PHYSICS 521 - FALL 2007

Homework Set 5 - Solutions

Problem 5.1

(a) We have

\[ P(E > 2\hbar\omega) = 1 - P(E \leq 2\hbar\omega) = 1 - P_0 - P_1 = 1 - |c_0|^2 - |c_1|^2 \]

If this vanishes, then

\[ |c_0|^2 + |c_1|^2 = 1 \]

Therefore all other coefficients vanish (c_n = 0 for n ≥ 2).

(b) The normalization condition is written above. The mean value of the energy is

\[ \langle H \rangle = E_0 |c_0|^2 + E_1 |c_1|^2 = \hbar \omega \left[ \frac{1}{2} |c_0|^2 + \frac{3}{2} |c_1|^2 \right] \]

If this equals \( \hbar \omega \), we obtain

\[ |c_0|^2 + |c_1|^2 = 1 \quad , \quad |c_0|^2 + 3 |c_1|^2 = 2 \]

whose solution is

\[ |c_0|^2 = |c_1|^2 = \frac{1}{2} \]

(c) We have

\[ x|\psi(0)\rangle = \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) [c_0|0\rangle + c_1|1\rangle] = \sqrt{\frac{\hbar}{2m\omega}} \left\{ c_0|1\rangle + c_1[\sqrt{2}|2\rangle + |0\rangle] \right\} \]

Therefore

\[ \langle \psi(0)|x|\psi(0)\rangle = \sqrt{\frac{\hbar}{2m\omega}} [c_0^* c_1 + c_1^* c_0] = \sqrt{\frac{\hbar}{2m\omega}} \cos \theta_1 \]

If \( \langle x \rangle = \frac{1}{2} \sqrt{\frac{\hbar}{m\omega}} \), then

\[ \cos \theta_1 = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta_1 = \frac{\pi}{4} \]

(d)

\[ |\psi(0)\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} e^{\pi i/4} |1\rangle \]

so

\[ |\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_0 t/\hbar} |0\rangle + \frac{1}{\sqrt{2}} e^{\pi i/4} e^{-iE_1 t/\hbar} |1\rangle = \frac{1}{\sqrt{2}} e^{-i\omega t/2} |0\rangle + \frac{1}{\sqrt{2}} e^{\pi i/4} e^{-3i\omega t/2} |1\rangle \]
which gives
\[ \theta_1(t) = \frac{\pi}{4} - \omega t \]
(the phase difference of the two coefficients).
The average \( x \) at time \( t \) is still given by the expression derived above for \( t = 0 \) but with the new \( \theta_1 \),
\[ \langle \psi(t)|x|\psi(t) \rangle = \sqrt{\frac{\hbar}{2m\omega}} \cos \left( \frac{\pi}{4} - \omega t \right) \]
as expected classically (oscillation).

**Problem 5.2**

**a.** The total Hamiltonian is the sum \( H = H_1 + H_2 \) since the particles do not interact. Each has eigenvalues and corresponding eigenfunctions \( (i = 1, 2) \)
\[ E_i = \left( n_i + \frac{1}{2} \right) \hbar \omega , \quad \phi_{n_i}(x_i) \]
The combined system has energies
\[ E = E_1 + E_2 = (n_1 + n_2 + 1) \hbar \omega \]
and corresponding eigenfunctions
\[ \Phi_{n_1n_2}(x_1, x_2) = \phi_{n_1}(x_1)\phi_{n_2}(x_2) \]
Each energy level is labeled by the quantum number \( n = n_1 + n_2 \). For a given \( n \), there are \( n + 1 \) different eigenfunctions (because \( n_1 \) takes the values \( 0, 1, \ldots, n \)). Therefore, the \( n \)th level has degeneracy \( n + 1 \).

**b.** \( H \) does not form a C.S.C.O. because its eigenvalues are degenerate. \( H_1 \) and \( H_2 \) commute and their eigenvalues uniquely label the eigenfunctions of \( H \). Therefore, they form a C.S.C.O.

Orthonormalization:
\[ \langle \Phi_{n'_1n'_2}|\Phi_{n_1n_2} \rangle = \langle n'_1|n_1 \rangle \langle n'_2|n_2 \rangle = \delta_{n'_1n_1} \delta_{n'_2n_2} \]
Closure:
\[ \sum_{n_1, n_2} |\Phi_{n_1n_2} \rangle \langle \Phi_{n_1n_2} | = \mathbb{I} \]
c. Measurement of the energy will yield

\[ E_n = (n + 1)\hbar\omega \]

with corresponding probabilities

\[ P_0 = |\langle \Phi_{00} | \psi(0) \rangle|^2 = \frac{1}{4} \]

\[ P_1 = |\langle \Phi_{10} | \psi(0) \rangle|^2 + |\langle \Phi_{01} | \psi(0) \rangle|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \]

\[ P_2 = |\langle \Phi_{20} | \psi(0) \rangle|^2 + |\langle \Phi_{11} | \psi(0) \rangle|^2 + |\langle \Phi_{02} | \psi(0) \rangle|^2 = \frac{1}{4} \]

All other probabilities vanish.

Measurement of the energy \( E_1 \) will yield

\[ E_{n_1} = \left( n_1 + \frac{1}{2} \right)\hbar\omega \]

with corresponding probabilities

\[ P_0 = |\langle \Phi_{00} | \psi(0) \rangle|^2 + |\langle \Phi_{01} | \psi(0) \rangle|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \]

\[ P_1 = |\langle \Phi_{10} | \psi(0) \rangle|^2 + |\langle \Phi_{11} | \psi(0) \rangle|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \]

All other probabilities vanish.

Measurement of the position \( x_1 \) will yield a value \( x_1 \) with corresponding probability density

\[ P(x_1) = \int dx_2 |\langle x_1 x_2 | \psi(0) \rangle|^2 \]

\[ = \frac{1}{4} \int dx_2 |\Phi_{00}(x_1, x_2) + \Phi_{10}(x_1, x_2) + \Phi_{01}(x_1, x_2) + \Phi_{11}(x_1, x_2)|^2 \]

\[ = \frac{1}{4} \int dx_2 |(\phi_0(x_1) + \phi_1(x_1))(\phi_0(x_2) + \phi_1(x_2))|^2 \]

The integral over \( x_2 \) may be performed by using the orthogonality property

\[ \langle n'_2 | n_2 \rangle = \int dx_2 \phi_{n'_2}^*(x_2) \phi_{n_2}(x_2) = \delta_{n_2 n'_2} \]

We obtain

\[ P(x_1) = \frac{1}{2} |\phi_0(x_1) + \phi_1(x_1)|^2 \]
Notice that \( \int \mathcal{P}(x_1) = 1 \) (by using the same orthogonality relation). Explicitly,

\[
\mathcal{P}(x_1) = \frac{1}{2} \sqrt{\frac{m\omega}{\pi \hbar}} \left( 1 + \sqrt{\frac{2m\omega}{\hbar} x_1} \right)^2 e^{-m\omega x_1^2/\hbar}
\]

To find the probability distribution of velocity, it is convenient to work in the momentum representation in which the wavefunctions of a harmonic oscillator are obtained from those in the position representation by the replacement \( \sqrt{\frac{m\omega}{\hbar}} x \to p/\sqrt{\hbar m\omega} \),

\[
A_n(p) = (-)^n \phi_n \left( \frac{p}{m\omega} \right)
\]

The probability density of velocity is the same as the probability density of momentum (up to a multiplicative constant \( m \) because \( v = p/m \)). We obtain

\[
\mathcal{P}(p_1) = \frac{1}{2} |A_0(p_1) + A_1(p_1)|^2 = \frac{1}{2} \left| \phi_0 \left( \frac{p_1}{m\omega} \right) - \phi_1 \left( \frac{p_1}{m\omega} \right) \right|^2
\]

Explicitly,

\[
\mathcal{P}(p_1) \sim \left( 1 + \sqrt{\frac{2}{\hbar m\omega} p_1} \right)^2 e^{-\frac{p_1^2}{\hbar m\omega}}
\]

and the probability density of velocity is

\[
\mathcal{P}(v_1) = \mathcal{C} \left( 1 + \sqrt{\frac{2m}{\hbar \omega} v_1} \right)^2 e^{-\frac{mv_1^2}{\hbar \omega}}
\]

The constant \( \mathcal{C} \) is found by demanding \( \int dv_1 \mathcal{P}(v_1) = 1 \). We obtain

\[
\mathcal{C} = \frac{1}{2} \sqrt{\frac{m}{\pi \hbar \omega}}
\]

**Problem 5.3**

**a. \( \alpha \).** After the measurement, the wavefunction collapses to

\[
|\psi'(0)\rangle = \frac{1}{\sqrt{2}} \left( |\Phi_{10}\rangle + |\Phi_{01}\rangle \right)
\]

This is a stationary state, so all averages at \( t > 0 \) will be the same as at \( t = 0 \). We have

\[
\langle \psi'(0)|x_1|\psi'(0)\rangle = \frac{1}{2} \left( \langle 1|x|1\rangle_1 \langle 0|0\rangle_2 + \langle 0|x|1\rangle_1 \langle 1|0\rangle_2 + \langle 1|x|0\rangle_1 \langle 0|1\rangle_2 + \langle 0|x|0\rangle_1 \langle 1|1\rangle_2 \right) = 0
\]
where we used \( x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \). Similarly,

\[
\langle \psi'(0)|p_1|\psi'(0) \rangle = 0
\]

The average energy is

\[
\langle \psi'(0)|H_1|\psi'(0) \rangle = \frac{1}{2} (\langle 1|H_1|1 \rangle + \langle 0|H_1|0 \rangle) = \frac{1}{2} \left( \frac{3}{2}\hbar\omega + \frac{1}{2}\hbar\omega \right) = \hbar\omega
\]

i.e., half of the total, as expected (identical particles).

The results for the second particle are the same.

\( \beta. \) A measurement of \( H_1 \) will yield \( \frac{1}{2}\hbar\omega \) or \( \frac{3}{2}\hbar\omega \) with equal probabilities.

The probability distribution of position is

\[
P(x_1) = \frac{1}{2} \sqrt{\frac{m\omega}{\pi\hbar}} \left( 1 + \frac{2m\omega x_1^2}{\hbar} \right) e^{-m\omega x_1^2 / \hbar}
\]

\( b. \) This time the wavefunction collapses to

\[
|\psi''(0)\rangle = \frac{1}{\sqrt{2}} (|\Phi_{00}\rangle + |\Phi_{10}\rangle)
\]

Subsequently, it evolves as

\[
|\psi''(t)\rangle = \frac{1}{\sqrt{2}} \left( e^{-i\omega t} |\Phi_{00}\rangle + e^{-2i\omega t} |\Phi_{10}\rangle \right)
\]

\( \alpha. \) Averages:

\[
\langle \psi''(t)|x_1|\psi''(t) \rangle = \frac{1}{2} \left( e^{-i\omega t} \langle 0|x_1|1 \rangle + e^{+i\omega t} \langle 1|x_1|0 \rangle \right) = \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t)
\]

\[
\langle \psi''(t)|p_1|\psi''(t) \rangle = \frac{1}{2} \left( e^{-i\omega t} \langle 0|p_1|1 \rangle + e^{+i\omega t} \langle 1|p_1|0 \rangle \right) = -\sqrt{\frac{m\hbar\omega}{2}} \sin(\omega t)
\]

Notice that \( \langle p_1 \rangle = m\frac{d}{dt} \langle x_1 \rangle \).

\[
\langle \psi''(t)|H_1|\psi''(t) \rangle = \frac{1}{2} (\langle 0|H_1|1 \rangle + \langle 1|H_1|0 \rangle) = \hbar\omega
\]
For the second particle we obtain
\[ \langle \psi''(t)|x_2|\psi''(t)\rangle = \frac{1}{2} \left( \langle 0|x_2|0 \rangle + \langle 1|x_2|1 \rangle \right) = 0 \]
\[ \langle \psi''(t)|p^2|\psi''(t)\rangle = 0 \]
\[ \langle \psi''(t)|H_2|\psi''(t)\rangle = \frac{1}{2} \hbar \omega \]

\( \beta \). A measurement of \( H_1 \) will yield \( \frac{1}{2} \hbar \omega \) or \( \frac{3}{2} \hbar \omega \) with equal probabilities.

The probability distribution of position is
\[ P(x_1) = \int dx_2 |\langle x_1|x_2|\psi''(t)\rangle|^2 \]
\[ = \frac{1}{2} \int dx_2 |e^{-i\omega t} \phi_0(x_1) \phi_0(x_2) + e^{-2i\omega t} \phi_1(x_1) \phi_0(x_2)|^2 \]
\[ = \frac{1}{2} |\phi_0(x_1) + e^{-i\omega t} \phi_1(x_1)|^2 \]
Explicitly,
\[ P(x_1) = \frac{1}{2} \sqrt{\frac{m\omega}{\pi \hbar}} \left( 1 + 2 \sqrt{\frac{2m\omega}{\hbar}} x_1 \cos(\omega t) + \frac{2m\omega x_1^2}{\hbar} \right) e^{-m\omega x_1^2/\hbar} \]

**Problem 5.4**

a. We have
\[ U(t) = e^{-iHt/\hbar} = e^{-i(N+\frac{1}{2})\omega t} \]
Therefore
\[ \tilde{a}(t)|n\rangle = U^\dagger(t) a U(t)|n\rangle = e^{-i(n+\frac{1}{2})\omega t} U^\dagger(t)|n\rangle = e^{-i(n+\frac{1}{2})\omega t} \sqrt{n} U^\dagger(t)|n-1\rangle \]
\[ = e^{-i(n+\frac{1}{2})\omega t} \sqrt{n} e^{i(n-\frac{1}{2})\omega t}|n-1\rangle = e^{-i\omega t} a|n\rangle \]
It follows that
\[ \tilde{a}(t) = e^{-i\omega t} a \]
\[ \tilde{a}^\dagger(t) = e^{+i\omega t} a^\dagger \]

b.
\[ \tilde{x}(t) = U^\dagger(t)xU(t) = \sqrt{\frac{\hbar}{2m\omega}} U^\dagger(t)(a^\dagger + a)U(t) \]
\[ = \sqrt{\frac{\hbar}{2m\omega}} (\tilde{a}^\dagger(t) + \tilde{a}(t)) = \sqrt{\frac{\hbar}{2m\omega}} (e^{i\omega t} a^\dagger + e^{-i\omega t} a) \]
Expressing $a$ and $a^\dagger$ in terms of $x$ and $p$, we obtain

$$\tilde{x}(t) = x \cos(\omega t) + \frac{p}{m\omega} \sin(\omega t)$$

Similarly, we obtain

$$\tilde{p}(t) = p \cos(\omega t) - m\omega x \sin(\omega t)$$

Thus $(x, p)$ gets rotated in phase space.

c. Notice that

$$\tilde{x}(\frac{\pi}{2\omega}) = x \cos \frac{\pi}{2} + \frac{p}{m\omega} \sin \frac{\pi}{2} = \frac{p}{m\omega}$$

Therefore,

$$pU^\dagger(\frac{\pi}{2\omega})|x\rangle = m\omega \tilde{x}(\frac{\pi}{2\omega})U^\dagger(\frac{\pi}{2\omega})|x\rangle = m\omega U^\dagger(\frac{\pi}{2\omega})x|x\rangle = m\omega U^\dagger(\frac{\pi}{2\omega})|x\rangle$$

Therefore $U^\dagger(\frac{\pi}{2\omega})|x\rangle$ is an eigenfunction of $p$ with eigenvalue $m\omega x$.

Similarly,

$$\tilde{p}(\frac{\pi}{2\omega}) = -m\omega x$$

so

$$xU^\dagger(\frac{\pi}{2\omega})|p\rangle = -\frac{1}{m\omega} \tilde{p}(\frac{\pi}{2\omega})U^\dagger(\frac{\pi}{2\omega})|p\rangle = -\frac{1}{m\omega} U^\dagger(\frac{\pi}{2\omega})p|p\rangle = -\frac{p}{m\omega} U^\dagger(\frac{\pi}{2\omega})|p\rangle$$

Therefore $U^\dagger(\frac{\pi}{2\omega})|p\rangle$ is an eigenfunction of $x$ with eigenvalue $-\frac{p}{m\omega}$.

d. We have

$$\psi(x, t) = \langle x|\psi(t)\rangle = \langle x|U(t)|\psi(0)\rangle$$

For $t = t_1 = \frac{\pi}{2\omega}$,

$$\psi(x, t_1) = \langle x|U(t_1)|\psi(0)\rangle$$

We know that

$$U^\dagger(t_1)|x\rangle = C_1|p_1\rangle , \quad p_1 = m\omega x$$

The normalization constant is found from

$$\langle x'|U(t_1)U^\dagger(t_1)|x\rangle = |C_1|^2 \langle p'_1|p_1\rangle = |C_1|^2 \delta(p'_1 - p_1) = |C_1|^2 \frac{1}{m\omega} \delta(x - x')$$

Since this must be equal to $\langle x'|x\rangle = \delta(x' - x)$, it follows that

$$C_1 = \sqrt{m\omega} e^{i\theta_1}$$

where $\theta_1$ is a phase. I couldn’t deduce $\theta_1$ from the above results. Can you?

Therefore,

$$\psi(x, t_1) = C_1 \langle p_1|\psi(0)\rangle = e^{-i\theta_1} \sqrt{m\omega} \int \frac{dy}{\sqrt{2\pi\hbar}} e^{-im\omega x y/\hbar} \psi(y, 0)$$
This can be generalized to $t = t_q$ by generalizing the result of part c. above. We have

$$\tilde{x}(t_q) = x \cos \frac{q\pi}{2} + \frac{p}{m\omega} \sin \frac{q\pi}{2}$$

For odd $q = 2k + 1$, this becomes

$$\tilde{x}(t_{2k+1}) = (-)^k \frac{p}{m\omega}$$

therefore $U^\dagger(t_{2k+1})|x\rangle$ is an eigenfunction of $p$ with eigenvalue $(-)^km\omega x$. It follows that

$$\psi(x, t_{2k+1}) = \langle x|U(t_{2k+1})|\psi(0)\rangle = C_{2k+1}^* \langle (-)^k p_1 |\psi(0)\rangle$$

$$= e^{-i\theta_{2k+1}} \sqrt{m\omega} \int \frac{dy}{\sqrt{2\pi}} e^{-i(-)^km\omega y/\hbar} \psi(y, 0)$$

For even $q = 2k$, we have

$$\tilde{x}(t_{2k}) = (-)^k x$$

therefore $U^\dagger(t_{2k})|x\rangle$ is an eigenfunction of $x$ with eigenvalue $(-)^k x$. It follows that

$$\psi(x, t_{2k}) = \langle x|U(t_{2k})|\psi(0)\rangle = C_{2k}^* \langle (-)^k x |\psi(0)\rangle = e^{-i\theta_{2k}} \psi((-)^k x, 0)$$

e. As shown in part d. above,

$$\psi(x, t_1) = e^{-i\theta_1} \sqrt{m\omega} \tilde{\psi}(p, 0), \quad p = m\omega x$$

where $\tilde{\psi}$ is the Fourier transform of $\psi$. If $\psi(x, 0) = \phi_n(x)$, then

$$\psi(x, t) = e^{-i(n+\frac{1}{2})\omega t} \phi_n(x)$$

Therefore,

$$\tilde{\phi}_n(p) = \frac{e^{i\theta_1}}{\sqrt{m\omega}} \psi(x, t_1) = \frac{e^{i\theta_1}}{\sqrt{m\omega}} e^{-i(n+\frac{1}{2})\omega t} \phi_n\left( \frac{p}{m\omega} \right) = \frac{e^{i\theta_1}}{\sqrt{m\omega}} (-i)^n e^{-i\pi/4} \phi_n\left( \frac{p}{m\omega} \right)$$

It turns out (by an explicit calculation of the Fourier transform of $\phi_n$) that

$$\theta_1 = \frac{\pi}{4}$$

but it should be possible to show this using the above results.

f. (i) It oscillates between a wavetrain of momentum $p = \pm \hbar k$ and a particle localized at $x = \pm \frac{\hbar k}{m\omega}$.

(ii) It oscillates between $e^{-\rho x}$ (particle near $-\infty$) and $e^{\rho x}$ (particle near $+\infty$).

(iii) It oscillates between a flat distribution and one centered at $x = 0$ of the form

$$P(x) \sim \sin^2(\alpha x), \quad \alpha = \frac{m\omega a}{2\hbar}$$

(iv) This is the ground state. It does not change (stationary state).