PHYSICS 522 - SPRING 2011

Midterm Exam II - Solutions

Problem 1

(a) There are $2 \times 4 = 8$ states in this system, $|m_1 m_2\rangle$, with $m_1 = \pm\frac{1}{2}$, $m_2 = \pm\frac{3}{2}, \pm\frac{1}{2}$.
Let $\vec{S} = \vec{S}_1 + \vec{S}_2$ be the total spin. Then

$$H = \frac{a}{2}[\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2]$$

Evidently, the eigenstates are $|SM\rangle$ with corresponding eigenvalues

$$E_S = \frac{a\hbar^2}{2}\left[S(S+1) - \frac{9}{2}\right] \quad , \quad S = 1, 2$$

of degeneracies $2S+1$ (Check: $3 + 5 = 8$).

(b) Using the standard procedure to switch from the $|SM\rangle$ basis to the $|m_1 m_2\rangle$ basis, we obtain

$$\begin{aligned}
|22\rangle &= |\tfrac{1}{2}\tfrac{3}{2}\rangle \\
|21\rangle &= \frac{1}{2}|-\tfrac{1}{2}\tfrac{3}{2}\rangle + \frac{\sqrt{3}}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle \\
|20\rangle &= \frac{1}{\sqrt{2}}\left[|-\tfrac{1}{2}\tfrac{1}{2}\rangle + |\tfrac{1}{2}-\tfrac{1}{2}\rangle\right] \\
|2-1\rangle &= \frac{1}{2}|\tfrac{1}{2}-\tfrac{3}{2}\rangle + \frac{\sqrt{3}}{2}|-\tfrac{1}{2}-\tfrac{1}{2}\rangle \\
|2-2\rangle &= |-\tfrac{1}{2}-\tfrac{3}{2}\rangle \\
|11\rangle &= -\frac{1}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle + \frac{\sqrt{3}}{2}|-\tfrac{1}{2}\tfrac{3}{2}\rangle \\
|10\rangle &= \frac{1}{\sqrt{2}}\left[|-\tfrac{1}{2}\tfrac{1}{2}\rangle - |\tfrac{1}{2}-\tfrac{1}{2}\rangle\right] \\
|1-1\rangle &= \frac{1}{2}|-\tfrac{1}{2}-\tfrac{1}{2}\rangle - \frac{\sqrt{3}}{2}|\tfrac{1}{2}-\tfrac{3}{2}\rangle
\end{aligned}$$

Problem 2

(a) We have

$$0 = \det(H - E\mathbb{I}) = E^2 - (E_1 + E_2)E - \lambda^2 + E_1 E_2$$

therefore

$$E = E_\pm = \frac{1}{2}\left[E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4\lambda^2}\right]$$

The corresponding normalized eigenvectors are

$$|\pm\rangle = \frac{1}{\sqrt{(E_\pm - E_1)^2 + \lambda^2}} \begin{pmatrix} \lambda \\ E_\pm - E_1 \end{pmatrix}$$

(b) From perturbation theory,

$$\delta E_1 = \lambda \langle 1|W|1\rangle + \lambda^2 \frac{|\langle 2|W|1\rangle|^2}{E_1 - E_2} + \ldots$$

We have

$$\langle 1|W|1\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

showing that the first-order correction vanishes, and

$$\langle 2|W|1\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

Therefore

$$\delta E_1 = \frac{\lambda^2}{E_1 - E_2} + \ldots$$

to be compared with

$$\begin{aligned} E_- - E_1 &= \frac{1}{2}\left[E_2 - E_1 - \sqrt{(E_2 - E_1)^2 + 4\lambda^2}\right] \\ &= \frac{E_2 - E_1}{2}\left[1 - \sqrt{1 + \frac{4\lambda^2}{(E_2 - E_1)^2}}\right] \\ &= -\frac{\lambda^2}{E_2 - E_1} + \ldots \end{aligned}$$

They agree to order λ^2.

The corresponding eigenvector is

$$|1'\rangle = |1\rangle + \lambda \frac{\langle 2|W|1\rangle}{E_1 - E_2}|2\rangle + \ldots = |1\rangle + \frac{\lambda}{E_1 - E_2}|2\rangle + \ldots = \begin{pmatrix} 1 \\ \frac{\lambda}{E_1 - E_2} \end{pmatrix} + \ldots$$

to be compared with

$$|-\rangle = \frac{1}{\lambda}\begin{pmatrix} \lambda \\ \delta E_1 \end{pmatrix} + \ldots = \begin{pmatrix} 1 \\ \frac{\lambda}{E_1 - E_2} \end{pmatrix} + \ldots$$

They agree to order λ.

Similarly,

$$\delta E_2 = \lambda \langle 2|W|2\rangle + \lambda^2 \frac{|\langle 1|W|2\rangle|^2}{E_1 - E_2} + \ldots = \frac{\lambda^2}{E_2 - E_1} + \ldots$$

in agreement with

$$E_+ - E_2 = \frac{\lambda^2}{E_2 - E_1} + \dots$$

and the corresponding eigenvector

$$|2'\rangle = |2\rangle + \lambda \frac{\langle 1|W|2\rangle}{E_2 - E_1}|1\rangle + \dots = |2\rangle + \frac{\lambda}{E_2 - E_1}|1\rangle + \dots = \begin{pmatrix} \frac{\lambda}{E_2 - E_1} \\ 1 \end{pmatrix} + \dots$$

in agreement with

$$|+\rangle = \frac{1}{E_2 - E_1}\begin{pmatrix} \lambda \\ E_2 - E_1 \end{pmatrix} + \dots = \begin{pmatrix} \frac{\lambda}{E_2 - E_1} \\ 1 \end{pmatrix} + \dots$$

Problem 3

We have

$$\langle\alpha|H|\alpha\rangle = 2\int_0^\infty dx \left[\frac{\hbar^2}{2m}{\phi'_\alpha}^2 + V\phi_a^2\right] = 2\int_0^\alpha dx \left[\frac{\hbar^2}{2m} + gx(\alpha - x)^2\right] = \frac{\hbar^2}{m}\alpha + \frac{1}{6}g\alpha^4$$

where we used $\int_{-\infty}^{+\infty} = 2\int_0^\infty$, because all functions are even.
Also,

$$\langle\alpha|\alpha\rangle = 2\int_0^\infty dx\phi_\alpha^2 = 2\int_0^\alpha dx(\alpha - x)^2 = \frac{2}{3}\alpha^3$$

so

$$\langle H\rangle = \frac{\langle\alpha|H|\alpha\rangle}{\langle\alpha|\alpha\rangle} = \frac{3\hbar^2}{2m\alpha^2} + \frac{g\alpha}{4}$$

This is minimized when

$$0 = \frac{d\langle H\rangle}{d\alpha} = -\frac{3\hbar^2}{m\alpha^3} + \frac{g}{4}$$

which gives

$$\alpha = \left(\frac{12\hbar^2}{mg}\right)^{1/3}$$

and

$$E_0 = \frac{3^{4/3}}{4}\left(\frac{\hbar^2 g^2}{2m}\right)^{1/3} \approx 1.082\left(\frac{\hbar^2 g^2}{2m}\right)^{1/3}$$

which is very close to the actual value.

Problem 4

(a) The system is in the singlet state

$$|00\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

(b) If the electron is found in the state $|+\rangle$, then the system will be in $|++\rangle$ or $|+-\rangle$. However, $|++\rangle$ is orthogonal to the state of the system, so $|+-\rangle$ is the only possibility. The probability is

$$P = |\langle +-|00\rangle|^2 = \frac{1}{2}$$

This is also expected from symmetry (invariance under reflection in z-direction).

(c) If the spin of the positron is in the $\hat{n}$-direction, then the positron is in the eigenstate of

$$S_n = \hat{n}\cdot\vec{S} = \sin\theta S_x + \cos\theta S_z = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

with eigenvalue $+\frac{\hbar}{2}$,

$$|+\rangle_n = \frac{\sin\theta}{\sqrt{2(1-\cos\theta)}}|+\rangle + \sqrt{\frac{1-\cos\theta}{2}}|-\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$$

The probability that the spin of the positron is found in the $\hat{n}$ direction is the same as the probability that the spin of the positron is found in the $-\hat{n}$ direction (by symmetry), i.e.,

$$P_n = \frac{1}{2}$$

After A finds the spin of the electron to be in the positive z-direction, the system is in the state

$$|\psi\rangle = \cos\frac{\theta}{2}|++\rangle + \sin\frac{\theta}{2}|+-\rangle$$

The probability of finding the system in this state is

$$P_\psi = |\langle\psi|00\rangle|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$$

The probability of A finding the spin of the electron to be in the positive z-direction is

$$P = \frac{P_\psi}{P_n} = \sin^2\frac{\theta}{2}$$