World line Representation and Derivation of the Amplitude for Pair Creation in Vacuum in an External Electric Field using the Proper-Time Method

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Abstract

This project paper is of two section. First section explains the world line representation and how it helps in interpreting pair creation and annihilation. Second section deals with the derivation of the amplitude for pair creation in vacuum in the special case of a one dimensional external time dependent electric field using Schwinger proper time method.

World line Representation in Explaining Pair Creation

A world line is the spacetime line along which the point like particle evolves. Through the use of an invariant relativistic quantity this world line can be drawn. The squared spacetime length s^2 is invariant in all frames of reference. Using natural units $\hbar = c = 1$, the spacetime length is defined as follows

$$s^2 = t^2 - \vec{x}^2$$

and in differential form using Einstein notation with the metric $diag(\eta^{\mu\nu}) = (+\ -\ -\ -)$

$$ds^2 = dx_\mu dx^\mu = \eta^{\mu\nu} dx_\mu dx_\nu \tag{1}$$

where $dx_\mu = (dx_0, dx_1, dx_2, dx_3)$ are the spacetime variables. Particularly, since space like particles violates the speed limit c and light like particles are massless, the quantity of choice for a particle with mass is the proper time τ of the particle on a time like path ($dx_i = 0;\ i = 1, 2, 3$); hence, $ds^2 = d\tau^2 = dx_0^2$ and $d\tau = dx_0$. Physically, the proper time of a point like particle is the shortest time for a particle to evolve from one state to another in its own reference ("rest") frame; thus, making it a useful physical parameter. We can use the proper time (or an affinely transformed function of it) to parametrize our coordinates $x_\mu \equiv x_\mu(\tau)$. Given the 4-momentum vector in the rest frame of the particle with $p_\mu = (m, 0, 0, 0)$), and choosing an affine transform of the proper time $\lambda(\tau) = \tau/m$, where m is the mass of the particle, the momentum can be expressed in terms of the parametrized spacetime variables as

$$p_\mu = \frac{dx_\mu(\tau)}{d\lambda}. \tag{2}$$

dividing equation (1) by $d\lambda^2$ we obtain

$$\left(\frac{d\tau}{d\lambda}\right)^2 = \frac{dx_\mu}{d\lambda}\frac{dx^\mu}{d\lambda} = p_\mu p^\mu = m^2 \tag{3}$$

Now, we can see that

$$\left(\frac{d\tau}{d\lambda}\right)^2 = m^2 \qquad\qquad d\tau = \pm m\, d\lambda \tag{4}$$

This parametrization was employed by [Stückelberg 1941a]; he explained the signs $\pm$ in equation (4) as a representation of the charge of the particle. Stückelberg's reasoning is that, on its world line, a charged particle forwardly evolving, $(dx_0/d\lambda) > 0$, can be interpreted as a particle with opposite charge (antiparticle) backwardly evolving, $(dx_0/d\lambda) < 0$. In turn, this representation is used to explain pair creation and annihilation. In Figure 1, one can see a plot presenting three possible world lines (A, B and C) when a one dimensional electric field is applied for a short period of time $\delta t = t_2 - t_1$. This figure shows that around $t = 0$ when an electric field is applied, a

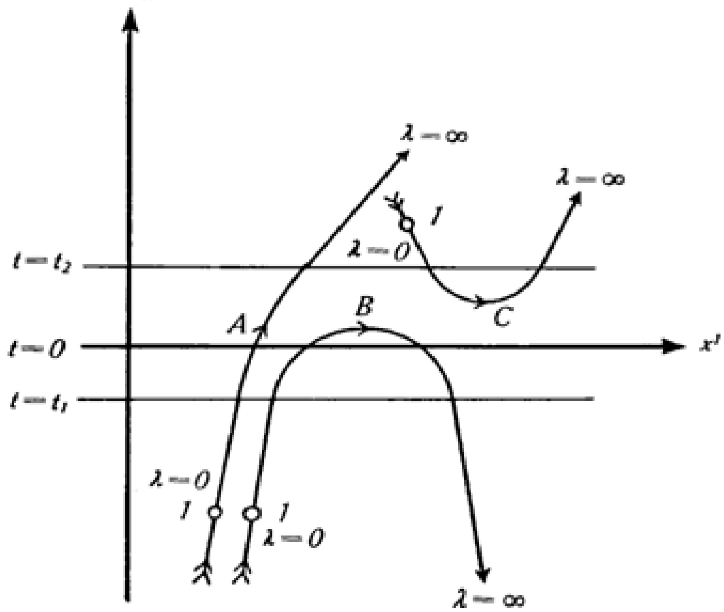

Figure 1: [Stückelberg 1941b] the world lines A, B and C on a plot of time $t = x^4$ vs. x^1, where $\lambda \equiv \lambda(\tau)$ is an affine parameter that depends on the proper time τ. A: The regular 1-1 graph that is for every x^4 there is one x^1. B: represents the annihilation of a particle in which there are two values for x^1 for each value x^4 for $x^4 < 0$. C: represents the creation of a particle in which there are two values for x^1 for each value x^4 for $x^4 > 0$.

charged particle can be accelerated on its world line A. It can be annihilated (accelerated backwards, deflected) on its world line B, which can be seen as an antiparticle, that started at $\lambda = \infty$ and annahilated the particle at time $t = 0$ by traveling backwards along its world line. Another option is of a particle-antiparticle pair being created on world line C, with the antiparticle traveling backwards (against the arrow) and the particle traveling forwards (along the arrow) on the world line.

This idea was further explained by Feynman using the a picture of paths of

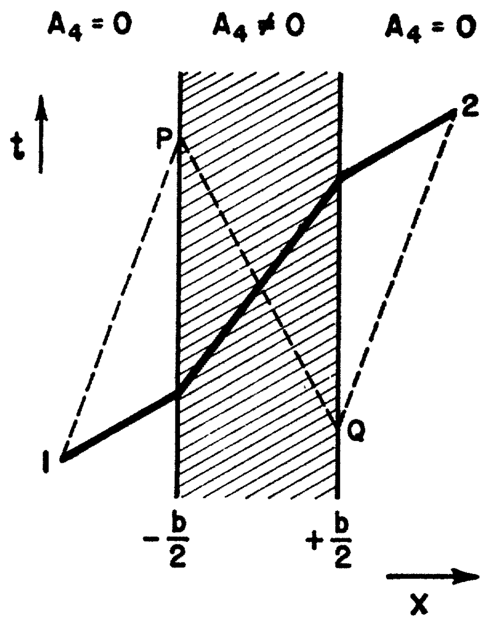

Figure 2: [Feynman 1948] This is a plot of time t vs position x with a potential barrier (shaded region). Two paths are drawn from starting point ① to end point ② that describe the electrons journey across the barrier. The dotted path is for the energetically weak electron annihilated at point P due to a positron created from an electron-positron pair at point Q and traveled along the line QP while the electron produced at Q gets detected at point ②. The solid path is that of an energetic electron that experienced deflection due to the potential.

an electron [Feynman 1948]. In Figure 2, one can see two possible paths an electron can take passing a potential barrier starting at point ① and detected at point ②. The paths depend on the energy of the electron; the solid path is what an energetic electron can take with a deflection when it passes the barrier and reaches point ②. Another path is the one when the weak electron starting at ① gets annihilated at point P by a positron produced from an electron-positron pair created at point Q with the created electron getting detected at (2).

This world line picture and the path integral machinery were tools that lead [Feynman 1950] and [Schwinger 1951] to obtain the quantum electrodynamics effective action. This action was used to calculate the corrections to the Maxwell Lagrangian via multi-loop correction method in a first quantization formulation. These corrections, celebrated in [Dunne 2012], were derived analytically by Heisenberg and Euler who reached the full effective action that can be used for vacuum polarization discussed in the following section.

Derivation of the Amplitude for Pair Production in Vacuum in an External Electric Field

Vacuum in a constant external electric field results in the creation (and annihilation) of virtual electrons and positrons, i.e. e^-e^+ pair production; hence, the vacuum is said to be polarized. To be able to theoretically calculate the amplitude of this production one needs to consider the perturbative effect of the electric field on the vacuum. We will follow the derivation found in [Dittrich *et al.* 1985] and [Dittrich *et al.* 2000] that uses the proper time method introduced by [Schwinger 1951]. Only considering up to one-loop correction, the Lagrangian density (onwards Lagrangian) will be formed of two parts:

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} \tag{5}$$

where $\mathcal{L}^{(0)}$ is the classical Maxwell Lagrangian with no corrections, and $\mathcal{L}^{(1)}$ is the one-loop correction, i.e. perturbation, which neglects radiative corrections. Now, we will focus on the one-loop Lagrangian $\mathcal{L}^{(1)}$.

For a vacuum state $|0\rangle$ prepared in the past at time $t \to -\infty$, $|0, t = -\infty\rangle$, the probability amplitude of the vacuum state to remain in the ground state when no electric field is applied, $A_\mu = 0$,

$$\langle 0, t = +\infty | 0, t = -\infty \rangle^{A=0} \equiv \langle 0_+ | 0_- \rangle^{A=0}$$

i.e. no pair production, is $| \langle 0_+|0_- \rangle^{A=0} |^2 = 1$. Nevertheless, when there is an electromagnetic field present, with electromagnetic $A_\mu \neq 0$, the amplitude will not be equal to unity, $| \langle 0_+|0_- \rangle^{A} |^2 \neq 1$, since pair production will occur. We can find that the amplitude is related to the effective action as follows:

$$\langle 0_+|0_- \rangle^{A} = e^{iW^{(1)}[A]} \tag{6}$$

where $W^{(1)}[A]$ is the effective action of the one-loop correction. Using Schwinger action principle as described by [Toms 2007], we can obtain

$$W^{(1)}[A] = \int d^4x \mathcal{L}^{(1)}(x) \tag{7}$$

We can then calculate the probability amplitude for pair production which would be equal to $1 - | \langle 0_+|0_- \rangle^{A} |^2$.

In order to work with fermions, we will need to work with Dirac field ψ, where $\bar{\psi} = \psi^\dagger \gamma^o$, and γ^μ are the 4×4 Dirac matrices (as stated in the course notes). The expectation value of the current (source) j^μ can be obtained via the functional derivative of the effective action with respect to the vector potential

$$\frac{\delta W^{(1)}[A]}{\delta A_\mu(x)} = \langle 0|j^\mu(x)|0\rangle^A \tag{8}$$

where the symmetrized current (normal ordered) is defined as

$$j^\mu = \frac{e}{2}[\bar{\psi}, \gamma^\mu \psi] = \frac{e}{2}{:}\bar{\psi}\gamma^\mu\psi{:} \tag{9}$$

with e taken as the charge of the fermion.
Now, the aim is to solve equation (8) to find an expression for the effective action $W^{(1)}[A]$ and in turn the one-loop Lagrangian correction $\mathcal{L}^{(1)}$, using the proper time method. We note that "Charge symmetrization translates into a time symmetrization [i.e. time ordering]" [Dittrich *et al.* 2000]. This is how Stückelberg's reasoning is used in this derivation.

$$\begin{aligned}\frac{\delta W^{(1)}[A]}{\delta A_\mu(x)} = \langle 0|j^\mu(x)|0\rangle^A &= -e \lim_{x' \to x} \gamma^\mu \langle 0|\mathbf{T}\left(\psi(x')\bar{\psi}(x)\right)|0\rangle \\ &= ie\, \mathbf{tr}_\gamma \left[\gamma^\mu G(x,x|A)\right]\end{aligned} \tag{10}$$

where $\mathbf{T}$ is our time ordering operator and $\mathbf{tr}_\gamma$ means the trace of the spinor index. $G(x,x|A)$ is the Green function. $G(x,x'|A)$ is a function of the initial and final spacetime variables $x^\mu(\tau)$ and $x'^\mu(\tau)$, respectively, in an external electromagnetic field $A_\mu(x)$. The Green function should satisfy the Dirac field equation

$$(-i\gamma^\mu D_\mu + m)\, G(x,x'|A) = \delta(x-x') \tag{11}$$

where $D_\mu = \partial_\mu + ieA_\mu$. In momentum (phase) space we can write

$$\left(-i\gamma^\mu \tilde{D}_\mu + m\right) G[A] = 1 \tag{12}$$

where $\tilde{D}_\mu = p_\mu + ieA_\mu$. From (12) we can formulate this Green function using the proper time method

$$\begin{aligned} G[A] &= \frac{1}{m - i\gamma^\mu \tilde{D}_\mu} = \frac{\left(m + i\gamma^\mu \tilde{D}_\mu\right)}{\left(m - i\gamma^\mu \tilde{D}_\mu\right)\left(m + i\gamma^\mu \tilde{D}_\mu\right)} = \frac{\left(m + i\gamma^\mu \tilde{D}_\mu\right)}{m^2 + \left(\gamma^\mu \tilde{D}_\mu\right)^2} \\ &= \left(m + i\gamma^\mu \tilde{D}_\mu\right) i \int_0^\infty d\tau e^{-i\tau\left[m^2 + \left(\gamma^\mu \tilde{D}_\mu\right)^2\right]} \end{aligned} \tag{13}$$

The last step was achieved using the identity shown in [de Albuquerque *et al.* 1998]

$$X^{-1} = i \int_0^\infty d\tau e^{-i\tau X} \tag{14}$$

where τ is the proper time on which the 4-spacetime variables depend, $x^\mu(\tau)$.

Therefore we need to find $W^{(1)}[A]$ that will satisfy equation (10)

$$\frac{\delta W^{(1)}[A]}{\delta A_\mu(x)} = ie\, \mathbf{tr}_\gamma \left[\gamma^\mu G(x,x|A)\right].$$

An ansatz that will fulfill the requirement is,

$$W^{(1)}[A] = \int d^4x \mathcal{L}^{(1)}(x) = \int d^4x \left\{ -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \mathbf{tr}_{x,\gamma} \left[e^{-i\tau\left(\gamma^\mu \tilde{D}_\mu\right)^2} \right] \right\} \tag{15}$$

where $\mathbf{tr}_{x,\gamma}$ is the trace of the gamma index and position in momentum space. Equation (15) is well defined as $\tau \to \infty$; however, as $\tau \to 0$ a cutoff τ_o is introduced in order to perform the integration. The logic behind this is that after integrating, unphysical quantities appearing are removed and afterwards by applying the limits $\tau_o \to 0$ the surviving physical quantities do not diverge. Hence, our expression is said to be renormalized [de Albuquerque *et al.* 1998].

Before we test the validity of the ansatz, let us list some functional derivative rules which will be useful. For a functional $F[Q(x)]$ [Dittrich *et al.* 2001]:

(a) Given an arbitrary function $f(x)$

$$\begin{aligned} F[Q(x)] &= \int dx\, Q(x) f(x) \\ \frac{\delta F[Q(x)]}{\delta Q(y)} &= f(y), \text{ Noting that: } \frac{\delta Q(x)}{\delta Q(y)} = \delta(x-y) \end{aligned} \tag{16}$$

(b) For an exponential single integral

$$\begin{aligned} F[Q(x)] &= exp\left[\int dx\, Q(x) f(x)\right] \\ \frac{\delta F[Q(x)]}{\delta Q(y)} &= f(y) exp\left[\int dx\, Q(x) f(x)\right] \end{aligned} \tag{17}$$

(c) For an exponential double integral

$$\begin{aligned} F[Q(x)] &= exp\left[\iint dx\, dx'\, Q(x) f(x,x') Q(x')\right] \\ \frac{\delta F[Q(x)]}{\delta Q(y)} &= \left[\int dx'\, f(y,x') Q(x') + \int dx\, Q(x) f(x,y)\right] F[Q(x)] \end{aligned} \tag{18}$$

We can see that $\delta \tilde{D}_\alpha(x) = ie\delta A_\alpha(x)$ by using the chain rule

$$\frac{\delta}{\delta A_\alpha(x)} = \frac{\delta \tilde{D}_\alpha(x)}{\delta A_\alpha(x)} \frac{\delta}{\delta \tilde{D}_\alpha(x)} = ie \frac{\delta}{\delta \tilde{D}_\alpha(x)}.$$

Therefore, by using equation (15) in equation (10)

$$\frac{\delta W^{(1)}[A(y)]}{\delta A_\alpha(x)} = ie\frac{\delta W^{(1)}[A(y)]}{\delta \tilde{D}_\alpha(x)}$$

$$= \frac{-ie}{2}\int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \frac{\delta}{\delta \tilde{D}_\alpha(x)} \mathbf{tr}_{x,\gamma}\left[\int d^4y\ exp\left(-i\tau\left(\gamma_\mu \tilde{D}^\mu(y)\gamma_\nu \tilde{D}^\nu(y)\right)\right)\right]$$

$$= \frac{-ie}{2}\int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2}$$

$$\times\ \mathbf{tr}_{x,\gamma}\left[\int d^4y(-i\tau)\gamma_\mu\gamma_\nu \frac{\delta\left(\tilde{D}^\mu(y)\tilde{D}^\nu(y)\right)}{\delta \tilde{D}_\alpha(x)} e^{-i\tau\left(\gamma_\mu \tilde{D}^\mu(y)\gamma_\nu \tilde{D}^\nu(y)\right)}\right]$$

where

$$\frac{\delta\left(\tilde{D}^\mu(y)\tilde{D}^\nu(y)\right)}{\delta \tilde{D}_\alpha(x)} = \delta^\mu_\alpha \delta(x-y)\tilde{D}^\nu(y) + \delta^\nu_\alpha \delta(x-y)\tilde{D}^\mu(y)$$

Hence,

$$\frac{\delta W^{(1)}[A(y)]}{\delta A_\alpha(x)}$$

$$= \frac{ie}{2}\int_0^\infty d\tau\ i\ e^{-i\tau m^2}$$

$$\times\ \mathbf{tr}_{x,\gamma}\left[\int d^4y\gamma_\mu\gamma_\nu\left(\delta^\mu_\alpha \delta(x-y)\tilde{D}^\nu(y) + \delta^\nu_\alpha \delta(x-y)\tilde{D}^\mu(y)\right) e^{-i\tau\left(\gamma_\mu \tilde{D}^\mu(y)\gamma_\nu \tilde{D}^\nu(y)\right)}\right]$$

$$= \frac{ie}{2}\int_0^\infty d\tau\ i\ e^{-i\tau m^2}\mathbf{tr}_{x,\gamma}\left[\left(\gamma^\alpha\gamma_\nu \tilde{D}^\nu(x) + \gamma_\mu \tilde{D}^\mu(x)\gamma^\alpha\right) e^{-i\tau\left(\gamma_\mu \tilde{D}^\mu(x)\gamma_\nu \tilde{D}^\nu(x)\right)}\right]$$

Now, we will swap the indices $\nu \leftrightarrow \mu$ and group the terms to reach the form of equation (13)

$$
\begin{aligned}
&\frac{\delta W^{(1)}[A(y)]}{\delta A_\alpha(x)} \\
&= \frac{ie}{2}\int_0^\infty d\tau\ i\ e^{-i\tau m^2}\mathbf{tr}_\gamma\left[\ \gamma^\alpha \langle x|\left(\gamma_\mu\tilde{D}^\mu(x)+\gamma_\mu\tilde{D}^\mu(x)\right)e^{-i\tau\left(\gamma^\mu\tilde{D}_\mu\right)^2}|x\rangle\right] \\
&= ie\int_0^\infty d\tau\ i\ e^{-i\tau m^2}\mathbf{tr}_\gamma\left[\gamma^\alpha \langle x|\gamma_\mu\tilde{D}^\mu(x)e^{-i\tau\left(\gamma^\mu\tilde{D}_\mu\right)^2}|x\rangle\right]
\end{aligned}
$$

using the γ matrices property, $\mathbf{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho ...\right]=0$ for odd numbers of γ matrices, we can rewrite the equation with an added term

$$
\begin{aligned}
&= ie\ \mathbf{tr}_\gamma\left[\gamma^\alpha \langle x|\left(m+\gamma_\mu\tilde{D}^\mu(x)\right)i\int_0^\infty d\tau e^{-i\tau\left(m^2+\left(\gamma^\mu\tilde{D}_\mu\right)^2\right)}|x\rangle\right] \\
&= ie\ \mathbf{tr}_\gamma\left[\gamma^\alpha G(x,x|A)\right]
\end{aligned}
$$

Therefore, the ansatz (15) satisfies equation (10) and we can write the one-loop correction of the Lagrangian

$$
\mathcal{L}^{(1)}(x) = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}e^{-i\tau m^2}\mathbf{tr}_\gamma\left[\langle x|e^{-i\tau\left(\gamma^\mu\tilde{D}_\mu\right)^2}|x\rangle\right] \tag{19}
$$

and using [Schwinger 1951] interpretation, we will define the coordinate representation of the proper time evolution operator as the transformation amplitude

$$
K(x,x';\tau|A) = \langle x|e^{-i\tau\left(\gamma^\mu\tilde{D}_\mu\right)^2}|x\rangle\,. \tag{20}
$$

Using the path integral formulation, we can solve the functional integral

$$
K(x',x'';\tau|A) = \langle x',\tau|x'',0\rangle = \int_{x(0)=x''}^{x(\tau)=x'}\mathcal{D}x(\tau)e^{iS[x(\tau)]}. \tag{21}
$$

However, for the simple case of one dimensional electric field there is a simpler way following [Itzykson *et al.* 1980].

The evolution operator interpreted by Schwinger in coordinate space as equation (20) has the form in momentum space as

$$U(\tau) = e^{-i\tau H} \tag{22}$$

where H is the Hamiltonian. Before writing the Hamiltonian we will need some tools in order to present it in a neat form. Using the following anticommutation $\{\ ,\ \}$ and commutation $[\ ,\ ]$ relations

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu}\mathbf{1}_{4\times 4} \\
\sigma^{\mu\nu} &= \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \\
F_{\mu\nu} &= \frac{i}{e}\left[D_\mu, D_\nu\right]
\end{aligned} \tag{23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor, we can write the multiplication of two gamma matrices as follows

$$\begin{aligned}
\gamma^\mu\gamma^\nu &= \frac{1}{2}\gamma^\mu\gamma^\nu + \frac{1}{2}\gamma^\mu\gamma^\nu + \frac{1}{2}\gamma^\nu\gamma^\mu - \frac{1}{2}\gamma^\nu\gamma^\mu \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) + \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \\
&= \frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} + \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right] \\
&= \eta^{\mu\nu}\mathbf{1}_{4\times 4} + \frac{2}{i}\sigma^{\mu\nu}.
\end{aligned} \tag{24}$$

Therefore, we can now write $\left(\gamma^\mu D_\mu\right)^2$ explicitly

$$\begin{aligned}
\left(\gamma^\mu D_\mu\right)^2 &= \gamma^\mu D_\mu \gamma^\nu D_\nu = \gamma^\mu\gamma^\nu D_\mu D_\nu \\
\text{using equation (24)} &= \left(\eta^{\mu\nu}\mathbf{1}_{4\times 4} - 2i\sigma^{\mu\nu}\right)\left(\frac{e}{i}F_{\mu\nu} + D_\nu D_\mu\right) \\
\text{since } F_{\mu\nu} \text{ is antisymmetric} &= D^2\mathbf{1}_{4\times 4} - 2e\sigma^{\mu\nu}F_{\mu\nu} \\
\&\ D_\nu D_\mu \text{ is symmetric} &= D^2 - 2e\sigma^{\mu\nu}F_{\mu\nu}
\end{aligned} \tag{25}$$

Therefore we can write the Hamiltonian of equation (22) using the expression obtained in (25) in momentum space

$$H = \left(\gamma^\mu \tilde{D}_\mu\right)^2 = \tilde{D}^2 - 2e\sigma^{\mu\nu}F_{\mu\nu} \tag{26}$$

In the case of pure one dimensional time dependent electric field along the x_3 axis choosing our gauge considered is $A_0 = A_1 = A_2 = 0$ and $A_3 = E_3 x_0$, then

$$-2e\sigma^{\mu\nu}F_{\mu\nu} = -2e\sigma^{0\,3}F_{0\,3} = -2e\frac{i}{4}\left[\gamma^0, \gamma^3\right]\partial_0 A_3 = -e\frac{i}{2}\left[\gamma^0, \gamma^3\right]E_3 \tag{27}$$

Therefore, we can calculate the trace of the exponential of the second term in the Hamiltonian (26). First the gamma commutation, where σ_i are the Pauli matrices (as stated in the course notes)

$$\begin{aligned}
\left[\gamma^0, \gamma^3\right] &= \gamma^0\gamma^3 - \gamma^3, \gamma^0 \\
&= \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} - \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \\
&= 2\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}
\end{aligned} \tag{28}$$

Now obtaining the trace

$$\begin{aligned}
\mathbf{tr}_\gamma\left[\exp\left(i\tau e\frac{i}{2}\left[\gamma^0, \gamma^3\right]E_3\right)\right] &= \mathbf{tr}_\gamma\left[e^{-e\tau\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}E_3}\right] \\
= e^{e\tau\sigma_3 E_3} + e^{-e\tau\sigma_3 E_3} = 2e^{e\tau E_3} &+ 2e^{-e\tau E_3} = 4\cosh\left(\tau e E_3\right)
\end{aligned} \tag{29}$$

What is left is the trace of the first term of the Hamiltonian (26) and according to [Itzykson *et al.* 1980] by reshuffling and then using correspondence to the harmonic oscillator's evolution operator, we obtain

$$\mathbf{tr}\left[e^{-i\tau\tilde{D}^2}\right] = \mathbf{tr}\left[e^{-i\tau(p_\mu + ieA_\mu)^2}\right] = \frac{eE_3}{\tau\left(2\pi\right)^2\sinh\left(\tau e E_3\right)} \tag{30}$$

Therefore our transformation amplitude will equal to

$$\begin{aligned}
K(x', x''; \tau|A) = \langle x', \tau|x'', 0\rangle &= \mathbf{tr}\left[\exp\left(-i\tau\tilde{D}^2 + 2ie\sigma^{\mu\nu}F_{\mu\nu}\right)\right] \\
&= \frac{4eE_3\cosh\left(\tau e E_3\right)}{\tau\left(2\pi\right)^2\sinh\left(\tau e E_3\right)} = \frac{1}{\tau\left(2\pi\right)^2}eE_3\coth\left(\tau e E_3\right)
\end{aligned} \tag{31}$$

and our one-loop correction Lagrangian

$$\mathcal{L}^{(1)}(x) = -\frac{1}{8\pi^2}\int\limits_0^\infty \frac{d\tau}{\tau^2} e^{-i\tau m^2} eE_3 \coth\left(\tau eE_3\right) \tag{32}$$

We need to check that if $E_3 = 0$ the $\mathcal{L}^{(1)}(x)$ should equal to zero

$$\mathcal{L}^{(1)}(E_3 = 0) = -\frac{1}{8\pi^2}\int\limits_0^\infty \frac{d\tau}{\tau^2} e^{-i\tau m^2} \neq 0 \tag{33}$$

Therefore, we need to subtract this value from (34) and therefore our one-loop lagrangian correction is equal to

$$\mathcal{L}^{(1)}(x) = -\frac{1}{8\pi^2}\int\limits_0^\infty \frac{d\tau}{\tau^2} e^{-i\tau m^2} \left[eE_3 \coth\left(\tau eE_3\right) - 1\right] \tag{34}$$

and the amplitude for pair production will equal to

$$\begin{aligned} 1 - \left| \langle 0_+ | 0_- \rangle^A \right|^2 &= 1 - \left| e^{iW^{(1)}[A]} \right|^2 = 1 - \left| e^{i\int d^4x \mathcal{L}^{(1)}(x)} \right|^2 \\ &= 1 - \left| \exp\left\{ -\frac{i}{8\pi^2} \int d^4x \int\limits_0^\infty \frac{d\tau}{\tau^2} e^{-i\tau m^2} \left[eE_3 \coth\left(\tau eE_3\right) - 1\right] \right\} \right|^2 \end{aligned} \tag{35}$$

Finally, our corrected Lagrangian (5), with the classic Maxwell Lagrangian for a one dimensional electric field $\mathcal{L}^{(0)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}E_3^2$, will equal to

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} = \frac{1}{2}E_3^2 - \frac{1}{8\pi^2}\int\limits_0^\infty \frac{d\tau}{\tau^2} e^{-i\tau m^2} \left[eE_3 \coth\left(\tau eE_3\right) - 1\right] \tag{36}$$

Endnotes

We have shown how the world line representation can interpret pair production. The proper time presentation is said to be of popular use in String theory in which the concept is extended from world lines to world sheets (or graphs). We have derived the amplitude for pair production using Schwinger proper time method which preserves gauge invariance and in our case offered a method of obtaining the Maxwell Lagrangian perturbative terms using one-loop approximation for the case of a one dimensional electric field.

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