PHYSICS 522 - SPRING 2011
Final Exam - Solutions

Problem 1

(a) We have $|\ell - \frac{1}{2}| \le j \le \ell + \frac{1}{2}$, so $j = \ell \pm \frac{1}{2}$.

(b) We have

$$\vec{L} = \frac{\langle \vec{L}\cdot\vec{J}\rangle}{j(j+1)\hbar^2}\,\vec{J} \quad , \quad \vec{S} = \frac{\langle \vec{S}\cdot\vec{J}\rangle}{j(j+1)\hbar^2}\,\vec{J}$$

Using

$$\vec{L}\cdot\vec{J} = \frac{1}{2}(\vec{J}^2 + \vec{L}^2 - \vec{S}^2) \quad , \quad \vec{S}\cdot\vec{J} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 + \vec{S}^2)$$

we obtain

$$\langle \vec{L}\cdot\vec{J}\rangle = \hbar^2 \frac{j(j+1) + \ell(\ell+1) - \frac{3}{4}}{2} \quad , \quad \langle \vec{S}\cdot\vec{J}\rangle = \hbar^2 \frac{j(j+1) - \ell(\ell+1) + \frac{3}{4}}{2}$$

and so $g_\ell \vec{L} + g_S \vec{S} = g_J \vec{J}$, where

$$g_J = \frac{g_\ell[j(j+1) + \ell(\ell+1) - \frac{3}{4}] + g_S[j(j+1) - \ell(\ell+1) + \frac{3}{4}]}{2j(j+1)} = \frac{g_\ell + g_S}{2} + \frac{g_\ell - g_S}{2}\,\frac{\ell(\ell+1) - \frac{3}{4}}{j(j+1)}$$

Problem 2

Let

$$H = H_0 + W \quad , \quad H_0 = a\vec{L}^2 \ , \ \ W = b\hbar^2\cos(2\phi)$$

Since $b \ll a$, we shall treat W as a perturbation.
The eigenstates of H_0 are $|\ell m\rangle$,

$$H_0|\ell m\rangle = E_\ell|\ell m\rangle \quad , \quad E_\ell = a\hbar^2\ell(\ell+1)$$

The S level ($\ell = 0$) is non-degenerate, and

$$\delta E_0 = \langle 00|W|00\rangle = \int d\Omega W|Y_0^0|^2 = \frac{b\hbar^2}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta\cos(2\phi) = 0$$

The P level ($\ell = 1$) is degenerate and we need to calculate the 3×3 matrix W. The only non-vanishing matrix elements are

$$\langle 11|W|1-1\rangle = \langle 1-1|W|11\rangle^* = -\frac{3b\hbar^2}{8\pi}\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^2\theta e^{-2i\phi}\cos(2\phi)$$

Using

$$\int_0^\pi d\theta \sin^2\theta = \frac{\pi}{2} \ , \quad \int_0^{2\pi} d\phi e^{-2i\phi}\cos(2\phi) = \pi$$

we obtain

$$\langle 11|W|1-1\rangle = \langle 1-1|W|11\rangle = -\frac{3\pi b\hbar^2}{16}$$

so for the P level,

$$W = -\frac{3\pi b\hbar^2}{16}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The eigenvalues are

$$\delta E_1^{\pm} = \pm\frac{3\pi b\hbar^2}{16}$$

with corresponding eigenvectors

$$\frac{1}{\sqrt{2}}\left(|11\rangle \mp |1-1\rangle\right)$$

Problem 3

A hydrogen atom is subjected to a constant electric field $\vec{E}_0$ that lasts for a time $0 < t < \tau$. If at $t = 0$ the atom is in the $2S$ state, use perturbation theory to determine the time dependence of the system in the interval $0 < t < \tau$.

What is the probability that it will be in the $2P$ state for $t > \tau$?

You may use the spherical harmonics given above, and

$$R_{20} = \frac{1}{\sqrt{2a_0^3}}\left(1 - \frac{r}{2a_0}\right)e^{-r/(2a_0)}\ , \quad R_{21} = \frac{1}{\sqrt{24a_0^3}}\frac{r}{a_0}e^{-r/(2a_0)}$$

The perturbation is

$$W = -q\mathcal{E}_0 z = -q\mathcal{E}_0 r\cos\theta = -\sqrt{\frac{4\pi}{3}}q\mathcal{E}_0 r Y_1^0$$

for $0 < t < \tau$, where we defined the z-axis along the external electric field.

The state of the system is

$$|\psi(t)\rangle = \sum_{n,\ell,m} e^{-iE_n t/\hbar} b_{n\ell m}(t)|n\ell m\rangle$$

First-order perturbation theory yields for $0 < t < \tau$,

$$b_{n\ell m}(t) = \frac{1}{i\hbar}\int_0^t dt' e^{i\omega_{n2}t'} W_{n\ell m;200}(t')$$

where we used the fact that at $t = 0$ the atome is in the $2S$ state ($|200\rangle$). We have $\hbar\omega_{n2} = E_n - E_0$ and

$$W_{n\ell m;200}(t') = -\sqrt{\frac{4\pi}{3}}q\mathcal{E}_0\langle n\ell m|rY_1^0|200\rangle$$

The angular part of the matrix element is proportional to

$$\langle \ell m | Y_1^0 | 00 \rangle \propto \langle \ell m | 10 \rangle$$

This vanishes unless $\ell = 1$ and $m = 0$. Therefore the non-vanishing coefficients are b_{n10} and the state for $0 < t < \tau$ is

$$|\psi(t)\rangle = \sum_n e^{-iE_n t/\hbar} b_{n10}(t) |n10\rangle$$

For $t > \tau$, the perturbation is switched off and the state evolves as

$$|\psi(t)\rangle = \sum_n e^{-iE_n t/\hbar} b_{n10}(\tau) |n10\rangle$$

The probability that the system is in the $2P$ state for $t > \tau$ is

$$P = |b_{210}(\tau)|^2 = \frac{|W_{210;200}|^2}{\hbar^2} \left| \int_0^\tau dt \right|^2 = \frac{|W_{210;200}|^2 \tau^2}{\hbar^2}$$

where

$$\begin{aligned}
W_{210;200} &= -\sqrt{\frac{4\pi}{3}} q\mathcal{E}_0 \langle 210 | r Y_1^0 | 200 \rangle \\
&= -\sqrt{\frac{4\pi}{3}} q\mathcal{E}_0 \int_0^\infty dr r^3 R_{21} R_{20} \int d\Omega |Y_1^0|^2 Y_0^0 \\
&= -\frac{q\mathcal{E}_0}{\sqrt{3}} \int_0^\infty dr r^3 R_{21} R_{20} \\
&= -\frac{q\mathcal{E}_0}{12 a_0^4} \int_0^\infty dr r^4 \left(1 - \frac{r}{2a_0}\right) e^{-r/a_0} \\
&= 3q\mathcal{E}_0 a_0
\end{aligned}$$

Therefore

$$P = \frac{9q^2 \mathcal{E}_0^2 a_0^2 \tau^2}{\hbar^2}$$

independent of time.

Problem 4

(i) We have

$$W_{fi} = \frac{1}{(2\pi)^3} \int d^3 r_a d^3 r_b |\phi(\vec{r}_a)|^2 e^{-i(\vec{k}_f - \vec{k}_i)\cdot \vec{r}_b} W(\vec{r}_b - \vec{r}_a)$$

Expressing W in terms of its Fourier transform and integrating over $\vec{r}_b$, we obtain

$$W_{fi} = \frac{1}{(2\pi)^{3/2}} \int d^3 r_a d^3 k |\phi(\vec{r}_a)|^2 e^{-i\vec{k}\cdot\vec{r}_a} \widetilde{W}(\vec{k}) \delta^3(\vec{k}_i + \vec{k} - \vec{k}_f)$$

Using the δ-function to integrate over $\vec{k}$, we obtain

$$W_{fi} = \frac{1}{(2\pi)^{3/2}} \int d^3 r_a |\phi(\vec{r}_a)|^2 e^{-i(\vec{k}_f - \vec{k}_i)\cdot \vec{r}_a} \widetilde{W}(\vec{k}_f - \vec{k}_i)$$

(ii) We have

$$w = \frac{2\pi}{\hbar}|W_{fi}|^2\rho(E_i) \quad , \qquad \rho(E_i) = m\sqrt{2mE_i}$$

The incoming beam current is

$$J_i = \frac{1}{(2\pi)^3}\frac{\hbar k_i}{m} = \frac{1}{(2\pi)^3}\sqrt{\frac{2E_i}{m}}$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \frac{w}{J_i} = \frac{(2\pi)^4 m^2}{\hbar}|W_{fi}|^2$$

Therefore,

$$\frac{d\sigma}{d\Omega} = \frac{2\pi m^2}{\hbar}|\widetilde{W}(\vec{k}_f - \vec{k}_i)|^2 \left| \int d^3r_a |\phi(\vec{r}_a)|^2 e^{-i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}_a} \right|^2$$

The Born cross section is

$$\frac{d\sigma_B}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}|\widetilde{W}(\vec{k}_f - \vec{k}_i)|^2$$

therefore,

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_B}{d\Omega}\mathcal{F}(\phi;\vec{q})$$

where

$$\mathcal{F}(\phi;\vec{q}) = (2\pi\hbar)^3 \left| \int d^3r_a |\phi(\vec{r}_a)|^2 e^{-i\vec{q}\cdot\vec{r}_a/\hbar} \right|^2$$

Problem 5

The energy levels and corresponding wavefunctions of a single particle are

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad , \quad \phi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \quad , \quad n = 1, 2, \ldots$$

where we placed the walls at $x = 0, L$.
To find the eigenvalues of W, let $\vec{S} = \vec{S}_1 + \vec{S}_2$. Then

$$W = \frac{a}{2}\left[\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right]$$

The eigenvalues are

$$W_S = \frac{a\hbar^2}{2}\left[S(S+1) - \frac{3}{2}\right]$$

where $S = 0, 1$, with corresponding eigenvectors $|00\rangle$ and $|1M\rangle$ ($M = -1, 0, +1$).
The energy levels of the system of two spinors are

$$E_{n_1 n_2 S} = E_{n_1} + E_{n_2} + W_S$$

The lowest level (ground state) has $n_1 = n_2 = 1$ and $S = 0$ (since $S = 1$ is not allowed, because it gives a symmetric wavefunction), therefore energy

$$E_{110} = \frac{\pi^2\hbar^2}{mL^2} - \frac{3a\hbar^2}{4}$$

degereracy 1 and corresponding wavefunction

$$\phi_1(x_1)\phi_1(x_2)|00\rangle = \frac{2}{L}\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}\frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

At the next level, $n_1 = 1$, $n_2 = 2$ and this time both $S = 0, 1$ are allowed. The one with lower energy has $S = 1$, because $W_1 < 0 < W_0$ (since $a < 0$), therefore energy

$$E_{121} = \frac{5\pi^2\hbar^2}{mL^2} + \frac{a\hbar^2}{4}$$

degeneracy 3 and corresponding wavefunctions

$$\frac{1}{\sqrt{2}}\left[\phi_1(x_1)\phi_2(x_2) - \phi_2(x_1)\phi_1(x_2)\right]|1M\rangle$$

Notice that the spatial part is antisymmetric, because $|1M\rangle$ is symmetric.