PHYSICS 521 - FALL 2007

Final Exam

Useful constants

- $\hbar = 1.05 \times 10^{-34}\ J \cdot s$
- $c = 3 \times 10^{8}\ m/s$
- $k_B = 1.38 \times 10^{-23}\ J/K$
- $1\ \text{eV} = 1.6 \times 10^{-19}\ J$

Problem 1

A particle moves in one dimension in the field of a potential

$$V(x) = V_0\theta(x) + \lambda\delta(x) \quad , \qquad \text{where} \quad \theta(x) = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x < 0 \end{cases} \quad , \qquad V_0\ , \lambda > 0$$

with energy $E > V_0$. Calculate the transmission coefficient if the particle is incident from the left.

Problem 2

The Hamiltonian representing an oscillating LC circuit can be expressed as

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

where the charge Q and magnetic flux Φ obey the commutation relation

$$[\ Q\ ,\ \Phi\ ] = i\hbar$$

(similar to $[x\ ,\ p] = i\hbar$).

(a) Find the eigenvalues and corresponding eigenfunctions of the Hamiltonian.

(b) Work out the Heisenberg relation for the product of the uncertainties in the current I and voltage V.

Recall $\Phi = LI$, $Q = CV$.

(c) If $L = 1\ \mu$H, $C = 1$ pF, how low must the temperature of the circuit be before quantum fluctuations become comparable to thermal energies (i.e., $E \sim k_B T$)?

Problem 3

The wavefunction of a particle subjected to a central potential $V(r)$ is given by

$$\psi(\vec{r}) = (x + y + 4z) f(r)$$

(a) Is ψ an eigenfunction of L^2? If so, what is the eigenvalue? If not, what are the possible outcomes of a measurement of L^2?

(b) What are the possible outcomes of a measurement of L_z and with what probability will each occur?

You may use

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \quad , \quad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta \quad , \quad Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

Problem 4

A particle in a spherically symmetric potential is known to be in an eigenstate of L^2 and L_z with eigenvalues $\hbar^2 \ell(\ell + 1)$ and $m\hbar$, respectively.

(a) Show that

$$\langle L_x \rangle = \langle L_y \rangle = 0 \quad , \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{\hbar^2}{2} [\ell(\ell + 1) - m^2]$$

(b) Show that ΔL_x and ΔL_y obey the Heisenberg uncertainty principle.

For which states does the inequality become equality?

Problem 5

Assuming the eigenfunctions for the Hydrogen atom to be of the form

$$\Psi(\vec{r}) = A r^{\beta} e^{-\alpha r} Y_\ell^m(\theta, \phi)$$

solve the Schrödinger equation and determine the parameters α, β as well as the corresponding energy levels.
Are all energy levels of the Hydrogen atom obtained this way?