PHYSICS 522 - SPRING 2008

Homework Set 5 - Solutions

Problem 5.1

a. For electrons, the energy levels are

\[ E_n = n\hbar \omega_0 \quad , \quad n = 1, 2, 3, 4, 5 \]

– For \( n = 1 \), two of the electrons are in the state \(|0\rangle\) and the third one in \(|1\rangle\). The two electrons in \(|0\rangle\) must form a singlet spin state and the third one can be in \(|+\rangle\) or \(|-\rangle\). Degeneracy: 2 (degeneracy will be lifted by interactions, so the ground state will be unique).

– For \( n = 2 \), either two of the electrons are in the state \(|0\rangle\) and the third one in \(|2\rangle\) (number of states: 2, as before), or two of the electrons are in the state \(|1\rangle\) and the third one in \(|0\rangle\) (number of states: 2, as before). Degeneracy: 2 + 2 = 4.

– For \( n = 3 \), we must have one electron in \(|0\rangle\), one in \(|1\rangle\) and one in \(|2\rangle\) (the other possibility, \( 3 = 1 + 1 + 1 \), is forbidden by the Pauli exclusion principle). Since they are all in different states, all spins are allowed. Degeneracy: \( 2 \times 2 \times 2 = 8 \).

– For \( n = 4 \), either two of the electrons are in the state \(|1\rangle\) and the third one in \(|2\rangle\) (number of states: 2), or two of the electrons are in the state \(|2\rangle\) and the third one in \(|0\rangle\) (number of states: 2). Degeneracy: \( 2 + 2 = 4 \).

– For \( n = 5 \), two of the electrons are in the state \(|2\rangle\) (forming a singlet) and the third one in \(|1\rangle\). Degeneracy: 2.

b. For spin-0 bosons, the energy levels are

\[ E_n = n\hbar \omega_0 \quad , \quad n = 0, 1, 2, 3, 4, 5, 6 \]

– For \( n = 0 \), all three bosons are in \(|0\rangle\). Degeneracy: 1 (unique vacuum).

– For \( n = 1 \), two of the bosons are in the state \(|0\rangle\) and the third one in \(|1\rangle\). Degeneracy: 1.

– For \( n = 2 \), either two of the bosons are in the state \(|0\rangle\) and the third one in \(|2\rangle\), or two of the bosons are in the state \(|1\rangle\) and the third one in \(|0\rangle\). Degeneracy: \( 1 + 1 = 2 \).

– For \( n = 3 \), either one boson is in \(|0\rangle\), one in \(|1\rangle\) and one in \(|2\rangle\), or all three are in \(|1\rangle\). Degeneracy: \( 1 + 1 = 2 \).

– For \( n = 4 \), either two of the bosons are in the state \(|1\rangle\) and the third one in \(|2\rangle\), or two of the bosons are in the state \(|2\rangle\) and the third one in \(|0\rangle\). Degeneracy: \( 1 + 1 = 2 \).
– For \( n = 5 \), two of the bosons are in the state \(|2\rangle\) and the third one in \(|1\rangle\). Degeneracy: 1.

– For \( n = 6 \), all three bosons are in \(|2\rangle\). Degeneracy: 1.

**Problem 5.2**

**a.** The wavefunction of the system of two electrons is

\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |\phi, +; \chi, -\rangle - |\chi, -; \phi, +\rangle \right]
\]

We have

\[
\rho_{II}(\bar{r}, \bar{r}') = \sum_{\epsilon_1, \epsilon_2 = \pm} |\langle \bar{r}, \epsilon_1; \bar{r}', \epsilon_2 | \psi \rangle|^2 + (\bar{r} \leftrightarrow \bar{r}') = 2 \sum_{\epsilon_1, \epsilon_2 = \pm} |\langle \bar{r}, \epsilon_1; \bar{r}', \epsilon_2 | \psi \rangle|^2
\]

\[
= |\langle \bar{r}, +; \bar{r}', -| \phi, +; \chi, -\rangle|^2 + |\langle \bar{r}, -; \bar{r}', +| \chi, -; \phi, +\rangle|^2
\]

\[
= |\phi(\bar{r}) \chi(\bar{r}')|^2 + |\chi(\bar{r}) \phi(\bar{r}')|^2
\]

and

\[
\rho_I(\bar{r}) = \int d^3 r' \rho_{II}(\bar{r}, \bar{r}') = \int d^3 r' \rho_{II}(\bar{r}, \bar{r}') = |\phi(\bar{r})|^2 + |\chi(\bar{r})|^2
\]

In the above argument, we didn’t use the orthogonality property of \(|\phi\rangle\) and \(|\chi\rangle\); we only used the orthogonality property of the spin states \(|\pm\rangle\) and the normalization condition \(\langle \phi | \phi \rangle = \langle \chi | \chi \rangle = 1\).

Integrating, we obtain

\[
\int d^3 r \int d^3 r' \rho_{II}(\bar{r}, \bar{r}') = \int d^3 r \rho_I(\bar{r}) = 2
\]

If instead we had two *distinguishable* spin-1/2 particles, the wavefunction would be

\[
|\psi\rangle' = |\phi, +; \chi, -\rangle
\]

We would have

\[
\rho_{II}(\bar{r}, \bar{r}') = \sum_{\epsilon_1, \epsilon_2 = \pm} |\langle \bar{r}, \epsilon_1; \bar{r}', \epsilon_2 | \psi' \rangle|^2 + (\bar{r} \leftrightarrow \bar{r}')
\]

\[
= |\langle \bar{r}, +; \bar{r}', -| \phi, +; \chi, -\rangle|^2 + |\langle \bar{r}, -; \bar{r}', +| \chi, -; \phi, +\rangle|^2
\]

\[
= |\phi(\bar{r}) \chi(\bar{r}')|^2 + |\chi(\bar{r}) \phi(\bar{r}')|^2
\]

and

\[
\rho_I(\bar{r}) = \int d^3 r' \rho_{II}(\bar{r}, \bar{r}') = |\phi(\bar{r})|^2 + |\chi(\bar{r})|^2
\]

same as before.
\[ b. \text{ In this case} \]
\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |\phi, +; \chi, +\rangle - |\chi, +; \phi, +\rangle \right]
\]

We have
\[
\rho_{II}(\vec{r}, \vec{r}') = \sum_{\epsilon_1, \epsilon_2 = \pm} |\langle \vec{r}, \epsilon_1; \vec{r}', \epsilon_2 | \psi\rangle|^2 + \langle \vec{r} \leftrightarrow \vec{r}' \rangle = 2 \sum_{\epsilon_1, \epsilon_2 = \pm} |\langle \vec{r}, \epsilon_1; \vec{r}', \epsilon_2 | \psi\rangle|^2
\]
\[
= 2|\langle \vec{r}, +; \vec{r}', + | \psi\rangle|^2 = |\phi(\vec{r})\chi(\vec{r}') - \chi(\vec{r})\phi(\vec{r}')|^2
\]

and
\[
\rho_I(\vec{r}) = \int d^3 r' \rho_{II}(\vec{r}, \vec{r}') = |\phi(\vec{r})|^2 + |\chi(\vec{r})|^2
\]

Integrating, we obtain
\[
\int d^3 r \int d^3 r' \rho_{II}(\vec{r}, \vec{r}') = \int d^3 r \rho_I(\vec{r}) = 2
\]

In the above, we used the orthogonality property \( \langle \chi | \phi \rangle = 0 \). If this is not true, then \( \rho_{II}(\vec{r}, \vec{r}') \) doesn’t change, but
\[
\rho_I(\vec{r}) = \int d^3 r' \rho_{II}(\vec{r}, \vec{r}') = |\phi(\vec{r})|^2 + |\chi(\vec{r})|^2 - 2Re [\langle \chi | \phi \rangle \phi^*(\vec{r}) \chi(\vec{r})]
\]

\[ c. \text{ For two bosons, one in state } |\phi, m_s\rangle \text{ and the other one in } |\chi, m'_s\rangle, \text{ the wavefunction is} \]
\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |\phi, m_s; \chi, m'_s\rangle + |\chi, m'_s; \phi, m_s\rangle \right]
\]

We have
\[
\rho_{II}(\vec{r}, \vec{r}') = \sum_{m_1, m_2} |\langle \vec{r}, m_1; \vec{r}', m_2 | \psi\rangle|^2 + \langle \vec{r} \leftrightarrow \vec{r}' \rangle = 2 \sum_{m_1, m_2} |\langle \vec{r}, m_1; \vec{r}', m_2 | \psi\rangle|^2
\]

For \( m_s \neq m'_s \), we obtain
\[
\rho_{II}(\vec{r}, \vec{r}') = |\langle \vec{r}, m_s; \vec{r}', m'_s | \phi, m_s; \chi, m'_s\rangle|^2 + |\langle \vec{r}, m'_s; \vec{r}', m_s | \chi, m'_s; \phi, m_s\rangle|^2
\]
\[
= |\phi(\vec{r})\chi(\vec{r}')|^2 + |\chi(\vec{r})\phi(\vec{r}')|^2
\]

and
\[
\rho_I(\vec{r}) = \int d^3 r' \rho_{II}(\vec{r}, \vec{r}') = |\phi(\vec{r})|^2 + |\chi(\vec{r})|^2
\]

For \( m_s = m'_s \), we obtain
\[
\rho_{II}(\vec{r}, \vec{r}') = 2|\langle \vec{r}, m_s; \vec{r}', m_s | \psi\rangle|^2 = |\phi(\vec{r})\chi(\vec{r}') + \chi(\vec{r})\phi(\vec{r}')|^2
\]

and
\[
\rho_I(\vec{r}) = \int d^3 r' \rho_{II}(\vec{r}, \vec{r}') = |\phi(\vec{r})|^2 + |\chi(\vec{r})|^2 + 2Re [\langle \chi | \phi \rangle \phi^*(\vec{r}) \chi(\vec{r})]
\]