Spinors in Curved Space

Erik Olsen

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Introduction: A Useful Method

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Bibliography
The problem: How to put gravity into a Lagrangian density?
Spinors in Curved Space

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- The solution: The Principle of General Covariance
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Spinors in Curved Space

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- 1. Ignoring gravity, find the equations of motion
- 2. Make the following substitutions:
  - Lorentz Tensors become Tensor-like objects
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  - Minkowski tensors ($\eta$ matrices) become the metric tensor for curved spacetime $g_{\mu\nu}$
Spinors in Curved Space

- A major issue: this method does not work for spinors
A major issue: this method does not work for spinors

No representations of GL(4) act like spinors under an infinitesimal Lorentz transformation
Tetrad Formalism

\[ g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \]  

where \( \xi \) represents a coordinate system under the influence of gravity.
Tetrad Formalism

\[ g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \]  

(1)

where \( \xi \) represents a coordinate system under the influence of gravity.

Let \( z_\alpha^X \) be normal local coordinates to each point in space-time \( X \)
Tetrad Formalism

\[ g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \quad (1) \]

where \( \xi \) represents a coordinate system under the influence of gravity.

Let \( z^\alpha_X \) be normal local coordinates to each point in space-time \( X \)

\[ g_{\mu\nu}(x) = \frac{\partial z^\alpha_X(x)}{\partial x^\mu} \frac{\partial z^\beta_X(x)}{\partial x^\nu} \eta_{\alpha\beta} \quad (2) \]
Tetrad Formalism

Let:

\[
\left( \frac{\partial z^\alpha_X(x)}{\partial x^\mu} \right)_{x=X} = V_\mu^\alpha(X) \tag{3}
\]

\[
\left( \frac{\partial z^\beta_X(x)}{\partial x^\nu} \right)_{x=X} = V_\nu^\beta(X) \tag{4}
\]
Let:

\[
\left( \frac{\partial z^\alpha_X(x)}{\partial x^\mu} \right)_{x=X} = V^\alpha_\mu(X) \tag{3}
\]

\[
\left( \frac{\partial z^\beta_X(x)}{\partial x^\nu} \right)_{x=X} = V^\beta_\nu(X) \tag{4}
\]

\[V^\alpha_\mu(X)\] is a tetrad.
Tetrad Formalism

Let:

\[
\left( \frac{\partial z_{X}^{\alpha}(x)}{\partial x^{\mu}} \right)_{x=X} = V_{\mu}^{\alpha}(X) \quad (3)
\]

\[
\left( \frac{\partial z_{X}^{\beta}(x)}{\partial x^{\nu}} \right)_{x=X} = V_{\nu}^{\beta}(X) \quad (4)
\]

\[ V_{\mu}^{\alpha}(X) \text{ is a tetrad.} \]

\[ \text{Fix } z_{X}^{\alpha}, \text{ change } x^{\mu} \text{ to } x^{\prime \mu} \]
Tetrad Formalism

Let:

\[ \left( \frac{\partial z^\alpha(x)}{\partial x^\mu} \right)_{x=X} = V_\mu^\alpha(X) \quad (3) \]

\[ \left( \frac{\partial z^\beta(x)}{\partial x^\nu} \right)_{x=X} = V_\nu^\beta(X) \quad (4) \]

- \( V_\mu^\alpha(X) \) is a tetrad.
- Fix \( z_\mu^\alpha \), change \( x^\mu \) to \( x'^\mu \)

\[ V_\mu^\alpha \rightarrow V'_\mu^\alpha \quad (5) \]
The Tetrad Formalism

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Spinors in Curved Space
Tetrad Formalism

\[ V'_{\mu}^\alpha = \frac{\partial z_\alpha^\mu}{\partial x'_{\mu}} \rightarrow \quad (6) \]

\[ V_{\mu}^{\prime \alpha} = \frac{\partial z_\alpha^\mu}{\partial x'_{\mu}} \frac{\partial x^\nu}{\partial x_\nu} \rightarrow \quad (7) \]

\[ V_{\mu}^{\prime \alpha} = \frac{\partial x^\nu}{\partial x'_{\mu}} \frac{\partial z_\alpha^\mu}{\partial x_\nu} \quad (8) \]

\[ V_{\mu}^{\prime \alpha} = \frac{\partial x^\nu}{\partial x'_{\mu}} V_{\nu}^\alpha \quad (9) \]
Tetrad Formalism

- Taking the Lorentz transform of $z_{\alpha}^{\chi}$
Taking the Lorentz transform of $z^\alpha_X$

$$z^\alpha_X \longrightarrow z'^\alpha_X = \Lambda^\alpha_\beta(X)z^\beta_X \quad (10)$$
Tetrad Formalism

Taking the Lorentz transform of $z_X^\alpha$

\[ z_X^\alpha \rightarrow z_X'^\alpha = \Lambda^\alpha_\beta (X) z_X^\beta \]  \hspace{1cm} (10)

\[ V_\mu^\alpha (X) = \frac{\partial }{\partial x_\mu } (\Lambda^\alpha_\beta z_X^\beta ) \rightarrow \]  \hspace{1cm} (11)

\[ V_\mu^\alpha (X) = \Lambda^\alpha_\beta \frac{\partial z_X^\beta }{\partial x_\mu } \]  \hspace{1cm} (12)

\[ V_\mu^\alpha (X) \rightarrow \Lambda^\alpha_\beta V_\mu^\beta (X) \]  \hspace{1cm} (13)
Tetrad Formalism

Let $B_\mu$ be a generally covariant vector.
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Let $B_\mu$ be a generally covariant vector.

\[ V^\mu_\alpha B_\mu = B_\alpha \]  \hspace{1cm} (14)
Tetrad Formalism

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\[ V_\alpha^\mu B_\mu = B_\alpha \]  

(1) Under a local Lorentz transformation it will behave as a vector.
Let $B_\mu$ be a generally covariant vector.

\[ V^\mu_\alpha B_\mu = B_\alpha \]  \hspace{1cm} (14)

(1) Under a local Lorentz transformation it will behave as a vector.

(2) Under a general coordinate transformation it will transform as four scalars.
Let $\nabla_\alpha$ be a covariant derivative and $\psi$ be a field.
Let $\nabla_\alpha$ be a covariant derivative and $\psi$ be a field

\[ \nabla_\alpha \psi \rightarrow \Lambda_\alpha^\beta(x)D(\Lambda(x))\nabla_\beta \psi(x) \quad (15) \]
Let $\nabla_\alpha$ be a covariant derivative and $\psi$ be a field

$$\nabla_\alpha \psi \to \Lambda_\alpha^\beta(x) D(\Lambda(x)) \nabla_\beta \psi(x) \quad (15)$$

$D(\Lambda)$ is the matrix representation of the infinitesimal Lorentz group
Covariant Derivatives

- Let $\nabla_\alpha$ be a covariant derivative and $\psi$ be a field

\[
\nabla_\alpha \psi \rightarrow \Lambda_\alpha^\beta (x) D(\Lambda(x)) \nabla_\beta \psi (x)
\]  

(15)

- $D(\Lambda)$ is the matrix representation of the infinitesimal Lorentz group

\[
\Gamma_\mu (x) = \frac{1}{2} \Sigma^{\alpha\beta} V_\alpha^\nu (\nabla_\mu V_{\beta\nu}(x))
\]

(16)
Covariant Derivatives

Where $\Sigma^{\alpha\beta}$ is the group generator for the Lorentz group
Covariant Derivatives

➤ Where $\Sigma^{\alpha\beta}$ is the group generator for the Lorentz group

$$V_{\beta\nu} = g_{\mu\nu} V_{\beta}^{\mu}$$  \hspace{1cm} (17)
The Spin $\frac{1}{2}$ Field

\[ L(x) = \frac{1}{2} i (\overline{\psi} \gamma^\alpha \partial_\alpha \psi - \partial_\alpha \overline{\psi} \gamma^\alpha \psi) - m \overline{\psi} \psi \]  

(18)
The Spin $\frac{1}{2}$ Field

\[ L(x) = \frac{1}{2} i (\overline{\psi} \gamma^\alpha \partial_\alpha \psi - \partial_\alpha \overline{\psi} \gamma^\alpha \psi) - m \overline{\psi} \psi \] (18)

\[ \Sigma_{\alpha\beta} = \frac{1}{4} [\gamma_\alpha, \gamma_\beta] \] (19)
The Spin $\frac{1}{2}$ Field

\[ L(x) = \frac{1}{2} i (\bar{\psi} \gamma^\alpha \partial_\alpha \psi - \partial_\alpha \bar{\psi} \gamma^\alpha \psi) - m \bar{\psi} \psi \]  

\[ \Sigma_{\alpha\beta} = \frac{1}{4} [\gamma_\alpha, \gamma_\beta] \]

where the $\gamma$ terms are the Dirac matrices
The Spin $\frac{1}{2}$ Field

\[ \Gamma_{\mu}(x) = \frac{1}{8}[\gamma_\alpha, \gamma_\beta]V_{\alpha}^\nu(x)(\nabla_\mu g_{\mu\nu}(x)V_{\beta}^\mu(x)) \]  (20)
The Spin $\frac{1}{2}$ Field

\[ \Gamma_\mu(x) = \frac{1}{8} [\gamma_\alpha, \gamma_\beta] V^\nu_\alpha(x) (\nabla_\mu g_{\mu\nu}(x) V^\mu_\beta(x)) \] (20)

\[ L(x) = \text{det} V \left\{ \frac{1}{2} i (\overline{\psi} \gamma^\mu \nabla_\mu \psi - (\nabla_\mu \overline{\psi}) \gamma^\mu \psi) - m \overline{\psi} \psi \right\} \] (21)
The Spin $\frac{1}{2}$ Field

\[ \Gamma_{\mu}(x) = \frac{1}{8} [\gamma_{\alpha}, \gamma_{\beta}] V^\nu_{\alpha}(x) (\nabla_{\mu} g_{\mu\nu}(x) V^\mu_{\beta}(x)) \]  \hspace{1cm} (20)

\[ L(x) = \det V \left\{ \frac{1}{2} i(\overline{\psi} \gamma^{\mu} \nabla_{\mu} \psi - (\nabla_{\mu} \overline{\psi}) \gamma^{\mu} \psi) - m \overline{\psi} \psi \right\} \]  \hspace{1cm} (21)

\[ \text{where } \gamma_{\mu} = V_{\alpha}^{\mu} \gamma^{\alpha} \]
The Spin $\frac{1}{2}$ Field

\[
\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \rightarrow (22)
\]

\[
V_\alpha^\mu \gamma^\alpha V_\beta^\nu \gamma^\beta + V_\beta^\nu \gamma^\beta V_\alpha^\mu \gamma^\alpha \rightarrow (23)
\]

\[
V_\alpha^\mu V_\beta^\nu \{\gamma^\alpha, \gamma^\beta\} (24)
\]
The Spin $\frac{1}{2}$ Field

\[
\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \rightarrow (22)
\]

\[
V^\mu_\alpha \gamma^\alpha V^\nu_\beta \gamma^\beta + V^\nu_\beta \gamma^\beta V^\mu_\alpha \gamma^\alpha \rightarrow (23)
\]

\[
V^\mu_\alpha V^\nu_\beta \{\gamma^\alpha, \gamma^\beta\} \rightarrow (24)
\]

\[
\{\gamma^\alpha, \gamma^\beta\} = 2\eta_{\alpha\beta} \rightarrow (25)
\]
The Spin $\frac{1}{2}$ Field

\[
\{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \quad \rightarrow \quad (22)
\]

\[
V_\alpha^\mu \gamma^\alpha V_\beta^\nu \gamma^\beta + V_\beta^\nu \gamma^\beta V_\alpha^\mu \gamma^\alpha \quad \rightarrow \quad (23)
\]

\[
V_\alpha^\mu V_\beta^\nu \{ \gamma^\alpha, \gamma^\beta \} \quad (24)
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\{ \gamma^\alpha, \gamma^\beta \} = 2\eta_{\alpha\beta} \quad (25)
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The Spin $\frac{1}{2}$ Field

\[ \{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \rightarrow \]  
\[ V_\alpha^\mu \gamma^\alpha V_\beta^\nu \gamma^\beta + V_\beta^\nu \gamma^\beta V_\alpha^\mu \gamma^\alpha \rightarrow \]  
\[ V_\alpha^\mu V_\beta^\nu \{ \gamma^\alpha, \gamma^\beta \} \] (24)

\[ \{ \gamma^\alpha, \gamma^\beta \} = 2 \eta_{\alpha \beta} \] (25)

\[ \{ \gamma^\mu, \gamma^\nu \} = 2 V_\alpha^\mu V_\beta^\nu \eta_{\alpha \beta} \] (26)

\[ \{ \gamma^\mu, \gamma^\nu \} = 2 g_{\mu \nu} \] (27)
Spinors do not work with the Principle of General Covariance
Conclusion

- Spinors do not work with the Principle of General Covariance
- Contracting the spinor into the tetrad solves this dilemma
Spinors do not work with the Principle of General Covariance
Contracting the spinor into the tetrad solves this dilemma
All an approximation; quantum effects neglected