PHYSICS 521 - FALL 2010
Midterm Exam II - Solutions

Problem 1

The potential is $V(x) = -q\mathcal{E}_0 x$ and the Hamiltonian is

$$H = \frac{p^2}{2m} - q\mathcal{E}_0 x$$

From Ehrenfest's theorem

$$\frac{d}{dt}\langle X\rangle = \frac{\langle P\rangle}{m} \quad , \quad \frac{d}{dt}\langle P\rangle = -\langle V'(X)\rangle$$

or using

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle [A \, , \, H]\rangle$$

for $A = X, P$, we obtain

$$\frac{d^2}{dt^2}\langle X\rangle = \frac{1}{m}\frac{d}{dt}\langle P\rangle = \frac{q\mathcal{E}_0}{m}$$

Classically, the force is $F = q\mathcal{E}_0$ and $F = ma$ implies

$$\frac{d^2x}{dt^2} = \frac{F}{m} = \frac{q\mathcal{E}_0}{m}$$

in agreement with the quantum mechanical result.

Problem 2

The Hamiltonian is

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma B_0\left[\frac{3}{5}S_y + \frac{4}{5}S_z\right] = \frac{\hbar\omega}{10}\begin{pmatrix} 4 & -3i \\ 3i & -4 \end{pmatrix} \quad , \quad \omega = -\gamma B_0$$

The energy levels are found from

$$\det\left[\frac{\hbar\omega}{10}\begin{pmatrix} 4 & -3i \\ 3i & -4 \end{pmatrix} - E\mathbb{I}\right] = 0$$

We obtain

$$E = E_\pm = \pm\frac{\hbar\omega}{2}$$

Corresponding normalized eigenvectors

$$|E_+\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 3i \\ -1 \end{pmatrix} \quad , \quad |E_-\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3i \end{pmatrix}$$

At $t = 0$ the state is

$$|\psi(0)\rangle = |+, z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To see how it evolves, we need to express it in terms of the energy eigenstates. We have

$$|\psi(0)\rangle = -\frac{3i}{\sqrt{10}}|E_+\rangle - \frac{1}{\sqrt{10}}|E_-\rangle$$

Therefore,

$$|\psi(t)\rangle = -\frac{3i}{\sqrt{10}}e^{-iE_+t/\hbar}|E_+\rangle - \frac{1}{\sqrt{10}}e^{-iE_-t/\hbar}|E_-\rangle$$

For the x-component of the spin, the eigenvalues are $\pm\hbar/2$, so these are the possible outcomes. The corresponding eigenstates are

$$|+, x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad |-, x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Probabilities:

$$\begin{aligned}
P_+ &= |\langle +, x|\psi(t)\rangle|^2 \\
&= \frac{1}{10}| - 3ie^{-i\omega t/2}\langle +, x|E_+\rangle - e^{+i\omega t/2}\langle +, x|E_-\rangle|^2 \\
&= \frac{1}{200}| - 3i(3i - 1) - (-1 + 3i)e^{+i\omega t}|^2 \\
&= \frac{1}{200}[(9 + \cos(\omega t) + 3\sin(\omega t))^2 + (3 - 3\cos(\omega t) + \sin(\omega t))^2] \\
&= \frac{5 + 3\sin(\omega t)}{10}
\end{aligned}$$

$$\begin{aligned}
P_- &= |\langle -, x|\psi(t)\rangle|^2 \\
&= \frac{1}{10}| - 3ie^{-i\omega t/2}\langle -, x|E_+\rangle - e^{+i\omega t/2}\langle +, x|E_-\rangle|^2 \\
&= \frac{1}{200}| - 3i(3i + 1) - (-1 - 3i)e^{+i\omega t}|^2 \\
&= \frac{1}{200}[(9 + \cos(\omega t) - 3\sin(\omega t))^2 + (3 - 3\cos(\omega t) - \sin(\omega t))^2] \\
&= \frac{5 - 3\sin(\omega t)}{10}
\end{aligned}$$

Evidently, $P_+ + P_- = 1$.