PHYSICS 611 - FALL 2018

Homework Set 4

due date: Mon., October 29, 2018

Show all your work for full credit.

Problem 4.1

The left-handed *Weyl* spinor ψ_L is *massless*. We wish to add a mass term to the Lagrangian, turning ψ_L into a *Majorana* spinor.

(a) Show that

$$\psi_L^T \sigma^2 \psi_L$$

is a Lorentz scalar. Show that, if the components of ψ_L are commuting numbers, then

$$\psi_L^T \sigma^2 \psi_L = 0$$

identically. Therefore, the components must be *anti*-commuting (*Grassmann*) numbers.

(b) Modify the Weyl Lagrangian by adding a term proportional to the *real* quantity $i\psi_L^T \sigma^2 \psi_L + c.c.$,

$$\mathcal{L}_M = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + \tfrac{1}{2} im(\psi_L^T \sigma^2 \psi_L - c.c.)$$

(Majorana Lagrangian density). Show that the new field equation is

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = im\sigma^2 \psi_L^*$$

(c) Derive the Majorana Lagrangian from the Dirac Lagrangian by choosing the Dirac field to be of the form

$$\psi = \begin{pmatrix} \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix}$$

(d) Quantize the Majorana theory by imposing anti-commutation relations

$$\{ \; \psi_{La}(\vec{x},t) \; , \; \psi_{Lb}^\dagger(\vec{y},t) \; \} = \delta_{ab} \delta^3(\vec{x}-\vec{y})$$

Construct the Hamiltonian and express it in terms of creation and annihilation operators.

Problem 4.2

Supersymmetry. Consider the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + F^* F$$

This is a system consisting of a complex scalar massless field ϕ, a left-handed spinor ψ_L and the auxiliary complex scalar field F whose equation of motion is trivial,

$$F = 0$$

Show that under the **(supersymmetry)** transformation

$$\begin{aligned} \delta\phi &= -i\epsilon^T\sigma^2\psi_L \\ \delta\psi_L &= \epsilon F + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^* \\ \delta F &= -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L \end{aligned}$$

where ϵ is a two-component spinor (whose components are *Grassmann* (anti-commuting) numbers), the Lagrangian density changes by a total divergence.
Find the Noether current for supersymmetry.

Problem 4.3
Supersymmetry with mass. Show that the system of a massive complex field ϕ and a Majorana fermion ψ_L given by the Lagrangian density

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L - m^2\phi^*\phi + \tfrac{1}{2}im(\psi_L^T\sigma^2\psi_L - c.c.)$$

possesses the supersymmetry

$$\begin{aligned} \delta\phi &= -i\epsilon^T\sigma^2\psi_L \\ \delta\psi_L &= m\epsilon\phi + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^* \end{aligned}$$

Note that this is not a symmetry if the two fields ϕ and ψ_L have different masses. Thus, supersymmetry forces bosons and fermions to have the same mass, which shows that it cannot be a symmetry of Nature (at present).

Problem 4.4
Recall

$$T_S^{\mu\nu} = \bar{\psi}S^{\mu\nu}\mathcal{P}_R\psi \ , \ \ T_A^{\mu\nu} = \bar{\psi}S^{\mu\nu}\mathcal{P}_L\psi \ , \ \ S^{\mu\nu} = \frac{i}{4}[\gamma^\mu \ , \ \gamma^\nu] \ , \ \ \mathcal{P}_{R,L} = \frac{1}{2}(1 \pm \gamma_5)$$

Define the *dual* of a tensor $T^{\mu\nu}$ by

$$\tilde{T}^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}T^{\rho\sigma}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in four dimensions (totally anti-symmetric in its indices with $\epsilon^{0123} = +1$).

(a) Show that $T_S^{\mu\nu}$ is *self-dual* ($T_S^{\mu\nu} = \tilde{T}_S^{\mu\nu}$) whereas $T_A^{\mu\nu}$ is *anti-self-dual* ($T_A^{\mu\nu} = -\tilde{T}_A^{\mu\nu}$).

(b) Find the action of Lorentz transformations on $T_S^{\mu\nu}$ and $T_A^{\mu\nu}$ and show that they transform in the $(1,0)$ and $(0,1)$ representations of the Lorentz group, respectively.