PHYSICS 611 - FALL 2018

Homework Set 5

due date: Wed., November 21, 2018

Show all your work for full credit.

Problem 5.1

The action for electromagnetism without charges is

$$S = -\frac{1}{4} \int d^4x \; F_{\mu\nu} F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(a) Derive the Maxwell equations in terms of the electric ($E_i = F_{0i}$) and magnetic ($B_i = \frac{1}{2}\epsilon_{ijk} F^{jk}$) fields.

(b) Construct the conserved energy-momentum tensor $T^{\mu\nu}$.

(c) Notice that $T^{\mu\nu}$ does not lead to the standard expressions for energy and momentum densities,

$$\varepsilon = \frac{1}{2}(E^2 + B^2) \quad , \quad \vec{S} = \vec{E} \times \vec{B}$$

To remedy this, define the modified energy-momentum tensor

$$\widehat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda (F^{\mu\lambda} A^\nu)$$

Show that $\widehat{T}^{\mu\nu}$ is a symmetric and conserved tensor leading to the standard expressions for energy and momentum.

Problem 5.2

Consider a massive vector field whose Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$

The field equations may be written in the form

$$\partial_\mu \partial^\mu A_\nu + m^2 A_\nu = 0$$

together with the *constraint*

$$\partial_\mu A^\mu = 0$$

(a) Show that the most general solution may be written in the form

$$A_\mu = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{r=1}^{3} \left\{ a^{(r)}(\vec{k}) e_\mu^{(r)}(\vec{k}) e^{-ik\cdot x} + a^{(r)\dagger}(\vec{k}) e_\mu^{(r)*}(\vec{k}) e^{-k\cdot x} \right\}$$

By normalizing the polarization vectors to

$$e^{(r)}(\vec{p}) \cdot e^{(s)*}(\vec{p}) = -\delta^{rs}$$

show that they satisfy

$$p \cdot e^{(r)}(\vec{p}) = 0 \ \ , \ \ \sum_{r=1}^{3} e_\mu^{(r)}(\vec{p}) e_\nu^{(s)*}(\vec{p}) = A\eta_{\mu\nu} + Bp_\mu p_\nu$$

and find A and B.

(b) Expand the "electric field"

$$\vec{E} = \partial_0 \vec{A} - \vec{\nabla} A_0$$

in modes. Show that the commutation relations

$$[a^{(r)}(\vec{p}), a^{(s)\dagger}(\vec{p}')] = \delta^{rs}\delta^3(\vec{p} - \vec{p}')$$

lead to the equal-time commutators ($i, j = 1, 2, 3$)

$$[A_i(\vec{x}, t), E_j(\vec{y}, t)] = i\delta_{ij}\delta^3(\vec{x} - \vec{y})$$

$$[A_i(\vec{x}, t), A_j(\vec{y}, t)] = [E_i(\vec{x}, t), E_j(\vec{y}, t)] = 0$$

(c) Show that the Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 + \frac{1}{2}m^2\vec{A}^2 + \frac{1}{2m^2}(\vec{\nabla} \cdot \vec{E})^2$$

and write the Hamiltonian

$$H = \int d^3x \mathcal{H}$$

in terms of modes.

Problem 5.3

(a) Derive the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \vec{A} \cdot \vec{J}$$

for electromagnetism from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

working in the axial gauge

$$A_3 = 0$$

Write $\mathcal{H}$ explicitly in terms of two degrees of freedom.

(b) Show that in the *Coulomb gauge*,

$$\vec{\nabla} \cdot \vec{A} = 0$$

the Lagrangian density leads to the Hamiltonian

$$H = \int d^3x (\frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 + \vec{J} \cdot \vec{A}) + \frac{1}{8\pi} \int d^3x \int d^3y \frac{J^0(x) J^0(y)}{|\vec{x} - \vec{y}|}$$

What is the physical meaning of the last term contributing to H?

Problem 5.4

For a complex scalar field ϕ with Lagrangian density

$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi$$

show that the divergence of the stress-energy tensor

$$T_\mu^\nu = D_\mu \phi (D^\nu \phi)^* + (D_\mu \phi)^* D^\nu \phi - \delta_\mu^\nu [(D_\alpha \phi)^* D^\alpha \phi - m^2 \phi^* \phi]$$

is given by

$$\partial_\nu T_\mu^\nu = F_{\mu\nu} J^\nu$$

Deduce the Lorentz force law

$$\vec{F} = \frac{d\vec{P}}{dt} = \int d^3x (\rho \vec{E} + \vec{J} \times \vec{B})$$