PHYSICS 521 - FALL 2007

Midterm Exam

Useful constants

- $\hbar = 1.05 \times 10^{-34}\ J \cdot s$
- $c = 3 \times 10^{8}\ m/s$
- $1\ \text{eV} = 1.6 \times 10^{-19}\ J$

Problem 1

A high-resolution neutron interferometer narrows the energy spread of thermal neutrons of kinetic energy $E = 0.02$ eV to a wavelength dispersion level of $\Delta\lambda/\lambda \approx 10^{-9}$. A neutron has mass $m_n = 1.67 \times 10^{-27}$ kg.

(a) Estimate the length of the wave packets in the direction of motion.

(b) Over what length of time will the wave packets spread appreciably ($\Delta E \sim E$)?

Problem 2

For a wavefunction $\psi(x,t)$ satisfying the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

we define the probability charge and current densities, respectively, by

$$\rho = |\psi|^2 \quad , \qquad j = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

(a) Derive the continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

By integrating, show that the total probability $P = \int dx\, \rho$ is conserved.

(b) If we add a constant imaginary term to the potential, i.e., replace $V(x)$ by $V(x) - iV_0$, where $V_0 > 0$, how does the continuity equation change?

Integrate the new continuity equation and find dP/dt in terms of V_0, $\hbar$ and P. Hence show that P decays exponentially with time.

Explain why V_0 represents *absorption*.

Problem 3

The Schrödinger equation for a rigid body that is constrained to rotate about a fixed axis and has a moment of inertia I about this axis is

$$-\frac{\hbar^2}{2I}\frac{\partial^2\psi}{\partial\theta^2} = i\hbar\frac{\partial\psi}{\partial t}$$

where $\psi(\theta, t)$ is a periodic function of the angle $\theta \in (0, 2\pi]$.
Find the eigenvalues and normalized eigenfunctions of the corresponding time-independent Schrödinger equation.
Is there any degeneracy?

Problem 4

The spin operator $\vec{S}$ acting on a spin-1/2 particle has components

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Find the eigenvalues and corresponding normalized eigenstates of the component $\hat{n} \cdot \vec{S}$ where

$$\hat{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

is the unit vector in the xy-plane forming an angle θ with the x-axis.

(b) Suppose the spin in the z-direction is measured and is found to be $+\hbar/2$. Subsequently the spin along $\hat{n}$ is measured. What are the possible outcomes of the second measurement and with what probability will each outcome occur?