2$ so E_1 and E_2 are less than or equal to $\frac{m}{2}$. Thus integration is over the upper triangle in the Dalitz plot below.

In the case where all three particles have mass, the area of integration becomes a first quadrant shape (its Dalitz plot).

Example 5 *Two in, two out.*

We have the formula for the transition probability. We now calculate the cross-section. Let $\mathbf{k}_2 = 0$ (rest frame of particle 2); The volume is area times length $V = AL$; $v = \frac{L}{t}$ so $flux = \frac{\#particles}{area \cdot time} = \frac{1}{At} = \frac{v}{AL} = \frac{v}{V}$. The cross-section is

$$
\begin{aligned}
\sigma &= \frac{\Gamma}{flux} \\
&= \frac{V}{v}\Gamma \\
&= \frac{V}{v}\int \frac{d^3k_3}{(2\pi)^3\,2E_3}\frac{d^3k_4}{(2\pi)^3\,2E_4}|A|^2 V(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\frac{1}{2E_1V}\frac{1}{2E_2V} \\
&= \frac{1}{vE_1E_2}\int \frac{d^3k_3}{(2\pi)^3\,2E_3}\frac{d^3k_4}{(2\pi)^3\,2E_4}|A|^2(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\frac{1}{2}\frac{1}{2}
\end{aligned}
$$

The integral is clearly Lorentz invariant as is the factor in front of it since the following is Lorentz invariant

$$
\begin{aligned}
\sqrt{(k_1\cdot k_2)^2 - m_1^2m_2^2} &= \sqrt{(E_1m_2)^2 - m_1^2m_2^2} \\
&= m_2\sqrt{E_1^2 - m_1^2} \\
&= m_2E_1v \\
&= E_1E_2v
\end{aligned}
$$

where the left hand side has been evaluated in the rest frame of particle 2. The third equality comes from $E^2 = \gamma^2m^2$ so $E^2(1-\mathbf{v}^2) = m^2$ which gives $E^2 - m^2 = E^2\mathbf{v}^2$ or $v = \frac{|\mathbf{k}|}{E}$.
We also have, in the center of mass frame, $-\mathbf{k}_2 = \mathbf{k}_1 = \mathbf{k}$ and $-\mathbf{k}_4 = \mathbf{k}_3 = \mathbf{k}'$,

$$
\begin{aligned}
\sqrt{(k_1\cdot k_2)^2 - m_1^2m_2^2} &= \sqrt{(E_1E_2+\mathbf{k}^2)^2 - m_1^2m_2^2} \\
&= \sqrt{E_1^2E_2^2 + \mathbf{k}^4 + 2E_1E_2\mathbf{k}^2 - m_1^2m_2^2} \\
&= \sqrt{(\mathbf{k}^2+m_1^2)\,(\mathbf{k}^2+m_2^2) + \mathbf{k}^4 + 2E_1E_2\mathbf{k}^2 - m_1^2m_2^2} \\
&= \sqrt{2\mathbf{k}^4 + (m_1^2+m_2^2)\mathbf{k}^2 + 2E_1E_2\mathbf{k}^2} \\
&= \sqrt{(2\mathbf{k}^2 + m_1^2 + m_2^2 + 2E_1E_2)\mathbf{k}^2} \\
&= \sqrt{(E_1^2 + E_2^2 + 2E_1E_2)\mathbf{k}^2} \\
&= (E_1+E_2)\,|\mathbf{k}| \\
&= E_{CM}\,|\mathbf{k}|
\end{aligned}
$$

The relative velocity of the particles is $v = \frac{(E_1+E_2)}{E_1E_2}\,|\mathbf{k}| = \frac{|\mathbf{k}|}{E_1} + \frac{|\mathbf{k}|}{E_2} = |\mathbf{v}_1 - \mathbf{v}_2|$ so

$$
\begin{aligned}
\sigma &= \frac{1}{4vE_1E_2}\int \frac{d^3k_3}{(2\pi)^3\,2E_3}\frac{d^3k_4}{(2\pi)^3\,2E_4}|A|^2(2\pi)^4\delta^4(k_1+k_2-k_3-k_4) \\
&= \frac{1}{4E_{CM}\,|\mathbf{k}|}\int \frac{d^3k_3}{(2\pi)^3\,2E_3}\frac{d^3k_4}{(2\pi)^3\,2E_4}|A|^2(2\pi)^4\delta^4(k_1+k_2-k_3-k_4) \\
&= \frac{1}{4E_{CM}\,|\mathbf{k}|}\int \frac{d^3k_3}{(2\pi)^3\,2E_32E_4}|A|^2 2\pi\delta(E_1+E_2-E_3-E_4) \\
&= \frac{1}{16\pi^2E_{CM}\,|\mathbf{k}|}\int \frac{d\Omega}{E_3E_4}k_3^2\frac{1}{|f'(k_3)|}\,|A|^2
\end{aligned}
$$

where $f(k_3) = E_1 + E_2 - E_3 - E_4 = E_{CM} - \sqrt{m_3^2 + k_3^2} - \sqrt{m_4^2 + k_3^2}$ so $f'(k_3) = \frac{k_3}{E_3} + \frac{k_3}{E_4} = \frac{E_3+E_4}{E_3 E_4} k_3 = \frac{E_{CM}}{E_3 E_4} k_3$. Therefore

$$\sigma = \frac{k_3}{k} \frac{1}{64\pi^2 E_{CM}^2} \int d\Omega \, |A|^2$$

where A is theory dependent.
In φ^3 theory, we have Feynman diagrams
so $iA = (-i\lambda)^2 \left(\frac{i}{(k_1+k_2)^2 - m^2} + \frac{i}{(k_1-k_3)^2 - m^2} + \frac{i}{(k_1-k_4)^2 - m^2} \right) + \ldots$ Define the Mandelstam variables $s = (k_1 + k_2)^2$, $t = (k_1 - k_3)^2$, $u = (k_1 - k_4)^2$. We refer to these diagrams by the dependence of the propagators: the first diagram is the s-channel, the second is the t-channel, and the third is the u-channel. Cyclic permutations of the k_i take s to t to u to s. Also we have crossing symmetry $iA(s, t, u) = iA(u, s, t)$.

$$\begin{aligned} s + t + u &= k_1^2 + k_2^2 + 2k_1 \cdot k_2 + k_1^2 + k_3^2 - 2k_1 \cdot k_3 + k_1^2 + k_4^2 - 2k_1 \cdot k_4 \\ &= 6m^2 + 2k_1 \cdot (k_2 - k_3 - k_4) \\ &= 6m^2 - 2k_1^2 \\ &= 4m^2 \end{aligned}$$

which implies

$$iA = (-i\lambda)^2 \left(\frac{i}{s - m^2} + \frac{i}{t - m^2} + \frac{i}{u - m^2} \right)$$

Poles develop as $s, t, u \to m^2$ so we expect large cross sections at those values. Now let all particles be incoming so $k_3 \to -k_3$ and $k_4 \to -k_4$ so their energies will be negative going back in time (antiparticles). We must change the field to a complex field: two particles $(b, b^\dagger)$ and two antiparticles $(c, c^\dagger)$. $1 + 2 \to \overline{3} + \overline{4}$, $1 + 3 \to \overline{2} + \overline{4}$, $1 + 4 \to \overline{2} + \overline{3}$ (different crossing symmetries). In the center of mass frame, particles 1 - 4 have 4-momentums $(E, \mathbf{k})$, $(E, -\mathbf{k})$, $(E, \mathbf{k}')$, $(E, -\mathbf{k}')$ respectively. $1 + 2 \to \overline{3} + \overline{4}$: $s = (2E)^2 = E_{CM}^2 \geq 4m^2 \geq 0$; $t = -(\mathbf{k} - \mathbf{k}')^2 \leq 0$; $u = -(\mathbf{k} + \mathbf{k}')^2 \leq 0$. Permute to go to another channel $1 + 3 \to \overline{2} + \overline{4}$: $u \geq 4m^2$, $s, t \leq 0$; permute again to yet another channel $1 + 4 \to \overline{2} + \overline{3}$: $t \geq 4m^2$, $s, u \leq 0$.
If we reverse everything, there is no effect on A. In fact $A_{|i\rangle \to |f\rangle} = A_{|f\rangle \to |i\rangle}$ is a consequence of the CPT theorem. The CPT transformation U_{CPT} is not unitary but anti-unitary because time reversal is anti-unitary. $[U_{CPT}, H] = [U_{CPT}, H_0] = 0$. Recall $S = \lim_{t_\pm \to \pm\infty} U(t_+) U^\dagger(t_-)$ where $U(t) = e^{iH_0 t} e^{-iHt}$. Now $U_{CPT} U(t) U_{CPT}^{-1} = U(-t)$ so a change of variables shows $U_{CPT} S U_{CPT}^{-1} = \lim_{t_\pm \to \pm\infty} U(-t_+) U^\dagger(-t_-) = S^\dagger$. Under the CPT transformation $(2\pi)^4 \delta(p_f - p_i) \, iA_{|i\rangle \to |f\rangle} = \langle f \,|S - 1|\, i \rangle$

$$\begin{aligned} \langle f \,|S - 1|\, i \rangle &= \langle U_{CPT} f \,|U_{CPT}(S - 1)|\, i \rangle^* \\ &= \left\langle U_{CPT} f \,\middle|U_{CPT}(S - 1) U_{CPT}^{-1}\middle|\, U_{CPT} i \right\rangle^* \\ &= \left\langle \overline{f} \,\middle|S^\dagger - 1\middle|\, \overline{i} \right\rangle^* \\ &= \left\langle \overline{i} \,\middle|S - 1\middle|\, \overline{f} \right\rangle \end{aligned}$$

which relates to $A_{|\overline{f}\rangle \to |\overline{i}\rangle}$.

1.4 Unitarity

S is unitary. Define $iT = S-1$ so $\langle f|T|i\rangle = -i\langle f|S-1|i\rangle = (2\pi)^4\delta^4(p_f-p_i)A_{fi}$. Then $1 = (1+iT)^\dagger(1+iT) = 1 - iT^\dagger + iT + T^\dagger T$ so $T - T^\dagger = iT^\dagger T$.

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\langle f|T^\dagger T|i\rangle = i\sum_n \langle f|T^\dagger|n\rangle\langle n|T|i\rangle$$

which translates to amplitudes

$$\left(A_{fi} - A^*_{if}\right)(2\pi)^4\delta^4(p_f-p_i) = i\sum_n (2\pi)^4\delta^4(p_f-p_n)A^*_{nf}(2\pi)^4\delta^4(p_n-p_i)A_{ni}$$

The second delta function means $p_n = p_i$ so

$$A_{fi} - A^*_{if} = i\sum_n (2\pi)^4\delta^4(p_i-p_n)A^*_{nf}A_{ni}$$

When $|f\rangle = |i\rangle$, we then obtain a form of the Optical Theorem

$$\operatorname{Im} A_{ii} = \frac{1}{2}\sum_n (2\pi)^4\delta^4(p_i-p_n)|A_{ni}|^2$$

Fock space gives a complete set of states $|k_1k_2...k_n\rangle = a^\dagger(k_1)a^\dagger(k_2)...a^\dagger(k_n)|0\rangle$, $n = 1,...\infty$ so $\sum\limits_{n=1}^{\infty}\int \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}...\frac{d^3k_2}{(2\pi)^3 2E_2}|k_1k_2...k_n\rangle\langle k_1k_2...k_n|$ is the identity. If two identical particles scatter to two particles that are identical to the incoming particles, the first term of $\operatorname{Im} A_{ii}$ is $\frac{1}{2}\int\frac{d^3k_1}{(2\pi)^3 2E_1}(2\pi)^4\delta^4(p_i-k_1)$ (>--<); the second term is $\frac{1}{2}\int\frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}(2\pi)^4\delta(p_i-k_1-k_2)$ (>= =<) and so on. This generalizes to the Optical Theorem

$$\operatorname{Im} A_{ii} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int\frac{d^3k_1}{(2\pi)^3 2E_1}...\frac{d^3k_n}{(2\pi)^3 2E_n}|A_{ni}|^2\delta^4(p_i-k_1-...-k_n)$$

Example 6 *A single incoming Klein-Gordon particle of momentum p transitioning to itself.*

The only Lorentz invariant parameter is $p^2 \equiv s$ which should not be identified with the mass of the particle because intermediate particles may not be physical. Now A is a function of s: $\operatorname{Im} A(s) = \frac{1}{2}\int\frac{d^3k_1}{(2\pi)^3 2E_1}(2\pi)^4\delta^4(p-k_1)|A|^2 + ... = \frac{1}{2}2\pi\frac{\delta(E-E_1)}{2E_1}|A|^2$ which is Lorentz Invariant because the Lorentz invariant quantity $\delta\left(k_1^2-p^2\right) = \delta\left(E_1^2 - \mathbf{k}_1^2 - p_0^2 + \mathbf{p}^2\right) = \delta\left(E_1^2 - p_0^2\right) = \frac{1}{2E_1}\delta(E_1 - p_0)$. Hence

$$\operatorname{Im} A(s) = \frac{1}{2}2\pi\frac{\delta(E-E_1)}{2E_1}|A|^2 = \pi\delta\left(k_1^2-p^2\right)|A|^2$$

to first order $(-A-)$. The higher order corrections come from diagrams $(-A = A-)$ which contributes a factor $\frac{i}{p^2-m^2+i\varepsilon}$ and $(-A = A^*-)$ which

Example 7 *One particle decays into two particles of the same species*

The decay rate calculated earlier is $\Gamma = \frac{1}{2m}\int\frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}(2\pi)^4\delta(p-k_1-k_2)|A|^2$. In the rest frame of the incoming particle $p = (\sqrt{s},\mathbf{0})$, $k_1 = (E',\mathbf{k})$, and $k_2 = (E',-\mathbf{k})$.

$$\Gamma = \frac{k}{32\pi^2\sqrt{s}}\int d\Omega\,|A|^2$$

which is Lorentz invariant if $\frac{k}{\sqrt{s}}$ is. $E'^2 = m^2 + k^2 = \left(\frac{\sqrt{s}}{2}\right)^2$ by conservation of energy. Solve for k in terms of s and divide by $\sqrt{s}$ to get

$$\frac{2k}{2\sqrt{s}} = \sqrt{\frac{s-4m^2}{4s}}$$

which is Lorentz invariant. Then

$$m\Gamma = \frac{1}{128\pi^2}\sqrt{1-\frac{4m^2}{s}}\int d\Omega\,|A|^2$$

If $s < 4m^2$, decay is impossible (the initial energy is less than the sum of the masses of the decay products); $\Gamma = 0$ so $\operatorname{Im} A = 0$ and $A(s) = A^*(s^*)$.When two analytic function agree on an interval, the agree everywhere except singularities. so $\operatorname{Im} A = 0$ except at singularities. Graphically there is a pole at m^2 and a cut at parallel and slightly below the real axis for $s > 4m^2$. For $n = 3$, the pole is the same and there is still a single cut but for $s > 9m^2$. The physical content of the theory must approach from above.

1.5 Path Integrals

In one-dimensional non-relativistic quantum mechanics, we have position and momentum operators $\widehat{x}$ and $\widehat{p}$ with $[\widehat{x}, \widehat{p}] = i$. In the Heisenberg picture, $\widehat{x}(t) = e^{iHt}\widehat{x}(0)e^{-iHt}$ and $\widehat{p}(t) = e^{iHt}\widehat{p}(0)e^{-iHt}$. The eigenstates of $\widehat{x}(t)$ ($\widehat{p}(t)$) are $|x,t\rangle = e^{i\widehat{H}t}|x,0\rangle$ ($|p,t\rangle = e^{i\widehat{H}t}|p,0\rangle$) and both collections form a complete set of states with $\langle x|x'\rangle = \delta(x-x')$, $\langle p|p'\rangle = 2\pi\delta(p-p')$, and $\langle x|p\rangle = e^{ixp}$. To compare momentum and position at different times $\langle p,t_1|x,t_2\rangle = \left\langle p,t_1\left|e^{iH(t_2-t_1)}\right|x,t_1\right\rangle \approx \left\langle p,t_1\left|1+i\widehat{H}\,(t_2-t_1)\right|x,t_1\right\rangle$ for small time intervals. We can regard H as a function of p and x and write

$$\begin{aligned}\langle p,t_1\,|1+iH\,(t_2-t_1)|\,x,t_1\rangle &= (1+iH(p,x)\,(t_2-t_1))\,e^{-ixp}\\ &= e^{-iH(p,x)(t_2-t_1)}e^{-ixp}\end{aligned}$$

Look at the amplitude of transition from $|x_0,t_0\rangle$ to $|x',t'\rangle$

$$A = \langle x',t'|x_0,t_0\rangle \equiv \psi(x',t')$$

which is a solution to the Schrödinger equation; at $t' = t_0$, $\psi(x',t_0) = \langle x'|x_0\rangle = \delta(x'-x_0)$. At a time $t_1$between t_0 and t', insert the identity

$$\int\frac{dp'}{2\pi}\,|p',t'\rangle\,\langle p',t'|\int dx_1\,|x_1,t_1\rangle\,\langle x_1,t_1|\int\frac{dp_1}{2\pi}\,|p_1,t_1\rangle\,\langle p_1,t_1|$$

to get

$$\begin{aligned}A &= \int\frac{dp'}{2\pi}dx_1\frac{dp_1}{2\pi}\,\langle x',t'|p',t'\rangle\,\langle p',t'|x_1,t_1\rangle\,\langle x_1,t_1|p_1,t_1\rangle\,\langle p_1,t_1|x_0,t_0\rangle\\ &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}e^{ix'p}\,\langle p',t'|x_1,t_1\rangle\,e^{ix_1p_1}\,\langle p_1,t_1|x_0,t_0\rangle\\ &= \int\frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}e^{ix'p'}e^{ix_1p_1}\,\langle p_1,t_1|x_0,t_0\rangle\,\langle p',t'|x_1,t_1\rangle\end{aligned}$$

but the remaining brackets are hard to evaluate since t_1 is not necessarily close to either t_0 or t'. We can iterate by choosing t_2 between t_1 and t' and inserting the identity

$$\int dx_2\,|x_2,t_2\rangle\,\langle x_2,t_2|\int\frac{dp_2}{2\pi}\,|p_2,t_2\rangle\,\langle p_2,t_2|$$

into the last bracket to get

$$
\begin{aligned}
A &= \int \frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}\frac{dx_2dp_2}{2\pi}e^{ix'p'}e^{ix_1p_1}\langle p_1,t_1|x_0,t_0\rangle\langle p',t'|x_2,t_2\rangle\langle x_2,t_2|p_2,p_2\rangle\langle p_2,t_2|x_1,t_1\rangle\\
&= \int \frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}\frac{dx_2dp_2}{2\pi}e^{ix'p'}e^{ix_1p_1}e^{ix_2p_2}\langle p_1,t_1|x_0,t_0\rangle\langle p_2,t_2|x_1,t_1\rangle\langle p',t'|x_2,t_2\rangle
\end{aligned}
$$

Now divide the interval $[t_0,t']$ into equally spaced intervals $[t_i,t_{i+1}]$ of length ε with $t_0<t_1<...<t_n<t'$, then multiple iteration yields

$$
\begin{aligned}
A &= \int \frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}...\frac{dx_ndp_n}{2\pi}e^{ix'p'}e^{ix_1p_1}..e^{ix_np_n}\langle p_1,t_1|x_0,t_0\rangle...\langle p_n,t_n|x_{n-1},t_{n-1}\rangle\langle p',t'|x_n,t_n\rangle\\
&= \int \frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}...\frac{dx_ndp_n}{2\pi}e^{ix'p'}e^{ix_1p_1}..e^{ix_np_n}e^{-iH(p_1,x_0)(t_1-t_0)}e^{-ix_0p_1}...e^{-iH\left(p',x_n\right)\left(t'-t_n\right)}e^{-ix_np'}\\
&= \int \frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}...\frac{dx_ndp_n}{2\pi}e^{i(x'-x_n)p'}e^{i(x_1-x_0)p_1}...e^{i(x_n-x_{n-1})p_n}e^{-i\varepsilon(H(p_1,x_0)+...+H(p_n,x_{n-1})+H(p',x_n))}
\end{aligned}
$$

where we have chosen n large enough to make ε small enough to apply the approximation of the first paragraph of this section. As $\varepsilon\to0$, the choices of $x_i=x(t_i)$ ultimately give a path $x(t)$; similarly we get a path $p(t)$. We can write $H(p_i,x_{i-1})=H(t_i)=H(t)$. $x_i-x_{i-1}=x(t_i)-x(t_{i-1})=\varepsilon\dot{x}(t_i)$. Then

$$
\begin{aligned}
A &= \int \frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}...\frac{dx_ndp_n}{2\pi}e^{i\varepsilon\dot{x}(t')p(t')}e^{i\varepsilon\dot{x}(t_1)p(t_1)}...e^{i\varepsilon\dot{x}(t_n)p(t_n)}e^{-i\varepsilon(H(t_1)+...+H(t_n)+H(t'))}\\
&= \int \frac{dp'}{2\pi}\frac{dx_1dp_1}{2\pi}...\frac{dx_ndp_n}{2\pi}e^{i\varepsilon((p\dot{x}-H)(t_1)+...(p\dot{x}-H)(t_n))}\\
&\to e^{i\int dt(p\dot{x}-H)}\\
&= e^{i\int dtL}\\
&= e^{iS}
\end{aligned}
$$

where $S=\int_{t_0}^{t'}dtL$ is the action. This a solution of the Schrödinger equation. Note that, in the limit $\varepsilon\to0$, we are actually integrating over all possible paths.

If $H=\frac{p^2}{2m}+V(x)$ and ε has a small imaginary part

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{dp_i}{2\pi}e^{i\varepsilon(p\dot{x}-H)} &= \int_{-\infty}^{\infty}\frac{dp_i}{2\pi}e^{i\varepsilon\left(p_i\dot{x}_i-\frac{p_i^2}{2m}-V(x)\right)}\\
&= \int_{-\infty}^{\infty}\frac{dp_i}{2\pi}e^{i\varepsilon\left(-\frac{(p_i-m\dot{x})^2}{2m}+\frac{m}{2}\dot{x}^2-V(x)\right)}\\
&= e^{i\varepsilon\left(\frac{m}{2}\dot{x}^2-V(x)\right)}\int_{-\infty}^{\infty}\frac{dp_i}{2\pi}e^{-i\varepsilon\frac{(p_i-m\dot{x})^2}{2m}}\\
&= e^{i\varepsilon L(x,\dot{x})}\sqrt{\frac{m}{2\pi i\varepsilon}}
\end{aligned}
$$

If we define $[dx]=\lim_{\varepsilon\to0}\prod\frac{dx_i}{\sqrt{2\pi i\varepsilon/m}}$, we have

$$
\langle x',t'|x_0,t_0\rangle=\int[dx]\,e^{i\int dtL(x,\dot{x})}
$$

To look at the classical limit, divide i by $\hbar$ and let $\hbar\to0$ so $\frac{i}{\hbar}\int dtL\to\infty$. If we look at S and $S+\delta S$, the difference will be very large (complete destructive interference) unless $\delta S=0$ which implies the classical equations of motion.

Green function
To look at the Green function $\langle x', t' | x(t_1) x(t_2) | x_0, t_0 \rangle$ with $t_0 < t_2 < t_1 < t'$, insert the identity $\int dx_1 | x_1, t_1 \rangle \langle x_1, t_1 | \int dx_2 | x_2, t_2 \rangle \langle x_2, t_2 |$ to get

$$\begin{aligned}
\langle x', t' | x(t_1) x(t_2) | x_0, t_0 \rangle &= \int dx_1 dx_2 \langle x', t' | x(t_1) | x_1, t_1 \rangle \langle x_1, t_1 | x_2, t_2 \rangle \langle x_2, t_2 | x(t_2) | x_0, t_0 \rangle \\
&= \int dx_1 dx_2 x_1 x_2 \langle x', t' | x_1, t_1 \rangle \langle x_1, t_1 | x_2, t_2 \rangle \langle x_2, t_2 | x_0, t_0 \rangle \\
&= \int dx_1 dx_2 x_1 x_2 \int_{x_1, t_1 \to x', t'} [dx] \, e^{iS} \int_{x_2, t_2 \to x_1, t_1} [dx] \, e^{iS} \int_{x_0, t_0 \to x_2, t_2} [dx] \, e^{iS} \\
&= \int_{x_0, t_0 \to x', t'} dx_1 dx_2 x(t_1) x(t_2) e^{iS}
\end{aligned}$$

Note that $x_1(t_1)$ and $x_2(t_2)$ are operators on the left hand side but are just numbers in the result; numbers commute but operators may not. This is not a contradiction since we started with a time ordered product. We generalize to

$$\langle x', t' | T \left(x(t_1) \ldots x(t_2) \right) | x_0, t_0 \rangle = \int [dx] \, x(t_1) \ldots x(t_n) e^{iS}$$

(operators on the left, numbers on the right). This is all non-relativistic quantum mechanics, but it is easily brought into QFT. (x, t) is a 4-vector, φ is an operator and $|\varphi(x)\rangle$ denotes the eigenstate of φ.

$$\langle \varphi(x') | T(\varphi(x_1) \ldots \varphi(x_n)) | \varphi(x_0) \rangle = \int [d\varphi] \, \varphi(x_1) \ldots \varphi(x_n) e^{iS}$$

is a non-perturbative way of calculating any Green function.

Example 8 *To compare to scattering results we have already obtained, let* $t_0 \to -\infty$ *and* $t' \to \infty$. *In any theory* $\mathbf{1} = |\Omega\rangle \langle \Omega| + \int dE \, |E\rangle \langle E|$ *where* Ω *is the vacuum.*

Let φ be a Klein-Gordon field. $H|0\rangle = 0$ and $H|k\rangle = \sqrt{k^2 + m^2}\, |k\rangle$. The spectrum of H has one discrete energy (the vacuum) and all others are continuous starting at m.

$$\begin{aligned}
|\varphi(x_0)\rangle &= \langle \Omega | \varphi(x_0) | \Omega \rangle + \int dE \, \langle E | \varphi(x_0) | E \rangle \\
&= e^{iE_0 t_0} | \varphi(\mathbf{x}_0, 0) \rangle + \int dE g(E) e^{iEt_0} \langle E | \varphi(\mathbf{x}_0, 0) | E \rangle
\end{aligned}$$

because $|\varphi(\mathbf{x}_0, t_0)\rangle = e^{iHt_0} |\varphi(\mathbf{x}_0, 0)\rangle$ and Ω and E are eigenvalues of H. The factor $g(E)$ is the degeneracy at E. We have expressed $|\varphi(x_0)\rangle$ as a function $f(t_0)$; it may be impossible to calculate, but we only need its value as $t_0 \to -\infty$. $f(t_0)$ is the Fourier transform of the function $g(E) \langle E | \varphi(\mathbf{x}_0, 0) | E \rangle$; the Riemann-Lesbegue lemma shows that the Fourier transform of a function approaches 0 as $t \to \pm\infty$. If E is large, the phase differences are large and, as $t_0 \to \pm\infty$, there is full destructive interference.

$$\begin{aligned}
|\varphi(x_0, t_0)\rangle &\to e^{-iE_0 t'} |\varphi(\mathbf{x}, 0)\rangle \\
\langle \varphi(x', t')| &\to e^{-iE_0 t'} |\varphi(\mathbf{x}', 0) |\Omega\rangle \langle \Omega|
\end{aligned}$$

$$\int [d\varphi] \, \varphi(x_1) \ldots \varphi(x_n) e^{iS} \to e^{iE_0(t_0 - t')} \langle \varphi(\mathbf{x}', 0) | \Omega \rangle \langle \Omega | \varphi(\mathbf{x}_0, 0) \rangle \, \varphi(\mathbf{x}', 0) \, \langle \Omega | T \left(\varphi(x_1) \ldots \varphi(x_n) \right) | \Omega \rangle$$

The last bracket is definitely a physical object, the propagator for $n = 2$ or the scattering amplitude. Thus

$$\langle \Omega | T(\varphi(x_1)...\varphi(x_n)) | \Omega \rangle = \frac{\int [d\varphi]\, \varphi(x_1)...\varphi(x_n) e^{iS}}{\int [d\varphi]\, e^{iS}}$$

In terms of diagrams, the components of $\int [d\varphi]\, e^{iS}$ have no external legs (the diagram is a bubble) because it is a vacuum to vacuum transition. Looking at $\int [d\varphi]\, \varphi(x_1)\varphi(x_2)e^{iS}$, the simplest graph is just a line segment, the K-G propagator. We have the connected graphs

The disconnected graphs contribute which can be written as (1+bubbles)(all graphs with no bubbles). When divided by (1+bubbles), we have the propagator. Bubbles represent vacuum to vacuum transitions, but physical processes involve excitations of the vacuum. Therefore bubbles do not contribute to physical reality although they presumably make up the cosmological constant.

Let $G^{(n)}(x_1, ..., x_n) \equiv \langle \Omega | T(\varphi(x_1)...\varphi(x_n)) | \Omega \rangle = \frac{\int [d\varphi] \varphi(x_1)...\varphi(x_n) e^{iS}}{\int [d\varphi] e^{iS}}$ (graphs without bubbles). Add an external source J so $S \to S + \int J\varphi$. $\int J\varphi$ is shorthand for $\int d^4x J(x)\varphi(x)$. Define

$$Z[J] = \frac{\int [d\varphi]\, e^{i(S+\int J\varphi)}}{\int [d\varphi]\, e^{iS}}$$

Note

$$e^{i\int J\varphi} = 1 + i\int J\varphi + \frac{i^2}{2!}\left(\int J\varphi\right)^2 + \frac{i^3}{3!}\left(\int J\varphi\right)^3 + ...$$

so

$$Z[J] = 1 + \frac{i\int [d\varphi] \int J\varphi e^{iS}}{\int [d\varphi]\, e^{iS}} + \frac{\frac{i^2}{2!}\int [d\varphi]\, e^{iS} \int J\varphi \int J\varphi}{\int [d\varphi]\, e^{iS}} + ...$$

Note that

$$\frac{\delta Z}{\delta J(x)}|_{J=0} = \frac{i\int [d\varphi]\, \varphi e^{iS}}{\int [d\varphi]\, e^{iS}} = G^{(1)}(x)$$

Similarly

$$\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}|_{J=0} = \frac{i\int [d\varphi]\, \varphi(x_1)\varphi(x_2) e^{iS}}{\int [d\varphi]\, e^{iS}} = G^{(2)}(x_1, x_2)$$

etc.