Higgs decay to two gluons

Eleftherios Moschandreou

(a) **The decay rate**
An unstable particle has a probability to decay to a certain number of final states during a given time interval. The process of decaying is described by the decay rate as

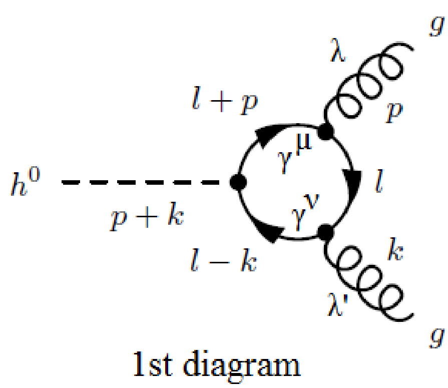
$$\Gamma \equiv \frac{Number\ of\ decays\ per\ unit\ time}{Number\ of\ undecayed\ particles\ present}$$

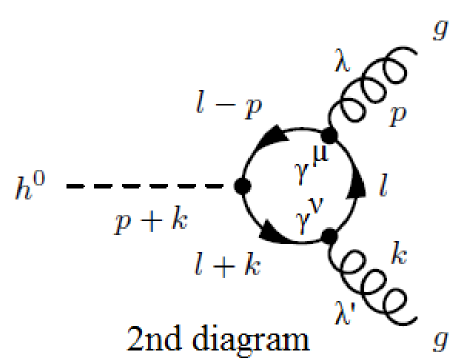
One of the channels, the scalar Higgs decays through is $h \to 2g$ (two gluons) In the Higgs rest frame, the decay rate in the neighborhood of the final momenta $\vec{p}$ and $\vec{k}$ is:

$$d\Gamma = \frac{1}{2m_A}\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\cdot\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\left|M(m_h \to p,k)\right|^2 (2\pi)^4\ \delta^{(4)}(m_h - p - k)$$

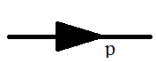
where $M(m_h \to p, k)$ is the the decay amplitude, a quantity that describes the interactions that take place during the decay.

(b) **From the Feynman diagram to the decay amplitude.**
The decay involves a fermion (quark) loop. For each quark there are two possible Feynman diagrams that contribute to the decay amplitude.



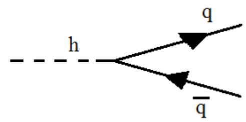


For the first diagram. We recognize three vertices and three fermion propagators:
Each fermion propagator contributes:



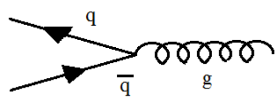
$$= \frac{i(\tilde{p} + m)}{p^2 - m^2 + i\epsilon}$$

The Higgs-fermion-antifermion (quark-antiquark) vertex contribution to the amplitude is:



$$= -i \ \frac{m_q}{\upsilon} \delta^{ab}$$

υ is the vacuum expectation value for the Higgs and the Kronecker delta ensures color conservation. The gluon-quark-anti quark vertex contribution is:



$$= ig\gamma^{\mu}t^{\alpha}$$

The diagram contailns a loop momentum l . We need to intergrate over this loop momentum. So finally the matrix element can be written as:

$$\begin{aligned} M &= -i \ \frac{m_q}{\upsilon} \ (-1) \times \\ & \int \frac{d^4 l}{(2\pi)^4} \ Tr[\frac{i(\widetilde{l} - \widetilde{k} + m_q)}{(l-k)^2 - m_q^2 + i\epsilon} \ ig_s\gamma^{\nu} \ \frac{i(\widetilde{l} + m_q)}{l^2 - m_q^2 + i\epsilon} \ ig_s\gamma^{\mu} \ \frac{i(\widetilde{l} + \widetilde{p} + m_q)}{(l+p)^2 - m_q^2 + i\epsilon} \times \\ & Tr[t^b t^a] \cdot \epsilon_{\nu}^{\lambda' *}(k) \cdot \epsilon_{\mu}^{\lambda *}(p) \end{aligned}$$

where $\epsilon_{\nu}^{\lambda' *}(k)$, $\epsilon_{\mu}^{\lambda *}(p)$ the polarization vectors of the outcoming gluons.

(c) **Calculating the decay amplitude:**
newline The numerator N of the loop momentum integral contains $3 \times 2 \times 3 = 18$ terms. Using trace technology we can eventually simplify it:

$$\begin{aligned} Tr[N] &= (i)^5 \cdot g_s^2 \cdot Tr[(\widetilde{l} - \widetilde{k} + m_q)\gamma^{\nu}(\widetilde{l} + m_q)\gamma^{\mu}(\widetilde{l} + \widetilde{p} + m_q)] \\ &= i \cdot g_s^2 \cdot Tr[(\widetilde{l} - \widetilde{k} + m_q) \cdot (\gamma^{\nu}\widetilde{l} + \gamma^{\nu} m_q)(\gamma^{\mu}\widetilde{l} + \gamma^{\mu}\widetilde{p} + \gamma^{\mu} m_q)] \\ &= i \cdot g_s^2 \cdot Tr\{(\widetilde{l} - \widetilde{k} + m_q) \cdot [\gamma^{\nu}\widetilde{l}(\gamma^{\mu}\widetilde{l} + \gamma^{\mu}\widetilde{p} + \gamma^{\mu} m_q) + \gamma^{\nu} m_q(\gamma^{\mu}\widetilde{l} + \gamma^{\mu}\widetilde{p} + \gamma^{\mu} m_q)]\} \\ &= i \cdot g_s^2 \cdot Tr\{(\widetilde{l} - \widetilde{k} + m_q) \cdot [\gamma^{\nu}\widetilde{l}\gamma^{\mu}\widetilde{l} + \gamma^{\nu}\widetilde{l}\gamma^{\mu}\widetilde{p} + \gamma^{\nu}\widetilde{l}\gamma^{\mu} m_q \\ & \quad + \gamma^{\nu} m_q \gamma^{\mu}\widetilde{l} + \gamma^{\nu} m_q \gamma^{\mu}\widetilde{p} + \gamma^{\nu} m_q \gamma^{\mu} m_q]\} \\ &= i \cdot g_s^2 \cdot Tr\{\widetilde{l} \cdot [\gamma^{\nu}\widetilde{l}\gamma^{\mu}\widetilde{l} + \gamma^{\nu}\widetilde{l}\gamma^{\mu}\widetilde{p} + \gamma^{\nu}\widetilde{l}\gamma^{\mu} m_q + \gamma^{\nu} m_q \gamma^{\mu}\widetilde{l} + \gamma^{\nu} m_q \gamma^{\mu}\widetilde{p} + \gamma^{\nu} m_q \gamma^{\mu} m_q] \\ & \quad - \widetilde{k}[\gamma^{\nu}\widetilde{l}\gamma^{\mu}\widetilde{l} + \gamma^{\nu}\widetilde{l}\gamma^{\mu}\widetilde{p} + \gamma^{\nu}\widetilde{l}\gamma^{\mu} m_q + \gamma^{\nu} m_q \gamma^{\mu}\widetilde{l} + \gamma^{\nu} m_q \gamma^{\mu}\widetilde{p} + \gamma^{\nu} m_q \gamma^{\mu} m_q] \\ & \quad + m_q[\gamma^{\nu}\widetilde{l}\gamma^{\mu}\widetilde{l} + \gamma^{\nu}\widetilde{l}\gamma^{\mu}\widetilde{p} + \gamma^{\nu}\widetilde{l}\gamma^{\mu} m_q + \gamma^{\nu} m_q \gamma^{\mu}\widetilde{l} + \gamma^{\nu} m_q \gamma^{\mu}\widetilde{p} + \gamma^{\nu} m_q \gamma^{\mu} m_q]\} \end{aligned}$$

To simplify things we drop all the terms with odd number of gamma matrices using the property

$$Tr[odd \ number \ of \ \gamma' s] = 0$$

So

$$\begin{aligned} Tr[N] &= i \cdot g_s^2 \cdot Tr\{\widetilde{l}\gamma^\nu \widetilde{l}\gamma^\mu m_q + \widetilde{l}\gamma^\nu m_q \gamma^\mu \widetilde{l} + \widetilde{l}\gamma^\nu m_q \gamma^\mu \widetilde{p} \\ &-\widetilde{k}\gamma^\nu \widetilde{l}\gamma^\mu m_q - \widetilde{k}\gamma^\nu m_q \gamma^\mu \widetilde{l} - \widetilde{k}\gamma^\nu m_q \gamma^\mu \widetilde{p} \\ &+m_q[\gamma^\nu \widetilde{l}\gamma^\mu \widetilde{l} + \gamma^\nu \widetilde{l}\gamma^\mu \widetilde{p} + \gamma^\nu m_q \gamma^\mu m_q]\} \end{aligned}$$

Grouping some terms:

$$\begin{aligned} Tr[N] &= i \cdot g_s^2 \cdot m_q \cdot Tr[\widetilde{l}\gamma^\nu \{\widetilde{l}, \gamma^\mu\} + \{\widetilde{l}, \gamma^\nu\}\gamma^\mu \widetilde{p} \\ &-\widetilde{k}\gamma^\nu \{\widetilde{l}, \gamma^\mu\} - \widetilde{k}\gamma^\nu \gamma^\mu \widetilde{p} \\ &+\gamma^\nu \widetilde{l}\gamma^\mu \widetilde{l}] + i \cdot g_s^2 \cdot m_q^3 \cdot Tr[\gamma^\nu \gamma^\mu] \end{aligned}$$

Furthermore using

$$Tr[\gamma^\nu \gamma^\mu] = 4g^{\mu\nu}$$

and

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

For the first term in the square bracket the calculations yield:

$$\begin{aligned} Tr[\widetilde{l}\gamma^\nu \{\widetilde{l}, \gamma^\mu\}] &= Tr[\widetilde{l}\gamma^\nu \{l_\alpha \gamma^\alpha, \gamma^\mu\}] = Tr[\widetilde{l}\gamma^\nu l_\alpha \{\gamma^\alpha, \gamma^\mu\}] = Tr[\widetilde{l}\gamma^\nu l_\alpha 2g^{\alpha\mu}] \\ &= Tr[\widetilde{l}\gamma^\nu l^\mu 2] = Tr[l_\rho \gamma^\rho \gamma^\nu l^\mu 2] \\ &= 2l^\mu l_\rho \cdot Tr[\gamma^\rho \gamma^\nu] = 8l^\mu l_\rho g^{\rho\nu} = 8l^\mu l^\nu \end{aligned}$$

So the trace can now be written:

$$\begin{aligned} Tr[\mathrm{N}] &= i \cdot g_s^2 \cdot m_q \cdot [8l^\mu l^\nu + 8l^\nu p^\mu - 8k^\nu l^\mu + Tr[-k_\alpha \gamma^\alpha \gamma^\nu \gamma^\mu p_\beta \gamma^\beta + \gamma^\nu l_\alpha \gamma^\alpha \gamma^\mu l_\beta \gamma^\beta]] \\ &+i \cdot g_s^2 \cdot m_q^3 \cdot 4g^{\mu\nu} \\ &= i \cdot g_s^2 \cdot m_q \cdot [8l^\mu l^\nu + 8l^\nu p^\mu - 8k^\nu l^\mu + Tr[-k_\alpha p_\beta \gamma^\alpha \gamma^\nu \gamma^\mu \gamma^\beta + l_\alpha l_\beta \gamma^\nu \gamma^\alpha \gamma^\mu \gamma^\beta]] \\ &+i \cdot g_s^2 \cdot m_q^3 \cdot 4g^{\mu\nu} \end{aligned}$$

Also an other identity that should be used:

$$Tr[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$$

Giving:

$$\begin{aligned} Tr[\mathrm{N}] &= i \cdot g_s^2 \cdot m_q \cdot [8l^\mu l^\nu + 8l^\nu p^\mu - 8k^\nu l^\mu - 4k_\alpha p_\beta (g^{\alpha\nu} g^{\mu\beta} - g^{\alpha\mu} g^{\nu\beta} + g^{\alpha\beta} g^{\mu\nu}) \\ &+4l_\alpha l_\beta (g^{\nu\alpha} g^{\mu\beta} - g^{\nu\mu} g^{\alpha\beta} + g^{\nu\beta} g^{\mu\alpha})] + i \cdot g_s^2 \cdot m_q^3 \cdot 4g^{\mu\nu} \\ &= i \cdot g_s^2 \cdot m_q \cdot [8l^\mu l^\nu + 8l^\nu p^\mu - 8k^\nu l^\mu - 4k^\nu p^\mu + 4k^\mu p^\nu - 4 \cdot kp \cdot g^{\mu\nu} \\ &+4l^\nu l^\mu - 4l^2 g^{\nu\mu} + 4l^\mu l^\nu] + i \cdot g_s^2 \cdot m_q^3 \cdot 4g^{\mu\nu} \\ &= i \cdot g_s^2 \cdot m_q \cdot [16l^\mu l^\nu + 8l^\nu p^\mu - 8k^\nu l^\mu - 4k^\nu p^\mu + 4k^\mu p^\nu - 4 \cdot kp \cdot g^{\mu\nu} - 4l^2 g^{\nu\mu}] \\ &+i \cdot g_s^2 \cdot m_q^3 \cdot 4g^{\mu\nu} \\ &= i \cdot g_s^2 \cdot 4m_q \cdot [4(l^\mu l^\nu - l^2 g^{\nu\mu}) + 2(l^\nu p^\mu - k^\nu l^\mu) - k^\nu p^\mu + k^\mu p^\nu + (m^2 - \cdot kp) \cdot g^{\mu\nu}] \end{aligned}$$

The next task is to treat the denominator:

$$\frac{1}{((l-k)^2 - m_q^2 + i\epsilon)(l^2 - m_q^2 + i\epsilon)((l+p)^2 - m_q^2 + i\epsilon)}$$

The integration is over l. We see that we have a product $l^2 \times l^2 \times l^2$ in terms of powers of l . So we can rewrite the denominator in the form of a quadratic polynomial raised in the third power. This procedure is called Feynman parametrization.
We start from observing that the integral

$$\int_{x=0}^{x=1} dx \frac{1}{[xA + (1-x)B]^2} = \int_{x=0}^{x=1} \int dxdy \cdot \delta(x + y - 1) \frac{1}{[xA + yB]^2}$$

is easily calculated and is equal with $\frac{1}{AB}$. For three terms in the denominator the appropriate formula is:

$$\frac{1}{ABC} = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} dxdydz \cdot \delta(x + y + z - 1) \frac{2!}{[xA + yB + zC]^3}$$

With $A = ((l-k)^2 - m_q^2 + i\epsilon)$, $B = (l^2 - m_q^2 + i\epsilon)$, $C = ((l+p)^2 - m_q^2 + i\epsilon)$:

$$\begin{aligned}
\frac{1}{ABC} &= \int_{x,y,z=0}^{x=1} dxdydz \frac{2\delta(x+y+z-1)}{[xA + yB + zC]^3} \\
&= \int_{x,y,z=0}^{x=1} dxdydz \frac{2\delta(x+y+z-1)}{[x((l-k)^2 - m_q^2 + i\epsilon) + y(l^2 - m_q^2 + i\epsilon) + z((l+p)^2 - m_q^2 + i\epsilon)]^3} \\
&= \int_{x,y,z=0}^{x=1} \frac{2\delta(x+y+z-1) \quad dxdydz}{[(x+y+z)l^2 + xk^2 - 2xlk + 2zlp + zp^2 - (x+y+z)m_q^2 + (x+y+z)i\epsilon)]^3}
\end{aligned}$$

The delta function imposes the constraint $x + y + z = 1$ so

$$\frac{1}{ABC} = \int_{x=0}^{1} \int_{z=0}^{z=1-x} dxdz \frac{2}{[l^2 + xk^2 + 2l(zp - xk) + zp^2 - m_q^2 + i\epsilon)]^3}$$

We complete the square by adding and subtracting the term $(zp - xk)^2$

$$\begin{aligned}
\frac{1}{ABC} &= \int_{x=0}^{1} \int_{z=0}^{1-x} \frac{2dxdz}{[\underline{l^2} + xk^2 + \underline{2l(zp - xk)} + \underline{(zp - xk)^2} - (zp - xk)^2 + zp^2 - m_q^2 + i\epsilon)]^3} \\
&= \int_{x=0}^{1} \int_{z=0}^{1-x} \frac{2dxdz}{[(l + zp - xk)^2 + xk^2 - (zp - xk)^2 + zp^2 - m_q^2 + i\epsilon)]^3} \\
&= \int_{x=0}^{1} \int_{z=0}^{1-x} \frac{2dxdz}{[(l + zp - xk)^2 + xk^2 - (z^2p^2 + x^2k^2 - 2xzpk) + zp^2 - m_q^2 + i\epsilon)]^3}
\end{aligned}$$

k^2 and p^2 are zero since they express the masses of the gluons which are massless. So the denominator becomes:

$$\frac{1}{ABC} = \int\limits_{x=0}^{1}\int\limits_{z=0}^{1-x} dxdz \frac{2}{[(l+(zp-xk))^2+2xzkp-m_q^2+i\epsilon)]^3}$$

$p+k$ is the four momentum of the Higgs scalar and p and k are the four momenta of the gluons. Also the square of the four momentum expresses the mass squared. So we have

$$\begin{aligned} (p+k)^2 &= m_h^2 \\ p^2+k^2+2pk &= m_h^2 \\ 2pk &= m_h^2 \end{aligned}$$

Finally by setting $l+(zp-xk)=\ell$ and $xzm_h^2-m_q^2+i\epsilon=\Delta$ the denominator becomes

$$\frac{1}{ABC} = \int\limits_{x=0}^{x=1}\int dxdz \frac{2}{[\ell^2+\Delta)]^3}$$

Going back to the numerator:

$$N = i\cdot g_s^2\cdot 4m_q\cdot [\overbrace{4(l^\mu l^\nu - l^2 g^{\nu\mu})+2(l^\nu p^\mu - k^\nu l^\mu)} -k^\nu p^\mu + k^\mu p^\nu + (m_q^2 - \cdot kp)\cdot g^{\mu\nu}]$$

We will call the term under the overbrace *N(L)*, denoting the terms of the numerator that contain the loop momentum l. These terms will be shifted to the the new momentum $\ell = l+zp-xk \to l = \ell - zp + xk$

$$\begin{aligned} N(L) &= 4(l^\mu l^\nu - l^2 g^{\nu\mu}) + 2(l^\nu p^\mu - k^\nu l^\mu) \\ &= 4[(\ell - zp + xk)^\mu(\ell - zp + xk)^\nu - (\ell - zp + xk)^2 g^{\mu\nu}] \\ &\quad +2((\ell - zp + xk)^\nu p^\mu - k^\nu(\ell - zp + xk)^\mu) \\ &= 4[(\ell^\mu - zp^\mu + xk^\mu)(\ell^\nu - zp^\nu + xk^\nu) - (\ell^2 + 2\ell(xk - zp) + x^2k^2 + z^2p^2 + 2xzkp)g^{\mu\nu}] \\ &\quad +2[(\ell^\nu - zp^\nu + xk^\nu)p^\mu - k^\nu(\ell^\mu - zp^\mu + xk^\mu)] = \end{aligned}$$

Droping k^2 and p^2:

$$\begin{aligned} 4[(\ell^\mu\ell^\nu + \ell^\mu(xk^\nu - zp^\nu) + \ell^\nu(xk^\mu - zp^\mu) + (xk^\nu - zp^\nu)(xk^\mu - zp^\mu) \\ -(\ell^2 + 2\ell(xk - zp) - 2xzkp)g^{\mu\nu})] + \\ +2(\ell^\nu p^\mu - zp^\nu p^\mu + xk^\nu p^\mu - k^\nu\ell^\mu + zk^\nu p^\mu - xk^\nu k^\mu) \end{aligned}$$

To simplify things even more we observe that we have some linear terms with respect to ℓ . For example the term $\ell^\mu(xk^\nu - zp^\nu)$ inside the integral will be

$$\int\limits_{x=0}^{x=1}\int\limits_{z=0}^{1-x} dxdz \int d^4l \frac{2\ell^\mu(xk^\nu - zp^\nu)}{[\ell^2+\Delta)]^3}$$

Since the interval of integration is symmetric and the integrand is an odd function, the integral vanishes. So we can drop all linear (ℓ) terms from the numerator, which now becomes.

$$\begin{aligned} N(L) &= 4[(\ell^\mu \ell^\nu + (xk^\nu - zp^\nu)(xk^\mu - zp^\mu)] - [\ell^2 - 2xzkp)g^{\mu\nu}] \\ &\quad +2(-zp^\nu p^\mu + xk^\nu p^\mu + zk^\nu p^\mu - xk^\nu k^\mu) \\ &= 4[(\ell^\mu \ell^\nu + x^2 k^\nu k^\mu - zxk^\nu p^\mu - xzp^\nu k^\mu + z^2 p^\nu p^\mu] \\ &\quad -(\ell^2 - 2xzkp)g^{\mu\nu}) + 2(-zp^\nu p^\mu + xk^\nu p^\mu + zk^\nu p^\mu - xk^\nu k^\mu) \\ &= 4\ell^\mu \ell^\nu - \ell^2 g^{\mu\nu} + (4z^2 - 2z)p^\nu p^\mu + (4x^2 - 2x)k^\nu k^\mu \\ &\quad -4xzp^\nu k^\mu + (2x + 2z - 4zx)k^\nu p^\mu + xz \cdot m_h^2 g^{\mu\nu} \end{aligned}$$

Now we have everything that we need to calculate the integral

$$\int \frac{d^4 l}{(2\pi)^4}\ Tr[\frac{i(\widetilde{l} - \widetilde{k} + m_q)}{(l-k)^2 - m_q^2 + i\epsilon}\ ig_s\gamma^\nu\ \frac{i(\widetilde{l} + m_q)}{l^2 - m_q^2 + i\epsilon}\ ig_s\gamma^\mu\ \frac{i(\widetilde{l} + \widetilde{p} + m_q)}{(l+p)^2 - m_q^2 + i\epsilon}]$$

Which now becomes

$$\int_{x=0}^{1} \int dxdz \int \frac{d^4\ell}{(2\pi)^4} ig_s^2 4m_q \{ \frac{2 \cdot [4\ell^\mu \ell^\nu - \ell^2 g^{\mu\nu} + (4z^2 - 2z)p^\nu p^\mu + (4x^2 - 2x)k^\nu k^\mu - 4xzp^\nu k^\mu]}{[\ell^2 + \Delta)]^3} + \\ + \frac{2(2x + 2z - 4zx)k^\nu p^\mu + 2xz \cdot m_h^2 g^{\mu\nu} - 2k^\nu p^\mu + 2k^\mu p^\nu + 2(m^2 - kp) \cdot g^{\mu\nu}}{[\ell^2 + \Delta)]^3} \}$$

With $pk = \frac{1}{2} m_h^2$.

$$\int_{x=0}^{1} \int dxdz \int \frac{d^4\ell}{(2\pi)^4} \cdot ig_s^2 \cdot 4m_q \{ \frac{2 \cdot [\overbrace{4\ell^\mu \ell^\nu - \ell^2 g^{\mu\nu}} + (4z^2 - 2z)p^\nu p^\mu + (4x^2 - 2x)k^\nu k^\mu]}{[\ell^2 + \Delta)]^3} \\ + \frac{2[(1 - 4xz)p^\nu k^\mu + (2x + 2z - 1 - 4zx)k^\nu p^\mu + (m_q^2 - \frac{1}{2}m_h^2 + xz \cdot m_h^2)g^{\mu\nu}]}{[\ell^2 + \Delta)]^3} \}$$

Lets deal with the terms under the overbrace, containing ℓ^2 and $\ell^\mu \ell^\nu$ first.

$$I(\ell) = \int \frac{d^4\ell}{(2\pi)^4} \left(\frac{8\ell^\mu \ell^\nu}{[\ell^2 + \Delta)]^3} - \frac{2\ell^2 g^{\mu\nu}}{[\ell^2 + \Delta)]^3} \right)$$

First the integral $\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{[\ell^2 + \Delta)]^3}$ will vanish if $\mu \neq \nu$ this can be seen through the example

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^1 \ell^2}{[\ell^2 + \Delta)]^3} = \frac{1}{(2\pi)^4} \int d^3\ell d^4\ell \int d^2\ell [\int d^1\ell \frac{\ell^1}{[\ell^2 + \Delta)]^3}]\ell^2$$

The integral in the brackets has an odd integrand in a symmetric integration interval so it vanishes. So only the terms where $\mu = \nu$ really matter. A tensorial quantity that has the $\mu \neq \nu$ elements zero is $g^{\mu\nu}$. So assume that $\ell^\mu \ell^\nu$ is proportional to $g^{\mu\nu}$:

$$\ell^\mu \ell^\nu = \lambda g^{\mu\nu} \rightarrow g_{\mu\nu} \ell^\mu \ell^\nu = \lambda g_{\mu\nu} g^{\mu\nu}$$

But $g_{\mu\nu}g^{\mu\nu}$ is equal with d , the dimension of the space. So $\lambda = \frac{\ell^2}{d}$. Now the integral $I(\ell)$ can be written:

$$I(\ell) = \int \frac{d^4\ell}{(2\pi)^4} \frac{2(\dfrac{4}{d} - 1)g^{\mu\nu}\ell^2}{[\ell^2 + \Delta)]^3}$$

To proceed we employ the technique called dimensional regularization. First we perform a Wick rotation $\ell^0 \to i\ell^0$. By doing this we get rid of the minus sign of the Minkowski metric and now $d^4\ell = d\Omega^3 d^3\ell$ in "polare coordinates" can be written We start from the d-dimensional integral (in "polar" coordinates, dD Euklidean) :

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{i \cdot \ell^2}{[\ell^2 - \Delta)]^3} = i \int \frac{d\Omega^d}{(2\pi)^d} \int_0^\infty d\ell \frac{\ell^2 \ell^{d-1}}{[\ell^2 - \Delta)]^3}$$

Note that the factor i came from the replacement $\ell^0 \to i\ell^0$. Also notice that the sign of Δ has changed.

The area of a d-dimensional sphere is

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

The second integral is

$$\begin{aligned}
\int_0^\infty d\ell \frac{\ell^{d+1}}{[\ell^2 - \Delta)]^3} &= \frac{1}{2}\int_{\ell^2=0}^\infty d(\ell^2 - \Delta)\frac{(\ell^2)^{\frac{d}{2}}}{[\ell^2 - \Delta]^3} = \frac{1}{2}\int_{\ell^2=0}^\infty d(\ell^2 - \Delta)\frac{(\ell^2)^{\frac{d}{2}}}{[\ell^2 - \Delta]^2(\ell^2 - \Delta)} = \\
&= -\frac{1}{2}\int_{\ell^2=0}^\infty d(\frac{1}{\ell^2 - \Delta})\frac{(\ell^2)^{\frac{d}{2}}}{\ell^2 - \Delta} = -\frac{1}{2\Delta}\int_{-1}^0 d(\frac{\Delta}{\ell^2 - \Delta})\frac{(\ell^2 - \Delta + \Delta)^{\frac{d}{2}}}{\ell^2 - \Delta} = \\
&= -\frac{1}{2\Delta}\int_{-1}^0 d(\frac{\Delta}{\ell^2 - \Delta})\frac{(\ell^2 - \Delta + \Delta)^{\frac{d}{2}}}{\ell^2 - \Delta} = -\frac{1}{2\Delta}\int_{-1}^0 d(\frac{\Delta}{\ell^2 - \Delta})\frac{\Delta^{\frac{d}{2}}(\frac{\ell^2-\Delta}{\Delta} + 1)^{\frac{d}{2}}}{\Delta\frac{\ell^2-\Delta}{\Delta}} = \\
&= \frac{\Delta^{\frac{d}{2}-2}}{2}\int_0^{-1} d(\frac{\Delta}{\ell^2 - \Delta})\frac{(\frac{\ell^2-\Delta}{\Delta} + 1)^{\frac{d}{2}}}{\frac{\ell^2-\Delta}{\Delta}} = \frac{\Delta^{\frac{d}{2}-2}}{2}\int_0^{-1} d(\frac{\Delta}{\ell^2 - \Delta})\frac{(\frac{\ell^2-\Delta}{\Delta} + 1)^{\frac{d}{2}}}{\frac{\ell^2-\Delta}{\Delta}} = \\
&= \frac{\Delta^{\frac{d}{2}-2}}{2}\int_0^{-1} dy\frac{(\frac{1}{y} + 1)^{\frac{d}{2}}}{\frac{1}{y}} = \frac{\Delta^{\frac{d}{2}-2}}{2}\int_0^{-1} dy \cdot y \cdot y^{-\frac{d}{2}}(y+1)^{\frac{d}{2}} = \\
&= \frac{-\Delta^{\frac{d}{2}-2}}{2}\int_0^1 d(-y)\cdot(-1)^{1-\frac{d}{2}}\cdot(-y)^{1-\frac{d}{2}}(-1)^{\frac{d}{2}}(-y-1)^{\frac{d}{2}} = \\
&= \frac{\Delta^{\frac{d}{2}-2}}{2}\int_0^1 dx\cdot(x)^{1-\frac{d}{2}}(x-1)^{\frac{d}{2}} = \\
&= \frac{\Delta^{\frac{d}{2}-2}}{2}\int_0^1 dx\cdot x^{2-\frac{d}{2}-1}(x-1)^{1+\frac{d}{2}-1}
\end{aligned}$$

Comparing with definition of the beta function :

$$\int_0^1 dx \cdot (x)^{\alpha-1}(x-1)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

the last integral can be written:

$$\begin{aligned}\int_0^\infty d\ell \frac{\ell^{d+1}}{[\ell^2-\Delta)]^3} &= \frac{\Delta^{\frac{d}{2}-2}}{2}\frac{\Gamma(2-\frac{d}{2})\Gamma(1+\frac{d}{2})}{\Gamma(3)} \\ &= \frac{\Delta^{\frac{d}{2}-2}}{2}\frac{\Gamma(2-\frac{d}{2})\cdot\frac{d}{2}\Gamma(\frac{d}{2})}{\Gamma(3)}\end{aligned}$$

So the integral we wish to calculate can be written:

$$\begin{aligned}\int \frac{d^d\ell}{(2\pi)^d}\frac{2(\frac{4}{d}-1)\ell^2}{[\ell^2-\Delta)]^3} &= 2(\frac{4}{d}-1)\int\frac{d\Omega^d}{(2\pi)^d}\int_0^\infty d\ell\frac{\ell^2\ell^{d-1}}{[\ell^2-\Delta)]^3} \\ &= 2(\frac{4}{d}-1)\frac{2\pi^{d/2}}{\Gamma(d/2)}\cdot\frac{1}{(2\pi)^d}\frac{\Delta^{\frac{d}{2}-2}}{2}\frac{\Gamma(2-\frac{d}{2})\cdot\frac{d}{2}\Gamma(\frac{d}{2})}{\Gamma(3)} \\ &= 2(\frac{4}{d}-1)\frac{d}{2}\frac{1}{4^{\frac{d}{2}}\pi^{d/2}}\Delta^{\frac{d}{2}-2}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)}\end{aligned}$$

This is potentially divergent. We need to examine the behavior in the neighborhood of $d=4$ which the dimension of our Minkowski space. To do so we introduce $2\varepsilon = 4-d$. So

$$\Gamma(2-\frac{d}{2}) = \Gamma(\varepsilon)$$

This gives

$$\begin{aligned}\frac{2(\frac{4}{d}-1)\frac{d}{2}}{4^{\frac{d}{2}}\pi^{d/2}}\Delta^{\frac{d}{2}-2}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)} &= \frac{(4-d)}{(4\pi)^2}(\frac{\Delta}{4\pi})^{\frac{d}{2}-2}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)} \\ &= \frac{2\varepsilon}{(4\pi)^2}(\frac{\Delta}{4\pi})^{-\varepsilon}\frac{\Gamma(\varepsilon)}{\Gamma(3)}\end{aligned}$$

Using now the expansion of the Γ function:

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon)$$

the above becomes

$$\frac{2\varepsilon}{(4\pi)^2}(\frac{\Delta}{4\pi})^{-\varepsilon}\frac{\Gamma(\varepsilon)}{\Gamma(3)} = \frac{2\varepsilon}{(4\pi)^2}(\frac{4\pi}{\Delta})^{\varepsilon}\frac{\frac{1}{\varepsilon}-\gamma+O(\varepsilon)}{2}$$

where γ the Euler - Mascheroni constant and $(\frac{4\pi}{\Delta})^\varepsilon \approx 1$. So

$$\begin{aligned}\int \frac{d^d\ell}{(2\pi)^d}\,\frac{2(\frac{4}{d}-1)\ell^2}{[\ell^2-\Delta)]^3} &= \frac{\varepsilon}{(4\pi)^2}[\frac{1}{\varepsilon}-\gamma+O(\varepsilon)] \\ &= \frac{1-\varepsilon\gamma+\varepsilon O(\varepsilon)}{(4\pi)^2}\end{aligned}$$

Taking the limit $\varepsilon \to 0$ we see that for $d=4$ the integral does not diverge and its value is $\boxed{\frac{i}{16\pi^2}}$.

The part of the integral in M that does not contain any ℓ (we denote it as I):

$$\begin{aligned} I &= \int_{x=0}^{1}\int dxdz\cdot i\cdot g_s^2\cdot 4m_q\int\frac{d^4\ell}{(2\pi)^4}\,[\frac{2\cdot[(4z^2-2z)p^\nu p^\mu+(4x^2-2x)k^\nu k^\mu+(1-4xz)p^\nu k^\mu]}{[\ell^2+\Delta)]^3} \\ &\quad+\frac{2[(2x+2z-1-4zx)k^\nu p^\mu+(m_q^2-\frac{1}{2}m_h^2+xz\cdot m_h^2)g^{\mu\nu}]}{[\ell^2+\Delta)]^3}] \\ &= \int_{x=0}^{1}\int dxdz\cdot i\cdot g_s^2\cdot 4m_q\times 2\cdot[(4z^2-2z)p^\nu p^\mu+(4x^2-2x)k^\nu k^\mu \\ &\quad+(1-4xz)p^\nu k^\mu+(2x+2z-1-4zx)k^\nu p^\mu+(m_q^2-\frac{1}{2}m_h^2+xz\cdot m_h^2)g^{\mu\nu}] \\ &\quad\times\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+\Delta)]^3}\end{aligned}$$

Lets repeat the calculation for the integral:

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+\Delta)]^3} &= (\ell \to i\ell) \\
&= -i\int \frac{d\Omega^4}{(2\pi)^4}\int_0^\infty d\ell \frac{\ell^{4-1}}{[\ell^2-\Delta)]^3} \\
&= -i\int \frac{d\Omega^4}{(2\pi)^4}\int_0^\infty d\ell \frac{\ell^{4-1}}{[\ell^2-\Delta)]^3} \\
&= -i\frac{2\pi^2}{16\pi^4}\int_0^\infty d\ell \frac{\ell^3}{[\ell^2-\Delta]^3} \\
&= -i\frac{1}{16\pi^2}\int_0^\infty \frac{\ell^2 d\ell^2}{[\ell^2-\Delta]^3} = \frac{1}{16\pi^2}\int_0^\infty \frac{(\ell^2-\Delta+\Delta)d\ell^2}{[\ell^2-\Delta]^3} \\
&= -i\frac{1}{16\pi^2}[\int_0^\infty \frac{d\ell^2}{[\ell^2-\Delta]^2} + \int_0^\infty \frac{\Delta d\ell^2}{[\ell^2-\Delta]^3}] \\
&= -i\frac{1}{16\pi^2}[\int_{-\Delta}^\infty \frac{dx}{x^2} + \int_{-\Delta}^\infty \Delta\frac{dx}{x^3}] \\
&= -i\frac{1}{16\pi^2}[(\frac{-1}{x}|_{-\Delta}^\infty) + (\frac{-\Delta}{2x^2}|_{-\Delta}^\infty)] = \frac{1}{16\pi^2}[\frac{-1}{\Delta}+\frac{\Delta}{2\Delta^2}] \\
&= \frac{i}{2\cdot 16\pi^2\Delta}
\end{aligned}$$

This quantity will be multiplied with the polarization vectors of the gluons $\epsilon_\nu^{\lambda'*}(k)$, $\epsilon_\mu^{\lambda*}(p)$. But having in mind the tranversality of the polarization: $p^\mu\epsilon_\mu^{\lambda*}(p)=0$ and $k^\nu\epsilon_\nu^{\lambda*}(k)=0$, many terms vanish. The terms that will survive are $[(1-4xz)p^\nu k^\mu + (m_q^2 - \frac{1}{2}m_h^2 + xz\cdot m_h^2)g^{\mu\nu}]\frac{1}{2\cdot 16\pi^2\Delta}$. So the initial expression of the decay amplitude M can now be written:

$$
\begin{aligned}
M &= -i\ \frac{m}{\upsilon}\ (-1) \times \\
& i \cdot g_s^2 \cdot 4m_q \int\limits_{x=0}^{x=1} \int dxdz [\frac{ig^{\mu\nu}}{16\pi^2} + i \frac{2 \cdot [(1-4xz)p^\nu k^\mu + (m_q^2 - \frac{1}{2}m_h^2 + xz \cdot m_h^2)g^{\mu\nu}]}{2 \cdot 16\pi^2 \Delta}] \\
& \epsilon_\nu^{\lambda' *}(k) \cdot \epsilon_\mu^{\lambda *}(p) \cdot Tr[t^b t^a] \\
M &= -i\ \frac{m}{\upsilon}\ (-1) \times i \cdot g_s^2 \cdot 4m_q \int\limits_{x=0}^{x=1} \int dxdz [\frac{g^{\mu\nu}\Delta + [(1-4xz)p^\nu k^\mu + (m_q^2 - \frac{1}{2}m_h^2 + xz \cdot m_h^2)g^{\mu\nu}]}{16\pi^2 \Delta}] \times \\
& \epsilon_\nu^{\lambda' *}(k) \cdot \epsilon_\mu^{\lambda *}(p) \cdot Tr[t^b t^a] \\
M &= -\frac{m}{\upsilon} \cdot g_s^2 \cdot m_q \int\limits_{x=0}^{x=1} \int dxdz [\frac{g^{\mu\nu}(-m_q^2 + xzm_h^2) + [(1-4xz)p^\nu k^\mu + (m_q^2 - \frac{1}{2}m_h^2 + xz \cdot m_h^2)g^{\mu\nu}]}{4\pi^2 \Delta}] \times \\
& \epsilon_\nu^{\lambda' *}(k) \cdot \epsilon_\mu^{\lambda *}(p) \cdot Tr[t^b t^a] \\
M &= -\frac{m}{\upsilon} \cdot g_s^2 \cdot m_q \int\limits_{x=0}^{x=1} \int dxdz [\frac{g^{\mu\nu}(-\frac{1}{2}m_h^2 + 2xzm_h^2) + (1-4xz)p^\nu k^\mu}{4\pi^2(-m_q^2 + xzm_h^2)}] \times \epsilon_\nu^{\lambda' *}(k) \cdot \epsilon_\mu^{\lambda *}(p) \cdot Tr[t^b t^a] \\
M &= -\frac{m}{\upsilon} \cdot g_s^2 \cdot m_q \int\limits_{x=0}^{x=1} \int dxdz [\frac{(1-4xz)[\frac{1}{2}m_h^2 \cdot g^{\mu\nu} - p^\nu k^\mu]}{4\pi^2 m_q^2 (1 - xz\frac{m_h^2}{m_q^2})}] \times \epsilon_\nu^{\lambda' *}(k) \cdot \epsilon_\mu^{\lambda *}(p) \cdot Tr[t^b t^a] \\
M &= -\frac{m_q^2}{\upsilon} \cdot g_s^2 \cdot \frac{\frac{1}{2}m_h^2 \cdot g^{\mu\nu} - p^\nu k^\mu}{4\pi^2 m_q^2} \cdot \underbrace{\int\limits_{x=0}^{x=1} \int dxdz [\frac{1-4xz}{1 - xz\frac{m_h^2}{m_q^2}}]} \times \epsilon_\nu^{\lambda' *}(k) \cdot \epsilon_\mu^{\lambda *}(p) \cdot Tr[t^b t^a] \\
M &= -\frac{g_s^2}{\upsilon} \cdot Tr[t^b t^a] \cdot \frac{\frac{1}{2}m_h^2 \cdot g^{\mu\nu} - p^\nu k^\mu}{4\pi^2} \cdot I(\frac{m_h^2}{m_q^2}) \times \epsilon_\nu^{\lambda' *}(k) \cdot \epsilon_\mu^{\lambda *}(p)
\end{aligned}
$$

From the GWS theory of weak interactions we have that $\upsilon = \frac{2m_W}{g}$ with $g = \frac{e}{\sin\theta_W}$. So $\upsilon = \frac{2m_W \cdot \sin\theta_W}{e}$. Also the trace $Tr[t^b t^a]$:

$$Tr[t^b t^a] = \frac{1}{2}\delta^{\alpha b}$$

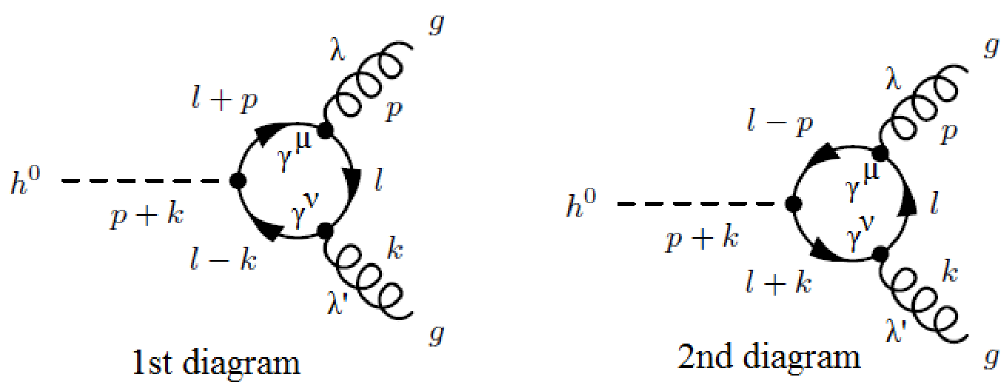
Plugging υ , g and $Tr[t^b t^a]$:

$$M = -\frac{e \cdot g_s^2}{2m_W \sin\theta_W} \cdot \frac{1}{2}\delta^{\alpha b} \cdot \frac{\frac{1}{2}m_h^2 \cdot g^{\mu\nu} - p^\nu k^\mu}{4\pi^2} \cdot I(\frac{m_h^2}{m_q^2}) \times \epsilon_\nu^{\lambda' *}(k) \cdot \epsilon_\mu^{\lambda *}(p)$$

This is our first achievement , the calculation of the transition amplitude for the first diagram:

$$\boxed{M_q^{\lambda\lambda'\alpha b} = -\frac{e \cdot g_s^2}{8\pi^2 m_W \sin\theta_W} \cdot \frac{1}{2}\delta^{\alpha b} \cdot (\frac{1}{2}m_h^2 \cdot g^{\mu\nu} - p^\nu k^\mu) \cdot I(\frac{m_h^2}{m_q^2}) \times \epsilon_\nu^{\lambda'*}(k) \cdot \epsilon_\mu^{\lambda*}(p)}$$

We need now to calculate the transition amplitude for the second diagram. Lets compare and observe the two diagrams:



We see that if we change in the first diagram $p \to k$, $\mu \to \nu$ and $\lambda \to \lambda'$ we get the second diagram. But these three transformations leave invariant the final (boxed) expression of $M_q^{\lambda\lambda'\alpha b}$. So we conclude that the two diagrams contribute equally to the total amplitude M which is $\times 2$ the boxed expression. Thus finally

$$\underline{M_q^{\lambda\lambda'\alpha b} = -\frac{e \cdot g_s^2}{4\pi^2 m_W \sin\theta_W} \cdot \frac{1}{2}\delta^{\alpha b} \cdot I(\frac{m_h^2}{m_q^2}) \cdot \overbrace{(\frac{1}{2}m_h^2 \cdot g^{\mu\nu} - p^\nu k^\mu) \times \epsilon_\nu^{\lambda'*}(k) \cdot \epsilon_\mu^{\lambda*}(p)}}$$

Now to find the decay rate we need to sum this expression over all possible gluon polarizations λ and λ' , over all colors α and b and over all possible intermediate loop quarks and then square it. To simplify things lets deal first with the tensorial structure under the overbrace (lets call it $T^{\lambda\lambda'}$).

$$\begin{aligned}
\sum_{\lambda,\lambda'}\left|T^{\lambda\lambda'}\right|^2 &= \sum_{\lambda,\lambda'}\left|(\frac{1}{2}m_h^2\cdot g^{\mu\nu}-p^\nu k^\mu)\times\epsilon_\nu^{\lambda' *}(k)\cdot\epsilon_\mu^{\lambda *}(p)\right|^2 = \\
& (\frac{1}{2}m_h^2\cdot g^{\mu_1\nu_1}-p^{\nu_1}k^{\mu_1})\cdot(\frac{1}{2}m_h^2\cdot g^{\mu_2\nu_2}-p^{\nu_2}k^{\mu_2}) \\
& \times\sum_{\lambda,\lambda'}\epsilon_{\nu_1}^{\lambda'}(k)\cdot\epsilon_{\mu_1}^{\lambda}(p)\cdot\epsilon_{\nu_2}^{\lambda' *}(k)\cdot\epsilon_{\mu_2}^{\lambda *}(p) = \\
&= [\frac{m_h^4}{4}\cdot g^{\mu_1\nu_1}g^{\mu_2\nu_2}-\frac{m_h^2}{2}\cdot g^{\mu_1\nu_1}p^{\nu_2}k^{\mu_2}-\frac{m_h^2}{2}\cdot p^{\nu_1}k^{\mu_1}g^{\mu_2\nu_2}+p^{\nu_1}k^{\mu_1}p^{\nu_2}k^{\mu_2}] \\
& \times\sum_{\lambda,\lambda'}\epsilon_{\nu_1}^{\lambda'}(k)\cdot\epsilon_{\mu_1}^{\lambda}(p)\cdot\epsilon_{\nu_2}^{\lambda' *}(k)\cdot\epsilon_{\mu_2}^{\lambda *}(p) = \\
&= \frac{m_h^4}{4}\cdot g^{\mu_1\nu_1}g^{\mu_2\nu_2}\times\sum_{\lambda}\epsilon_{\mu_1}^{\lambda}(p)\cdot\epsilon_{\mu_2}^{\lambda *}(p)\cdot\sum_{\lambda'}\epsilon_{\nu_1}^{\lambda'}(k)\cdot\epsilon_{\nu_2}^{\lambda' *}(k) \\
& -\frac{m_h^2}{2}\cdot g^{\mu_1\nu_1}p^{\nu_2}k^{\mu_2}\times\sum_{\lambda}\epsilon_{\mu_1}^{\lambda}(p)\cdot\epsilon_{\mu_2}^{\lambda *}(p)\cdot\sum_{\lambda'}\epsilon_{\nu_1}^{\lambda'}(k)\cdot\epsilon_{\nu_2}^{\lambda' *}(k) \\
& -\frac{m_h^2}{2}\cdot p^{\nu_1}k^{\mu_1}g^{\mu_2\nu_2}\times\sum_{\lambda}\epsilon_{\mu_1}^{\lambda}(p)\cdot\epsilon_{\mu_2}^{\lambda *}(p)\cdot\sum_{\lambda'}\epsilon_{\nu_1}^{\lambda'}(k)\cdot\epsilon_{\nu_2}^{\lambda' *}(k) \\
& +p^{\nu_1}k^{\mu_1}p^{\nu_2}k^{\mu_2}\times\sum_{\lambda}\epsilon_{\mu_1}^{\lambda}(p)\cdot\epsilon_{\mu_2}^{\lambda *}(p)\cdot\sum_{\lambda'}\epsilon_{\nu_1}^{\lambda'}(k)\cdot\epsilon_{\nu_2}^{\lambda' *}(k)
\end{aligned}$$

When we have a sum over the polarization vectors of massless gauge bosons, we can use the replacement originated from Ward identity:

$$\sum_{\lambda}\epsilon_\mu^\lambda(p)\cdot\epsilon_\nu^{\lambda *}(p) = g_{\mu\nu}$$

So the above can be written:

$$\begin{aligned}
\left|T^{\lambda\lambda'\alpha b}\right|^2 &= \frac{m_h^4}{4}\cdot g^{\mu_1\nu_1}g^{\mu_2\nu_2}g_{\mu_1\mu_2}g_{\nu_1\nu_2}-\frac{m_h^2}{2}\cdot g^{\mu_1\nu_1}p^{\nu_2}k^{\mu_2}g_{\mu_1\mu_2}g_{\nu_1\nu_2} \\
& -\frac{m_h^2}{2}\cdot p^{\nu_1}k^{\mu_1}g^{\mu_2\nu_2}g_{\mu_1\mu_2}g_{\nu_1\nu_2}+p^{\nu_1}k^{\mu_1}p^{\nu_2}k^{\mu_2}g_{\mu_1\mu_2}g_{\nu_1\nu_2} \\
&= \frac{m_h^4}{4}\cdot 4-\frac{m_h^2}{2}p^{\nu_2}k^{\mu_2}g_{\mu_2\nu_2}-\frac{m_h^2}{2}\cdot p^{\nu_1}k^{\mu_1}g_{\mu_1\nu_1}+p^2k^2 \\
&= m_h^4-m_h^2pk = m_h^4-m_h^2\frac{m_h^2}{2} \\
&= \frac{m_h^4}{2}
\end{aligned}$$

So the total amplitude $M_q^{\lambda\lambda'\alpha b}$ (underlined expression) can be squared:

$$\begin{aligned}
|M|^2 &= \left|\frac{-e\cdot g_s^2}{4\pi^2 m_W\sin\theta_W}\right|^2\cdot\sum_{a,b}\left|\frac{1}{2}\delta^{\alpha b}\right|^2\cdot\sum_q I(\frac{m_h^2}{m_q^2})\cdot\sum_{\lambda,\lambda'}\left|T^{\lambda\lambda'}\right|^2 \\
& \frac{e^2\cdot g_s^4}{16\pi^4 m_W^2\sin^2\theta_W}\cdot\frac{1}{4}\sum_{a,b}\left|\delta^{\alpha b}\right|^2\cdot\sum_q I(\frac{m_h^2}{m_q^2})\cdot\frac{m_h^4}{2}
\end{aligned}$$

The are 8 gluon colors so the indices $a, b \in \{1, 8\}$. Thus:

$$\sum_{a,b} \left|\delta^{ab}\right|^2 = 8$$

So finally

$$\boxed{|M|^2 = \frac{e^2 \cdot g_s^4 \cdot m_h^4}{16\pi^4 m_W^2 \sin^2\theta_W} \cdot \sum_q I(\frac{m_h^2}{m_q^2})}$$

Last thing is to go back to the decay rate formula:

$$\Gamma = \frac{1}{2m_h}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\cdot\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\left|M(m_h \to p, k)\right|^2 (2\pi)^4\ \delta^{(4)}(m_h - p - k)$$

In the rest frame of the scalar Higgs the energies of the two (massless) gluons are the same $E_p = E_k = |\vec{p}| = \left|\vec{k}\right|$. Also the delta function can be analyzed as:

$$\begin{aligned}\delta^{(4)}(m_h - p - k) &= \delta(m_h - E_p - E_k)\delta^{(3)}(\vec{p_h} - \vec{p} + \vec{k}) \\ &= \delta(m_h - 2|\vec{p}|)\delta^{(3)}(\vec{p} + \vec{k}) \\ &= \delta[-2(|\vec{p}| - \frac{m_h}{2})]\delta^{(3)}(\vec{p} + \vec{k})\end{aligned}$$

Using the identity of the delta function $\delta[a(x - x_0)] = \frac{1}{|a|}\delta(x - x_0)$. We have

$$\delta^{(4)}(m_h - p - k) = \frac{1}{2}\delta(|\vec{p}| - \frac{m_h}{2})\delta^{(3)}(\vec{p} + \vec{k})$$

Then Γ becomes:

$$\begin{aligned}\Gamma &= \frac{1}{2m_h}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{4|\vec{p}|^2}\left|M(m_h \to p, k)\right|^2 (2\pi)^4\frac{1}{2}\delta(|\vec{p}| - \frac{m_h}{2})\delta^{(3)}(\vec{p} + \vec{k}) \\ &= \frac{1}{64 m_h \pi^2}|M|^2 \int d^3p \frac{1}{|\vec{p}|^2}\delta(|\vec{p}| - \frac{m_h}{2}) \\ &= \frac{1}{64 m_h \pi^2}|M|^2 \int d\Omega \int_0^\infty |\vec{p}|^2\, d|\vec{p}|\, \frac{\delta(|\vec{p}| - \frac{m_h}{2})}{|\vec{p}|^2} \\ &= \frac{1}{64 m_h \pi^2}|M|^2\, 4\pi \cdot 1\end{aligned}$$

The final result for the decay rate is:

$$\begin{aligned}\Gamma &= \frac{|M|^2}{16 m_h \pi} \\ &= \frac{1}{16 m_h \pi} \frac{e^2 \cdot g_s^4 \cdot m_h^4}{16 \pi^4 m_W^2 \sin^2 \theta_W} \sum_q I(\frac{m_h^2}{m_q^2})\end{aligned}$$

If we introduce the fine structure constant $\alpha = \frac{e^2}{4\pi}$ and its analogue in strong interactions $\alpha_s = \frac{g_s^2}{4\pi}$:

$$\boxed{\Gamma = \frac{1}{4} \frac{\alpha \cdot \alpha_s^2 \cdot m_h^3}{\pi^2 m_W^2 \sin^2 \theta_W} \sum_q I(\frac{m_h^2}{m_q^2})}$$

Also the form factor implies that the heavier quarks contribute significantly to the decay rate. So we expect only the top quark to play an important role.

$$I_{(\frac{m_h^2}{m_q^2})} = \int\limits_{x=0}^{1} \int\limits_{z=0}^{1-x} dxdz [\frac{1 - 4xz}{1 - xz \frac{m_h^2}{m_q^2}}]$$