PHYSICS 522 - SPRING 2011
Midterm Exam I - Solutions

Problem 1

Using

$$\vec{J} = -\frac{i\hbar}{2m}\varphi^*\vec{\nabla}\phi + \text{c.c.}$$

we obtain

$$J_{S,r} = -\frac{i\hbar}{2m}|f|^2\frac{e^{-ikr}}{r}\left(\frac{e^{ikr}}{r}\right)' + \text{c.c.} = \frac{\hbar k}{m}\frac{1}{r^2}|f|^2$$

$$J_{S,\theta} = -\frac{i\hbar}{2m}\frac{1}{r^3}f^*\frac{\partial f}{\partial\theta} + \text{c.c.}$$

$$J_{S,\phi} = -\frac{i\hbar}{2m}\frac{1}{r^3\sin\theta}f^*\frac{\partial f}{\partial\phi} + \text{c.c.}$$

Total current:

$$I_S = \int d\Omega r^2 J_{S,r}\Big|_{r=R} = \int d\Omega R^2\frac{\hbar k}{m}\frac{1}{R^2}|f|^2 = \mathcal{C} \;\; , \quad \mathcal{C} = \frac{\hbar k}{m}\int d\Omega|f|^2$$

To relate this to the cross section, we need the incident current,

$$\vec{J}_{in} = J_{in}\hat{z} \; , \quad J_{in} = -\frac{i\hbar}{2m}e^{-ikz}\left(e^{ikz}\right)' + \text{c.c.} = \frac{\hbar k}{m}$$

The cross section is

$$\sigma = \frac{I_S}{J_{in}} = \int d\Omega|f|^2$$

and

$$\mathcal{C} = \frac{\hbar k}{m}\sigma$$

Problem 2

(a)

$$f_B = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r' e^{-i\vec{q}\cdot\vec{r}'}V(r')$$

Choose axes so that $\vec{q}$ is in the z'-direction. Then

$$\vec{q}\cdot\vec{r}' = qr'\cos\theta'$$

and

$$q^2 = (\vec{k}_i - \vec{k}_S)^2 = 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2}$$

We have

$$f_B = -\frac{1}{4\pi}\frac{2m}{\hbar^2}2\pi\int_0^\infty dr' r'^2\int_0^\pi d\theta'\sin\theta' e^{-iqr'\cos\theta'}V(r')$$

The integral over θ' is done by changing variables to $t = \cos\theta'$. We have

$$\int_0^\pi d\theta' \sin\theta' e^{-iqr'\cos\theta'} = \int_{-1}^{1} dt e^{-iqr't} = \frac{1}{-iqr'} e^{-iqr'} + \text{c.c.}$$

therefore

$$f_B = \frac{m}{iq\hbar^2} \int_0^\infty dr' r' e^{-iqr'} V(r') + \text{c.c.} = \frac{mV_0}{iq\hbar^2} \int_0^{r_0} dr' r' e^{-iqr'} + \text{c.c.}$$

The integral over r' is done by integrating by parts,

$$\int_0^{r_0} dr' r' e^{-iqr'} = \frac{1}{-iq} r' e^{-iqr'}\Big|_0^{r_0} - \frac{1}{-iq} \int_0^{r_0} dr' e^{-iqr'} = \frac{1}{-iq} r_0 e^{-iqr_0} + \frac{1}{q^2}\left(e^{-iqr_0} - 1\right)$$

therefore

$$f_B = \frac{mV_0 r_0}{\hbar^2 q^2}\left[e^{-iqr_0} + \frac{1}{iqr_0}\left(e^{-iqr_0} - 1\right) + \text{c.c.}\right] = \frac{2mV_0 r_0}{\hbar^2 q^2}\left[\cos qr_0 - \frac{\sin qr_0}{qr_0}\right]$$

(b) In the low energy limit, $qr_0 \ll 1$, so

$$f_B = \frac{2mV_0 r_0}{\hbar^2 q^2}\left[1 - \frac{1}{2}(qr_0)^2 - \frac{qr_0 - \frac{(qr_0)^3}{6}}{qr_0} + \ldots\right] \approx \frac{2mV_0 r_0}{\hbar^2 q^2}\left[-\frac{(qr_0)^2}{3}\right] = -\frac{2mV_0 r_0^3}{3\hbar^2}$$

The differential Born cross section is

$$\frac{d\sigma_B}{d\Omega} = |f_B|^2 \approx \frac{4m^2 V_0^2 r_0^6}{9\hbar^4}$$

So it is independent of q, therefore independent of the angles, therefore isotropic.

The total Born cross section is

$$\sigma_B = \int d\Omega \frac{d\sigma_B}{d\Omega} = 4\pi |f_B|^2 \approx \frac{16\pi m^2 V_0^2 r_0^6}{9\hbar^4}$$

Problem 3

(a) In an s-wave state, u is a function of r only and the Schrödinger equation is

$$-\frac{\hbar^2}{2m} u'' + V(r) u = E u$$

For $r > b$, this can be written as

$$u'' + k^2 u = 0 \quad , \quad k = \frac{\sqrt{2mE}}{\hbar}$$

We shall choose the solution

$$u = \sin(kr + \delta_0)$$

by setting the normalization constant arbitrarily to 1 and adjusting the phase to match the general asymptotic expression $\sin(kr - l\frac{\pi}{2} + \delta_l)$.

Alternatively, one can use the general solution for $R = u/r$,

$$R = Aj_0(kr) + Bn_0(kr) \qquad \text{with} \qquad \tan\delta_0 = -\frac{B}{A}$$

where $j_0(x) = \frac{\sin x}{x}$, $n_0(x) = -\frac{\cos x}{x}$.

For $a < r < b$, the Schrödinger equation is

$$u'' + k'^2 u = 0 \quad , \qquad k' = \frac{\sqrt{2m(V_0 + E)}}{\hbar}$$

The solution needs to satisfy $u = 0$ at $r = a$, so

$$u = B \sin k'(r - a)$$

or we can start with the general solution

$$u = B' \sin k'r + C' \cos k'r$$

and impose $u(a) = 0$ to get $\frac{C'}{B'} = -\tan k'a$.

Next, we match the two expressions at $r = b$ to obtain

$$\sin(kb + \delta_0) = B \sin k'(b - a) \quad , \qquad k \cos(kb + \delta_0) = Bk' \cos k'(b - a)$$

Dividing these two equations, we obtain

$$\tan(kb + \delta_0) = \frac{k}{k'} \tan k'(b - a)$$

which determines δ_0,

$$\delta_0 = -kb + \alpha \quad , \qquad \tan\alpha = \frac{k}{k'} \tan k'(b - a)$$

(b) In the low energy limit, k is small and

$$k' \approx k_0 = \frac{\sqrt{2mV_0}}{\hbar}$$

therefore

$$kb + \delta_0 \approx \tan(kb + \delta_0) \approx \frac{k}{k_0} \tan k_0(b - a)$$

and so

$$\delta_0 \approx kb\left[\frac{\tan k_0(b-a)}{k_0 b} - 1\right]$$

The above approximations are valid as long as we are not near a resonance. The cross section is

$$\sigma \approx \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi}{k^2}\delta_0^2 \approx 4\pi b^2\left[\frac{\tan k_0(b-a)}{k_0 b} - 1\right]^2$$

(c) We have a resonance when

$$\frac{\tan k_0(b-a)}{k_0 b} - 1$$

diverges. This occurs when $\tan k_0(b-a)$ diverges, i.e., when

$$k_0(b-a) = \left(n+\frac{1}{2}\right)\pi \quad , \qquad n = 0, 1, 2, \dots$$

therefore, for a resonance,

$$V_0 = \left(n+\frac{1}{2}\right)^2 \frac{\pi^2\hbar^2}{2m(b-a)^2}$$

Problem 4

We have

$$f_B = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r' e^{-i\vec{q}\cdot\vec{r}'}V(\vec{r}')$$

Switch variables to $\vec{r}'' = \vec{r}' - \vec{R}$. Then the above expression becomes

$$f_B = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r'' e^{-i\vec{q}\cdot\vec{r}''-i\vec{q}\cdot\vec{R}}V(\vec{r}''+\vec{R})$$

Using the translation invariance property of the potential, we obtain

$$f_B = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r'' e^{-i\vec{q}\cdot\vec{r}''}e^{-i\vec{q}\cdot\vec{R}}V(\vec{r}'')$$

The factor $e^{-i\vec{q}\cdot\vec{R}}$ is independent of $\vec{r}''$ and can be taken out of the integral. We deduce

$$f_B = -\frac{1}{4\pi}\frac{2m}{\hbar^2}e^{-i\vec{q}\cdot\vec{R}}\int d^3r'' e^{-i\vec{q}\cdot\vec{r}''}V(\vec{r}'')$$

Apart from the additional factor, this is the same expression we started with. Therefore

$$f_B = e^{-i\vec{q}\cdot\vec{R}}f_B$$

It follows that unless $e^{-i\vec{q}\cdot\vec{R}} = 1$, the amplitude must vanish. The additional factor is 1 when

$$\vec{q}\cdot\vec{R} = 2\pi n \quad , \qquad n \in \mathbb{Z}$$