Neutron Beta-Decay

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Abstract

A Detailed account of the V-A theory of neutron beta decay is presented culminating in a precise calculation of the neutron lifetime.

1 Fermi Theory of Beta Decay

The first theory of neutron beta decay was developed by Enrico Fermi in 1933. His approach was a design similar to the radiation theory of light, where light quanta are emitted from an excited atom. The theory could be described as a point-like four fermion vertex similar to the one shown in figure 1, and inclusive of the emission of a neutrino[1]. In general, the theory could be applied to any nucleus but it was also possible to apply it directly to either neutron beta decay or muon decay.

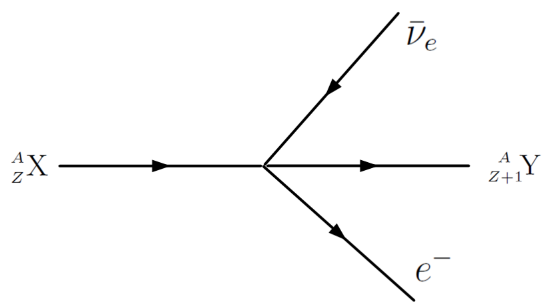

Figure 1: *Four-fermion vertex*

While Fermi's theory is now obsolete, it is considered an historical document for the simple reason that it was the first successful theory of the creation of massive particles. Among other things it also derived quantitative expressions for the lifetime of the decay as well as the shape of the electron emission spectrum.

2 Muon Decay

As an intermediate step towards the modern theory of neutron beta decay, it is important to review the related subject of muon decay. While both decays are four-fermion processes,

$$n \rightarrow p^+ + e^- + \nu_e \tag{2.1a}$$

$$\mu^- \rightarrow e^- + \nu_\mu + \nu_e \tag{2.1b}$$

the mathematics of muon decay are much easier for the simple reason that the proton and neutron in (2.1a) are composite particles, while all four particles in (2.1b) are elementary.

The tree level diagram for muon decay is shown in figure 2. For the weak interaction, the Feynman calculus uses the weak V-A vertex factor

$$\mathcal{K}^\mu \rightarrow -\frac{ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5) \tag{2.2}$$

[1] The inclusion of a neutrino as part of the interaction was major step forward since its existence was considered speculative at that time.

at both vertices. Due to the large mass of the W^- boson at approximately 80 GeV, the weak force propagator can be approximated by

$$\mathcal{D}_{\mu\nu} \rightarrow i\frac{g_{\mu\nu}}{M_W^2} \tag{2.3}$$

The first order amplitude $\mathcal{M}$ for the decay is then determined from

$$-i\mathcal{M} = [\bar{u}(\nu_\mu)\mathcal{K}^\mu u(\mu)] \cdot \mathcal{D}_{\mu\nu} \cdot [\bar{u}(e)\mathcal{K}^\nu v(\nu_e)] \tag{2.4}$$

where the quantities u and v are momentum space Dirac spinors with $\bar{u} = u^\dagger\gamma^0$

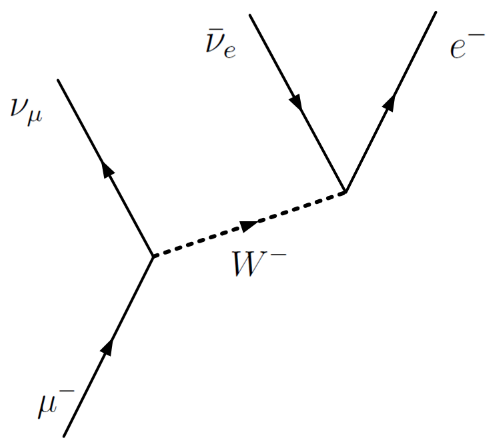

Figure 2: *Feynman diagram representing muon decay*

and $\bar{v} = v^\dagger\gamma^0$. Inserting the functional form of the propagator and the vertex factors shows that

$$\mathcal{M} = \frac{g_w^2}{8M_W^2}[\bar{u}(\nu_\mu)\gamma^\mu(1-\gamma^5)u(\mu)] \cdot [\bar{u}(e^-)\gamma_\mu(1-\gamma^5)v(\nu_e)] \tag{2.5}$$

To use this amplitude to calculate the decay rate of the muon requires a calculation of $< |\mathcal{M}|^2 >$ which is an average over initial spin of the muon of and a sum over over final spins of outgoing particles. Using trace theorems, and assuming massless neutrinos, one finds

$$< |\mathcal{M}|^2 >= \frac{g_w^4}{64M_W^4} Tr[\gamma^\mu(1-\gamma^5)(\not{p}_\mu + m_\mu)\gamma^\nu(1-\gamma^5)\not{p}_{\nu_\mu}] \\ \times Tr[\gamma_\mu(1-\gamma^5)(\not{p}_{\nu_e})\gamma_\nu(1-\gamma^5)(\not{p}_e + m_e)] \tag{2.6}$$

The first trace in this equation calculates to

$$Tr[1] = 8[p_\mu^\alpha p_{\nu_\mu}^\beta + p_\mu^\beta p_{\nu_\mu}^\alpha - g^{\alpha\beta}(p_\mu \cdot p_{\nu_\mu}) - i\epsilon^{\alpha\beta\lambda\rho}(p_\mu)_\lambda(p_{\nu_\mu})_\rho] \tag{2.7}$$

while the second trace can be determined by switching particles labels in the previous equation $\mu \rightarrow \nu_e$ and $\nu_\mu \rightarrow e^-$ and using covariant indices. A contraction on indices yields the final result

$$< |\mathcal{M}|^2 >= 2 \left[\frac{g_w}{M_W}\right]^4 (p_\mu \cdot p_{\nu_\mu}) (p_{\nu_e} \cdot p_e) \tag{2.8}$$

Calculation of the decay rate of the muon now proceeds using Fermi's golden rule which can be written

$$d\Gamma = \frac{|\mathcal{M}|^2}{2m_o\hbar} \prod_{i=1}^{n} \frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2E_i} \cdot (2\pi)^4 \delta^4(p_o - p_1 - \cdots - p_n) \tag{2.9}$$

where m_o is the mass of the decaying particle and n is the number of particles to which it decays. The calculated first order decay rate is

$$\Gamma = \frac{m_\mu^5 c^4}{192\pi^3\hbar^7} G_F^2 \tag{2.10}$$

where the Fermi coupling contant G_F is defined by

$$G_F \equiv \frac{\sqrt{2}}{8} \left[\frac{g_w}{M_W}\right]^2 \cdot (\hbar c)^3 \tag{2.11}$$

The value of the Fermi coupling constant follows from experimental measurements of the muon decay rate:

$$G_F/(\hbar c)^3 = 1.16637(1) \times 10^{-5} \mathrm{GeV}^{-2} \tag{2.12}$$

The precision with which G_F can be determined is based on the precision of the muon decay rate measurement.

3 First Order Calculation of Neutron Lifetime

A first approximation to neutron beta decay follows by assuming that both the neutron and the proton are pointlike Dirac particles which couple directly to a W^-. The four fermion Feynman diagram using this scheme is shown in figure 3. The matrix element for this decay is easily determined by simply replacing $u(\nu_\mu)$ and $u(\mu)$ in equation (2.5) with new spinors $u(n)$ and $u(p)$ representing the neutron and proton respectively.

Even with a massive proton as a new decay product, the form of the resulting matrix element $\mathcal{M}$ is identical to muon decay (except for particle labels) and reads

$$< |\mathcal{M}|^2 >= 2 \left[\frac{g_w}{M_W}\right]^4 (p_n \cdot p_p) (p_{\nu_e} \cdot p_e) \tag{3.1}$$

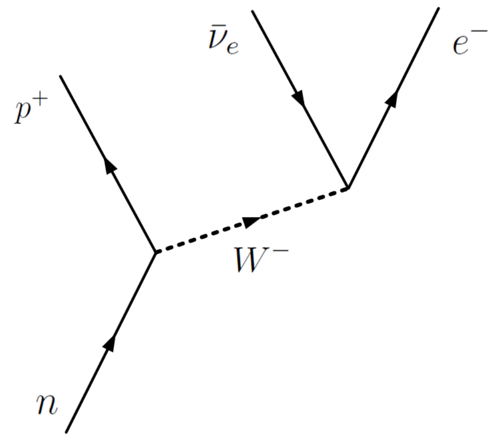

Figure 3: *Neutron Decay assuming a pointlike proton and neutron*

Assuming the neutron is initially at rest, conservation of energy and momentum can be used to write this in terms of the neutrino energy E_ν:

$$< |\mathcal{M}|^2 >= \left[\frac{g_w}{M_W}\right]^4 M_n E_\nu[(M_n^2 - M_p^2 - m_e^2) - 2M_n E_\nu] \tag{3.2}$$

This amplitude can now be inserted into Fermi's Golden rule to determine the neutron lifetime. Skipping the details of this lengthy calculation, the final decay rate formula indicating the distribution of electron energies is

$$\frac{d\Gamma}{dE_e} = \frac{1}{16\pi^3\hbar}\left[\frac{g_w}{M_W}\right]^4 p_e E_e(E_o - E_e)^2 \tag{3.3}$$

where

$$p_e = (E_e^2 - m_e^2)^{1/2} \qquad \text{and} \qquad E_o = M_n - M_p \tag{3.4}$$

A plot of decay rate vs electron energy is shown in figure 4. The minimum energy is simply the rest energy m_e of the electron while the maxmimum energy is approximately

$$E_{max} \approx M_n - M_p \tag{3.5}$$

Using these values as the limits of integration produces

$$I = \int p_e E_e(E_o - E_e)^2 dE_e = f^R m_e^5 \tag{3.6}$$

The quantity f^R is a phase space term. It can be written

$$f^R = \frac{1}{60}[2\xi^4 - 9\xi^2 - 8](\xi^2 - 1)^{1/2} + \frac{1}{4}\xi \ln[\xi + (\xi^2 - 1)^{1/2}] \tag{3.7}$$

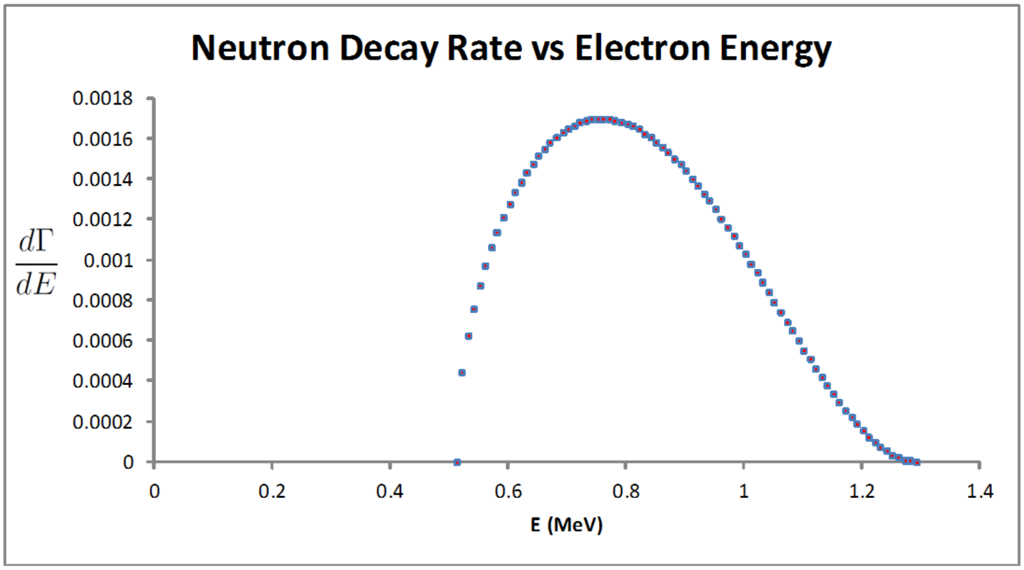

Figure 4: *Electron energy spectrum*

where ξ is simply the ratio of the integration limits

$$\xi \equiv \frac{M_n - M_p}{m_e} \tag{3.8}$$

The decay rate now follows as

$$\Gamma = \frac{f^R}{16\pi^3\hbar}\left[\frac{g_w}{M_W}\right]^4 m_e^5 \tag{3.9}$$

In terms of the Fermi coupling constant—and including appropriate factors of c—this is

$$\boxed{\Gamma = \frac{2f^R}{\pi^3\hbar^7}G_F^2\, m_e^5 c^4} \tag{3.10}$$

Using $f^R \sim 1.63329$ the first order decay rate is:

$$\tau = \frac{1}{\Gamma} = 1316.48s \tag{3.11}$$

Compare this calculation with the precision experimental value of $\tau = 881.5 \pm 1.5s$ which gives an error of approximately fifty percent.

4 Neutron Beta Decay With Spectator Quarks

Like muon decay, the theory of neutron beta decay based on the Feynman diagram of figure 3 is still a V-A theory and produces surprisingly good results. However, a more sophisticated theory will take into consideration the observed generational mixing of quarks associated with the weak interaction. Moreover, since a neutron is composite particle, the presence of spectator quarks must also be included in the theory. The Feynman diagram representing this decay is shown in 5.

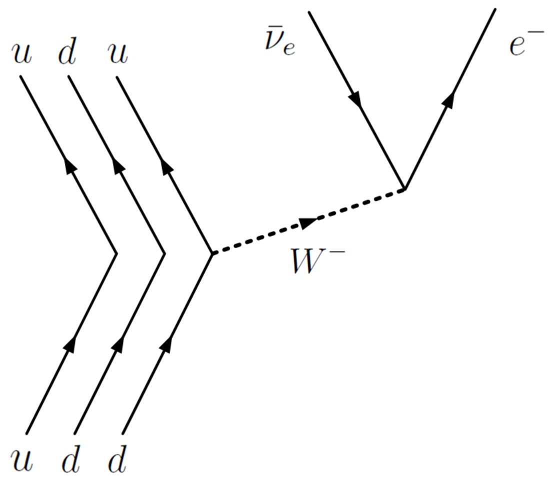

Figure 5: *Neutron decay with spectator quarks.*

4.1 Calculation of Matrix Element

Accounting for spectator u- and d- quarks is easily accomplished by requiring both the vector and axial vector portions of the quark-quark vertex factor to assume the more general form

$$\mathcal{K}^\mu \rightarrow -\frac{ig_w}{2\sqrt{2}}\gamma^\mu(g_V - g_A\gamma^5) \tag{4.1}$$

where quantities g_V and g_A are constants. Note that this is still a V-A theory except that the magnitudes of the vector and axial vector couplings have changed . If the vector portion of the coupling is conserved (the so-called Conserved Vector Current hypothesis), the value of g_v is exactly one. If this is not the case then the ratio of the constants is the important parameter which has an experimental particle data booklet value of

$$\lambda = g_A/g_V = -1.2695 \pm 0.0029 \tag{4.2}$$

Inclusion of the constants g_V and g_A is not the whole story however because electroweak theory also allows for generational mixing at the quark-quark vertex and this also modifies the vertex factor in (4.1) by multipliaction of a constant V_{ud}. This constant is an element of the 3×3 Cabibbo-Kobayashi-Maskawa (CKM) matrix.

$$U_{[CKM]} = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \tag{4.3}$$

and can be written as $V_{ud} = \cos\theta_c$ where θ_c is the Cabibbo angle. The experimental value for V_{ud} is currently known to be

$$V_{ud} = 0.97377 \pm 0.00027 \tag{4.4}$$

The tree level matrix element for the composite neutron now follows as

$$\mathcal{M} = \frac{g_w^2}{8M_W^2} V_{ud} \cdot [\bar{u}(p)\gamma^\mu(g_{\scriptscriptstyle V} - g_{\scriptscriptstyle A}\gamma^5)u(n)] \cdot [\bar{u}(e)\gamma_\mu(1-\gamma^5)v(\nu_e)] \tag{4.5}$$

Unfortunately, evaluation of this quantity is complicated. In general it has the form

$$< |\mathcal{M}|^2 > = < |\mathcal{M}_1|^2 > + < |\mathcal{M}_2|^2 > + < |\mathcal{M}_3|^2 > \tag{4.6}$$

where

$$< |\mathcal{M}_1|^2 > = \frac{1}{2}(1+\lambda)^2 |V_{ud}|^2 g_{\scriptscriptstyle V}^2 \left[\frac{g_w}{M_W}\right]^4 (p_n \cdot p_{\nu_e})\ (p_p \cdot p_e) \tag{4.7a}$$

$$< |\mathcal{M}_2|^2 > = \frac{1}{2}(1-\lambda)^2 |V_{ud}|^2 g_{\scriptscriptstyle V}^2 \left[\frac{g_w}{M_W}\right]^4 (p_n \cdot p_e)\ (p_{\nu_e} \cdot p_p) \tag{4.7b}$$

$$< |\mathcal{M}_3|^2 > = \frac{1}{2}(\lambda^2-1) |V_{ud}|^2 g_{\scriptscriptstyle V}^2 \left[\frac{g_w}{M_W}\right]^4 M_n M_p\, (p_{\nu_e} \cdot p_e) \tag{4.7c}$$

For $g_{\scriptscriptstyle V} \to 1$, $|V_{ud}| \to 1$ and $\lambda \to 1$ the decay amplitude reduces to that of equation (2.8) with appropriate label changes for the particles.

4.2 Calculation of Decay Rate

The amplitudes in (4.7) must now be inserted into Fermi's Golden rule to obtain the decay rate as function of electron energy. One possibility for accomplishing this is to write

$$\frac{d\Gamma}{dE_e} = \sum_{i=1}^{3} \frac{d\Gamma_i}{dE_e} \tag{4.8}$$

which can then be integrated to yield the total decay rate.

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \tag{4.9}$$

Another more useful approach is to consider the decay rate per unit electron energy per unit solid angles of electron and neutrino emission which can be written

$$\frac{d\Gamma}{dE_e d\Omega_e d\Omega_\nu} = \frac{1}{(4\pi)^5 \hbar M_n} \int_{\nu_e} |\mathcal{M}|^2 dE_\nu \tag{4.10}$$

where the integration is over all possible neutrino energies. The limits of integration are easily determined by straightforward application of conservation of energy and momentum. If $\theta_{e\nu}$ is the angle between the emitted electron and neutrino then the minimum and maximum values of E_ν will follow from

$$E_\nu = \frac{\frac{1}{2}(M_n^2 - M_p^2 + m_e^2) - M_n E_e}{M_n - E_e + p_e \cos\theta_{e\nu}} \tag{4.11}$$

Having identified the limits of integration the next step is to evaluate the dot products in equations (4.7):

$$(p_n \cdot p_{\nu_e})\,(p_p \cdot p_e) = \tfrac{1}{2} M_n E_e \left(M_n^2 - M_p^2 - m_e^2 - 2M_n E_\nu\right) \tag{4.12a}$$

$$(p_n \cdot p_e)\,(p_{\nu_e} \cdot p_p) = \tfrac{1}{2} M_n E_e \left(M_n^2 - 2M_n E_e + m_e^2 - M_p^2\right) \tag{4.12b}$$

$$(p_{\nu_e} \cdot p_e) = E_\nu E_e - \boldsymbol{p}_\nu \cdot \boldsymbol{p}_e \tag{4.12c}$$

Inserting these into the matrix elements of (4.7) gives contributions to the total decay rate per unit solid angle

$$\frac{d\Gamma_1}{dE_e d\Omega_e d\Omega_\nu} = (1+\lambda)^2 \frac{|V_{ud}|^2 g_V^2}{(4\pi)^5 \hbar} \left[\frac{g_w}{M_W}\right]^4 p_e E_e (E_o - E_e)^2 \tag{4.13a}$$

$$\frac{d\Gamma_2}{dE_e d\Omega_e d\Omega_\nu} = (1-\lambda)^2 \frac{|V_{ud}|^2 g_V^2}{(4\pi)^5 \hbar} \left[\frac{g_w}{M_W}\right]^4 p_e E_e (E_o - E_e)^2 \tag{4.13b}$$

$$\frac{d\Gamma_3}{dE_e d\Omega_e d\Omega_\nu} = (\lambda^2 - 1) \frac{|V_{ud}|^2 g_V^2}{(4\pi)^5 \hbar} \left[\frac{g_w}{M_W}\right]^4 p_e E_e (E_o - E_e)^2 \left(1 - \frac{\boldsymbol{p}_\nu \cdot \boldsymbol{p}_e}{E_\nu E_e}\right) \tag{4.13c}$$

Each of these terms can now be added together to yield the result

$$\frac{d\Gamma}{dE_e d\Omega_e d\Omega_\nu} = \frac{1+3\lambda^2}{(4\pi)^5 \hbar} |V_{ud}|^2 g_V^2 \left[\frac{g_w}{M_W}\right]^4 p_e E_e (E_o - E_e)^2 \left[1 + a\frac{\boldsymbol{p}_\nu \cdot \boldsymbol{p}_e}{E_\nu E_e}\right] \tag{4.14}$$

where a is the ***electron-neutrino angular correlation coefficient*** defined by:

$$a \equiv \frac{1 - |\lambda|^2}{1 + 3|\lambda|^2} \tag{4.15}$$

Equation (4.14) can now be integrated over electron energies and solid angles to determine the decay rate Γ. The integral over solid angles can be performed by inspection.

The term containing $\boldsymbol{p}_\nu \cdot \boldsymbol{p}_e$ integrates to zero while the first term introduces a factor of 4π since there is no angular dependence. Then

$$\frac{d\Gamma}{dE_e} = \frac{1+3\lambda^2}{64\pi^3\hbar}|V_{ud}|^2 g_V^2 \left[\frac{g_w}{M_W}\right]^4 p_e E_e (E_o - E_e)^2 \tag{4.16}$$

Now integrate over electron energies which has already been done in equation (3.6) to obtain

$$\Gamma = \frac{f^R m_e^5 c^4}{64\pi^3\hbar}\left[\frac{g_w}{M_W}\right]^4 |V_{ud}|^2 g_V^2 (1+3\lambda^2) \tag{4.17}$$

This formula can be compared with equation (3.10) which shows that the net effect of the V-A theory is to introduce an overal factor of

$$\tfrac{1}{4}|V_{ud}|^2 g_V^2 (1+3\lambda^2) \tag{4.18}$$

into the value of the decay rate. Once again the decay rate may be re-written in terms of the Fermi coupling constant

$$\Gamma = \frac{f^R m_e^5}{2\pi^3\hbar^7}|V_{ud}|^2 G_F^2 (1+3\lambda^2) \tag{4.19}$$

Defining

$$G_V = G_F \cdot V_{ud} \qquad \text{and} \quad G_A = \lambda G_F \cdot V_{ud} \tag{4.20}$$

allows for the final result

$$\boxed{\Gamma = \frac{f^R m_e^5 c^4}{2\pi^3\hbar^7}(G_V^2 + 3G_A^2)} \tag{4.21}$$

Actual numbers can now be inserted to calculate the decay rate. Using a somewhat more precise value of $f^R = 1.71482$ gives a lifetime of

$$\tau = 907.637s \tag{4.22}$$

Comparing this with the experimental value shows an error of less than 2.5 percent.

4.3 Additional Terms for the Decay Rate

The theory of the previous section uses only two independent parameters g_V and g_A, but it is possible that the correct formulation of the theory could involve as many as twenty. While this number may be too large, experimental data indicates that equation (4.14) is not the whole story. One possibility is the following decay rate formula

$$\frac{d\Gamma}{dE_e d\Omega_e d\Omega_\nu} = \frac{1}{\tau_n}F(E_e)\left[1 + a\frac{\boldsymbol{p}_e \cdot \boldsymbol{p}_\nu}{E_e E_\nu} + b\frac{m_e}{E_e} + <\boldsymbol{\sigma}> \cdot \left(A\frac{\boldsymbol{p}_e}{E_e} + B\frac{\boldsymbol{p}_\nu}{E_\nu} + D\frac{\boldsymbol{p}_e \times \boldsymbol{p}_\nu}{E_e E_\nu}\right)\right] \tag{4.23}$$

which shows four additional terms—three of which involve direct correlations with the neutron spin vector $\boldsymbol{\sigma}$. The term marked by the coefficient A represents a direct coupling between the direction of the emitted electron and the neutron spin, while term marked by the coefficient B represents a similar coupling involving the neutrino. Both coefficents can be written in terms of the more general complex[2] constant λ as

$$A \equiv -\frac{2(|\lambda|^2 + Re[\lambda])}{1+3|\lambda|^2} \qquad B \equiv \frac{2(|\lambda|^2 - Re[\lambda])}{1+3|\lambda|^2} \tag{4.24}$$

Officially, A and B are known as:

- A: ***neutron-spin electron-momentum correlation coefficient***
- B: ***neutron-spin antineutrino-momentum correlation coefficient***

and they may be referred to collectively as *asymmetry parameters*. Along with the definition of the parameter a defined in equation (4.15), the relations

$$a + B - A = 1 \qquad aB - A - A^2 = 0 \qquad a^2 + A^2 + B^2 = 1 \tag{4.25}$$

can be easily verified.

In addition to correlations involving A and B there also exists the possibility of a term involving the coefficient D known as the ***triple correlation coefficient***. This constant is defined in terms of the complex component of λ and may in fact be zero:

$$D \equiv \frac{2Im[\lambda]}{1+3|\lambda|^2} \tag{4.26}$$

Current estimates suggest that $D \sim (-4 \pm 6) \times 10^{-4}$. If a non-zero value can be confirmed this would be direct evidence of the violation of time reversal symmetry.

The only other term in (4.23) yet to be discussed involves the coefficient b known as the ***Fierz interference term***. This term arises as result of cross terms between the various couplings which appear in the interaction Hamiltonian, and does not correlate with the neutron spin. Unlike terms involving a, A, B and D, the presence of this term provides a small modification to the energy distribution spectrum of the electron and the final value of the decay rate. At this time no experimental data exists on the value of b.

[2]The addition of a small complex component to λ implies another parameter in addition to g_V and g_A.

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