PHYSICS 521 - FALL 2010
Homework Set 5

due date: Thu., October 21, 2010

Problem 5.1

Consider a harmonic oscillator of mass m and angular frequency ω. At time $t = 0$ the system is in the state

$$|\psi(0)\rangle = \sum_n c_n |n\rangle$$

where $|n\rangle$ is a stationary state of energy $E_n = (n + \frac{1}{2})\hbar\omega$.

(a) What is the probability $\mathcal{P}$ that a measurement of the energy performed at time $t > 0$ will yield a result greater than $2\hbar\omega$?

(b) From now on assume that the probability $\mathcal{P} = 0$. Moreover, the average energy is found to be

$$\langle H \rangle = \hbar\omega$$

Calculate $|c_n|$.

(c) Furthermore, suppose that $c_0 > 0$ and that the average position at $t = 0$ is found to be

$$\langle X \rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

Deduce c_n.

(d) What is the average poisition $\langle X \rangle$ at time $t > 0$?

Problem 5.2

Consider two non-interacting harmonic oscillators each of mass m and angular frequency ω.

(a) Write the Hamiltonian H of the system and find its eigenvalues and corresponding eigenvectors $|n_1, n_2\rangle$.

What is the degree of degeneracy of each energy level of H?

(b) At $t = 0$ the system is in the state

$$|\psi(0)\rangle = \frac{1}{2}\left(|0,0\rangle + |1,0\rangle + |0,1\rangle + |1,1\rangle\right)$$

If the total energy of the system is measured, what are the possible outcomes and with what probability will each occur?

(c) If instead one measures the energy of the first oscillator, <u>or</u> its position, <u>or</u> its velocity, what are the possible outcomes in each case and with what probability will each occur?

(d) At $t = 0$ the total energy is measured and found to be $2\hbar\omega$.

1. Calculate the mean values of the position, velocity and energy of each of the two harmonic oscillators at time $t > 0$.
2. If one measures the energy of the first oscillator, or its position at time $t > 0$, what are the possible outcomes in each case and with what probability will each occur?

(e) Repeat part *(d)* if instead of measuring the total energy at time $t = 0$, one measures the energy of the second harmonic oscillator and finds it to be $\frac{1}{2}\hbar\omega$.

Problem 5.3

Consider a harmonic oscillator. Recall that the Hamiltonian can be written as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

For an operator A, define its time evolution by the unitary transformation

$$A(t) = U^\dagger(t) A U(t) \quad , \qquad U(t) = e^{-iHt/\hbar}$$

(a) By acting on the eigenstates $|n\rangle$ of the Hamiltonian, calculate $a(t)$ and $a^\dagger(t)$.

(b) Calculate $X(t)$ and $P(t)$ in terms of X and P and interpret your results.

(c) Show that

$$U^\dagger(t_1)|x\rangle \quad , \qquad t_1 = \frac{\pi}{2\omega}$$

is an eigenfunction of P and

$$U^\dagger(t_1)|p\rangle$$

is an eigenfunction of X. Find the corresponding eigenvalues.

(d) If the system is initially described by the wavefunction $\psi(x, 0)$, show that at $t = t_1$ the wavefunction $\psi(x, t_1)$ can be written in terms of the Fourier transform of $\psi(x, 0)$.

By choosing for $\psi(x, 0)$ the eigenfunction of the Hamiltonian $\phi_n(x) = \langle x|n\rangle$, deduce a relation between $\phi_n(x)$ and its Fourier transform $\widetilde{\phi}_n(p) = \langle p|n\rangle$.

(e) Generalize the above result by deriving a relation between the wavefunction at subsequent times, $\psi(x, t_q)$ $(t_q = qt_1, \quad q = 1, 2, 3, \ldots)$ and the initial wavefunction $\psi(x, 0)$ and/or its Fourier transform.

(f) Describe *qualitatively* the evolution of the wave function in the following cases:

1. $\psi(x, 0) = Ce^{ikx}$ for a given k.
2. $\psi(x, 0) = Ce^{-\lambda x}$ for a given $\lambda > 0$.

3.

$$\psi(x,0) = \begin{cases} \frac{1}{\sqrt{a}} & , \quad |x| \leq \frac{a}{2} \\ 0 & , \quad \text{everywhere else} \end{cases}$$

4. $\psi(x,0) = Ce^{-x^2/b^2}$ for a given b.