PHYSICS 611 - FALL 2018
Homework Set 3

due date: Fri., October 12, 2018

Show all your work for full credit.

Problem 3.1

Consider a complex scalar field ψ with Lagrangian density

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - \lambda \left(\psi^* \psi - \frac{v^2}{2} \right)^2$$

(a) Show that the Lagrangian is symmetric under the $U(1)$ transformation

$$\psi \to e^{i\theta} \psi$$

and find the corresponding Noether current J^μ and charge Q.

(b) Express the field ϕ in terms of real fields ρ and σ as

$$\psi = \frac{1}{\sqrt{2}} \rho \, e^{i\sigma}$$

Find the conjugate momenta π_ρ and π_σ and express the Hamiltonian H and charge Q in terms of ρ, σ and their conjugate momenta.

(c) Define the ground state (vacuum) by

$$H|0\rangle = 0$$

Explain why

$$\langle 0|\rho|0\rangle = v$$

(d) Using standard commutation rules

$$[\rho(\vec{x}, t) \ , \ \pi_\rho(\vec{y}, t)] = [\sigma(\vec{x}, t) \ , \ \pi_\sigma(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y})$$

calculate the commutators $[Q, H]$, $[Q, \rho]$, and $[Q, \sigma]$.

Hence show that there exists a one-parameter set of ground states

$$|\theta\rangle = e^{-i\theta Q}|0\rangle$$

for which

$$H|\theta\rangle = 0 \quad , \quad \langle\theta|\rho|\theta\rangle = v \quad , \quad \langle\theta|\sigma|\theta\rangle = \theta$$

Problem 3.2

Consider the two-particle non-relativistic state

$$|\Psi(t)\rangle = \int d^3x_1 \int d^3x_2 \Phi(\vec{x}_1, \vec{x}_2; t)|\vec{x}_1, \vec{x}_2\rangle$$

where

$$|\vec{x}_1, \vec{x}_2\rangle = \psi^\dagger(\vec{x}_1)\psi^\dagger(\vec{x}_2)|0\rangle$$

Show that this state obeys the Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$$

where

$$H = -\frac{1}{2m}\int d^3x\psi^\dagger\nabla^2\psi + \frac{1}{2}\int d^3x\int d^3x'\psi^\dagger(\vec{x})\psi^\dagger(\vec{x}')V(\vec{x},\vec{x}')\psi(\vec{x}')\psi(\vec{x})$$

if and only if the wavefunction $\Phi(\vec{x}_1, \vec{x}_2; t)$ obeys the Schrödinger equation

$$i\frac{\partial\Phi}{\partial t} = -\frac{1}{2m}(\nabla^2_{x_1} + \nabla^2_{x_2})\Phi + V(\vec{x}_1, \vec{x}_2)\Phi$$

Problem 3.3

Consider a superfluid with contact interactions ($V(\vec{x}, \vec{x}') = g\delta^3(\vec{x} - \vec{x}')$) governed by the Hamiltonian

$$H = -\frac{1}{2m}\int d^3x\psi^\dagger\nabla^2\psi + \frac{g}{2}\int d^3x[\psi^\dagger(\vec{x})\psi(\vec{x})]^2$$

(a) Using the Heisenberg equation

$$\partial_\tau\mathcal{A} = [H - \mu N\ ,\ \mathcal{A}]\ \ ,\quad N = \int d^3x\psi^\dagger(\vec{x})\psi(\vec{x})$$

calculate $\partial_\tau\psi$ and $\partial_\tau\psi^\dagger$.

(b) Express ψ in terms of real fields ρ and σ as

$$\psi = \sqrt{\rho}\, e^{i\sigma}$$

Deduce equations for σ and ρ and find the dispersion relations for these two fields.

(c) Vortex. Using cylindrical coordinates (r, ϕ, z), assume that

$$\rho = f(r)\ \ ,\quad \sigma = \phi$$

where $f(r) \to C$ away from the center of the vortex ($r \to \infty$). Notice that there is no dependence on z.

Find an equation for $f(r)$ and solve it numerically (notice that $C = \mu/g$ – why?). Plot $f(r)$.

Can you find the approximate analytic form of $f(r)$ near the center ($r = 0$), and far away from it ($r \to \infty$)?