Calculation of the Lamb Shift

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One of the first motivating achievements of quantum field theory was the calculation of the Lamb shift by Hans Bethe[1] in agreement with experimental data taken by Lamb and Retherford[2] in 1947. This effect increases the energy of hydrogen’s 2S$_{1/2}$ state, destroying the degeneracy predicted by Dirac theory with the 2P$_{1/2}$ state. Bethe carried out this calculation on a train ride returning from a conference where he was inspired by the suggestions of Schwinger and Weisskopf. This work will follow his calculation in detail relying heavily on the original paper and the corresponding section of Itzykson’s Quantum Field Theory[3].

Bethe attributed this energy shift in the s-wave state to the difference in electromagnetic self-energy of the electron in the external field created by the nucleus and that of a free electron. Non-relativistic second order perturbation theory gives this self-energy to good approximation. The appropriate Schrödinger equation is

$$(E + \delta E)\psi = \left[ \frac{1}{2m} \left( \frac{1}{i} \nabla - e A_q \right)^2 + e(A^0 + A^0_q) \right] \psi$$

where $\psi$ is a wavefunction describing both the electron and the radiation field $A_q$:

$$A_q(x) = \int_{|k|<K} \frac{d^3k}{2k(2\pi)^3} \left[ \sum_{\lambda=1,2} (\mathbf{e}_\lambda(k)d^{(\lambda)}e^{-ik\cdot x} + e^{ik\cdot x}d^{(\lambda)*}e_{\lambda}(k)) \right]$$

$$A^0_q(x) = 0$$

$K$ represents the relativistic upper limit of the three-space momentum of virtual photons: $K \sim mc^2$. The following interaction is treated as a perturbation:

$$-\frac{1}{2m} \left( \frac{1}{i} \nabla e A_q + e A_q \frac{1}{i} \nabla - e^2 A^2_q \right).$$

The last term affects all levels in the same way, and is excluded from the calculation since it plays no part in the non-degeneracy. The first two terms give the following energy correction:

$$\delta E_m = \sum_{n\neq m} \int_{|k|<K} \frac{d^3k}{2k(2\pi)^3} \sum_{\lambda} e^2 \int d^3 x \left| \psi_m^*(x) \left[ \frac{(1/i)\nabla \cdot e_{\lambda}(k)e^{-ik\cdot x} + e^{ik\cdot x}e_{\lambda}(k) \cdot (1/i)\nabla} {E_m - E_n - k} \right] \psi_n(x) \right|^2$$
The limits of the integral with respect to $x$ are limited to the order of the Bohr radius $a = 1/(mZ\alpha)$. Thus $k \cdot x \sim K/(mZ\alpha)$ and $e^{ikx} \sim 1$. Defining

$$v = p = \frac{\nabla}{im}$$

and noting

$$\sum_{\lambda=1,2} \left| \langle m | v \cdot e(k) | n \rangle \right|^2 = \left( \delta_{ab} - \frac{k^a k^b}{k^2} \right) \langle m | v^a | n \rangle \langle n | v^b | m \rangle$$

and

$$\int \frac{d\Omega}{4\pi} \left( \delta_{ab} - \frac{k^a k^b}{k^2} \right) = \frac{2}{3} \delta_{ab},$$

we find the self-energy correction

$$\delta E_m = \frac{2\alpha}{3\pi} \sum_n \int_0^K \frac{kdk}{E_m - E_n - k} \left| \langle m | v | n \rangle \right|^2$$

for an electron in quantum state $m$ due to its interaction with transverse electromagnetic waves. Since it is the difference between this shift and the shift for an electron in vacuum that is observable, we subtract

$$\delta E_{\text{free}} = \frac{2\alpha}{3\pi} \sum_n \int_0^K \frac{kdk}{k} \left| \langle m | v | n \rangle \right|^2 = \frac{2\alpha}{3\pi} \int_0^K \frac{kdkv^2}{k}$$

from the bound state correction:

$$\delta E = \delta E_m - \delta E_{\text{free}} = \frac{2\alpha}{3\pi} \sum_n \int_0^K dk \left( \frac{1}{E_m - E_n - k} \right) \left| \langle m | v | n \rangle \right|^2.$$

Since $K \gg |E_n - E_m|$, integrating over $k$ results in

$$\delta E_m = \frac{2\alpha}{3\pi} \sum_n (E_n - E_m) \ln \left| \frac{K}{E_n - E_m} \right| \left| \langle m | v | n \rangle \right|^2.$$

Since the argument in the logarithm is nearly constant to first approximation, it suffices to calculate the sum using the average value of $E_n - E_m$. The sum becomes

$$\sum_n (E_n - E_m) \left| \langle m | v | n \rangle \right|^2 = \int \psi_n^* \nabla \cdot \nabla \psi_n d\tau = \frac{1}{2} \int \nabla^2 \psi_n^2 d\tau = 2\pi e^2 Z\psi_m^2(0).$$

The full result for the total energy shift is then found to be
\[ \delta E_m = \frac{Ze^2\alpha}{3}\psi_m^2(0)\ln \left| \frac{K}{\langle E_n - E_m \rangle_{Av}} \right|. \]

This result is zero for states \( m \) with angular momentum \( l > 0 \) because the wavefunctions vanish at the origin. For \( s \)-wave states,

\[ \psi_m^2(0) = \frac{1}{\pi} \left( \frac{Z}{na} \right)^3 \]

where \( a \) is the Bohr radius. Using the upper limit \( K \) proposed above and the numerical result 17.8 Ryd for \( \langle E_n - E_m \rangle_{Av} \), we find for the \( 2S_{1/2} \) state of hydrogen

\[ \delta E_{2s} = \frac{e^2\alpha}{3\pi} \left( \frac{1}{2a} \right)^3 \ln \left| \frac{m}{\langle E_n - E_m \rangle_{Av}} \right| = 1040 \text{ MHz}. \]

With no shift applicable to the \( 2P_{1/2} \) state due to its nonzero angular momentum, this is the calculated degeneracy between these states. This is near to agreement with the modern experimental data, which gives a value of 1057.862 \( \pm \) 0.020 MHz. The discrepancy can be attributed to non-relativistic limitation of the calculation.