AdS/CFT Correspondence

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LECTURE 1

Path Integral

1.1 Quantum Mechanics

Set $\hbar = 1$. Given a particle described by coordinates (q, t) with initial and final coordinates of (q_i, t_i) and (q_f, t_f), the amplitude of the transition is

$$\langle q_i, t_i \mid q_f, t_f \rangle = \langle \psi \mid q_f, t_f \rangle = \psi^* (q_f, t_f) \ .$$

where ψ satisfies the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \frac{\partial^2 \psi}{\partial q^2} \ .$$

Feynman introduced the path integral formalism

$$\langle q_i, t_i \mid q_f, t_f \rangle = \int_{q(t_i)=q_i, q(t_f)=q_f} [dq] \, e^{-iS}$$

where $S = \int dt L$ is the action and $L = L(q, \dot{q})$ is the Lagrangian, e.g.,

$$L = \frac{1}{2} m \dot{q}^2 - V(q)$$

We convert to imaginary time via the Wick rotation $\tau = it$ so

$$L \to -\frac{1}{2} m \dot{q}^2 - V(q) \equiv -L_E$$

where the derivative is with respect to τ, L_E is the total energy (which is bounded below), and the subscript E stands for Euclidean. The action becomes $iS \to S_E = \int d\tau L_E$ so the path integral becomes

$$\int [dq] \, e^{-S_E} \ .$$

The major contribution to the path integral's value comes from minimum S_E, i.e.,

$$\delta S = 0$$

i.e., from the classical trajectory $q = q_{cl}(\tau)$.
Consider a perturbation of the classical trajectory

$$\begin{aligned} q(\tau) &= q_{cl}(\tau) + \delta q(\tau) \\ S &= S_{cl} + \ldots \\ \int [dq]\, e^{-S_E} &= e^{-S_{cl}} \int [dq]\, e^{-\cdots} \\ \langle q_i, t_i \mid q_f, t_f \rangle &\approx e^{-S_{cl}} \end{aligned}$$

The correlators are

$$\langle q_i, \tau_i | T[q(\tau_1)\, q(\tau_2)\, \cdots] | q_f, \tau_f \rangle = \int_{q(\tau_i)=q_i, q(\tau_f)=q_f} [dq]\, q(\tau_1)\, q(\tau_2)\, \cdots e^{-S_E} \quad (1.1.1)$$

where T denotes a time ordered product. Notice that the path integral automatically takes care of time ordering. This can easily be seen by splitting the path integral into time intervals bounded by $\tau_1, \tau_2, \ldots$.
We are interested in the vacuum expectation values

$$G(\tau_1, \tau_2, \ldots) = \langle 0 | T[q(\tau_1)\, q(\tau_2)\, \cdots] | 0 \rangle \ . \quad (1.1.2)$$

To write them in terms of a path integral, notice that by using the time evolution operator, we have

$$|q_f, \tau_f\rangle = e^{-\tau_f H} |q_f, 0\rangle \quad (1.1.3)$$

where H is the Hamiltonian with eigenstates $|n\rangle$; $H|n\rangle = E_n |n\rangle$ and we take $E_0 = 0$ (not necessary). Inserting the identity $\mathbb{I} = \sum_{states} |n\rangle\langle n|$ in equation (1.1.3),

$$|q_f, \tau_f\rangle = \sum_{states} e^{-\tau_f E_n} |n\rangle \langle n \mid q_f, 0\rangle$$

we see that in the limit $\tau_f \to \infty$, all terms but one vanish. Therefore,

$$|q_f, \tau_f\rangle \to |0\rangle \langle 0 \mid q_f, 0\rangle$$

Similarly, $\langle q_i, \tau_i| \to \langle q_i, 0 \mid 0\rangle \langle 0|$ as $\tau_i \to -\infty$. Substituting these limiting expressions into equation (1.1.1) gives

$$\begin{aligned} \langle 0 | T[q(\tau_1)\, q(\tau_2) \cdots] | 0 \rangle \langle q_i, 0 \mid 0 \rangle \langle 0 \mid q_f, 0 \rangle &= \int [dq]\, q(\tau_1)\, q(\tau_2)\, \ldots e^{-S_E} \\ \langle 0 \mid 0 \rangle \langle q, 0 \mid 0 \rangle \langle 0 \mid q, 0 \rangle &= \int [dq]\, e^{-S_E} \\ G(\tau_1, \tau_2, \ldots) &= \frac{\int [dq] q(\tau_1) q(\tau_2) \ldots e^{-S_E}}{\int [dq] e^{-S_E}} \end{aligned}$$

where we integrate over all trajectories $q(\tau)$ with $\tau \in (-\infty, +\infty)$ and no specified end points.

1.2 Statistical Mechanics

Set the Boltzmann constant $k_B = 1$. Denote temperature by $T \geq 0$; At $T = 0$ the system settles in the ground state. Vacuum expectation values correspond to $T = 0$. To study finite T, introduce the partition function

$$\begin{aligned}
Z &= \sum_{states} e^{-E_n/T} \\
&= \sum_{states} \left\langle n \left| e^{-H/T} \right| n \right\rangle \\
&= \sum_{states} \mathrm{Tr}\left(e^{-H/T} \left| n \right\rangle \left\langle n \right| \right) \\
&= \mathrm{Tr}\left(e^{-H/T} \right) \\
&= \int dq \left\langle q, 0 \left| e^{-H/T} \right| q, 0 \right\rangle \\
&= \int dq \left\langle q, 0 \mid q, 1/T \right\rangle \\
&= \int dq \int_{q'(0)=q'(1/T)=q} [dq']\, e^{-S_E} \\
&= \int_{q(0)=q(1/T)} [dq]\, e^{-S_E}
\end{aligned}$$

Therefore, the partition function is given by a path integral over periodic orbits of period $1/T$.

Example 1 *Quantum mechanics: consider a free particle $V = 0$ and $L = \frac{1}{2} m\dot{q}^2$. Then using path integrals*

$$\langle q_i, t_i \mid q_f, t_f \rangle = \sqrt{\frac{m}{2\pi(\tau_f - \tau_i)}} e^{-m(q_f - q_i)^2/2(\tau_f - \tau_i)}$$

Example 2 *Statistical mechanics: Set $\tau_i = 0$, $\tau_f = 1/T$, and $q_i = q_f = q$.*

$$\begin{aligned}
Z &= \int dq \sqrt{\frac{mT}{2\pi}}\, which\ is\ infinite\ so\ put\ thesystem\ in\ a\ box \\
&= V\sqrt{\frac{mT}{2\pi}}\, where\ V\ is\ the\ volume\ (length).
\end{aligned}$$

which is the same as

$$\begin{aligned}
Z &= V \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-p^2/2m} \\
&= V\sqrt{\frac{mT}{2\pi}}\,.
\end{aligned}$$

1.3 Thermodynamics

1.3.1 Canonical ensemble

Expand the partition function around the classical trajectory to get

$$Z = e^{-S_{cl}+\cdots} = e^{-F/T}$$

where F is the Helmholtz free energy so

$$F = S_{cl}T\ .$$

The probability of a state $|n\rangle$ is $p_n = \frac{1}{Z}e^{-E_n/T}$. The internal energy is

$$\begin{aligned} U &= \langle E \rangle \\ &= \sum_{states} p_n E_n \\ &= T^2 \frac{\partial\,(\ln Z)}{\partial T} \end{aligned}$$

The entropy is

$$S = \sum_{states} p_n \ln p_n = T\frac{\partial\,(\ln Z)}{\partial T} - \ln Z = \frac{U}{T} - \frac{F}{T}$$

which means

$$F = U - TS\ .$$

1.3.2 Microcanonical ensemble

The microcanonical ensemble is isolated from its environment. The number of states with a given energy E is given by the multiplicity $g(E) = e^{S(E)}$, i.e., the entropy counts the number of different states.

$$\begin{aligned} Z &= \sum_{energy\ levels} e^{-E/T} g(E) \\ &\approx g(E_0)e^{-E_0/T} \\ &= e^{S(E_0)-E_0/T} \end{aligned}$$

where E_0 maximizes $S(E) - E/T$. We deduce $\frac{\partial S}{\partial E} = \frac{1}{T}$.

1.4 Field Theory

Set the speed of light $c = 1$. Let $x^\mu = (t, \mathbf{x})$ be the position 4-vector. Consider a real field $\varphi(x)$ with dynamics governed by the action $S = \int d^4x\mathcal{L}$ where $\mathcal{L}\left(\partial_\mu\varphi, \varphi\right)$ is the Lagrangian density; the Lagrangian is $L = \int d^3x\mathcal{L}$. Use the Wick rotation $\tau = it$ so

$ds^2 = -dt^2 + dx^2 = d\tau^2 + dx^2 = ds_E^2$ (Euclidean continuation). The correlation functions or Green functions are

$$\begin{aligned} G(x_1, x_2, \dots) &= \langle 0 \left| T\left(\varphi(x_1)\varphi(x_2)\cdots\right) \right| 0 \rangle \\ &= \frac{\int [d\varphi]\, \varphi\left(x_1\right) \varphi\left(x_2\right) \cdots\; e^{-S_E}}{\int [d\varphi]\, e^{-S_E}} \end{aligned}$$

as before. Taking an arbitrary function J as a source current, the generating functional is given by

$$Z\left[J\right] = \int [d\varphi] \left(e^{-S_E + \int d^4x J\varphi} \right)$$

and

$$G(x_1, x_2 \dots) = \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots\; Z\left[J\right] \Bigg|_{J=0} \tag{1.4.1}$$

Example 3 *For a free field of mass m,*

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + J\phi$$

The classical field equation is the Klein-Gordon equation

$$-\nabla^2 \phi_{cl} + m^2 \phi_{cl} = J$$

Writing a quantum field as

$$\phi = \phi_{cl} + \delta\phi$$

we obtain

$$S_E = S_{cl} + S^{(2)}(\delta\phi) \quad , \quad S_{cl} = \int d^4 \mathcal{L}_E(\partial_{cl\mu}, \phi_{cl})$$

Notice that there is no linear terms, because it is proportional to the field equation. Then

$$Z[J] = e^{-S_{cl}}$$

where we omitted a constant which was independent of J. We have

$$S_{cl} = \frac{1}{2} \int d^4x d^4x' J(x) D_E(x, x') J(x')$$

where D_E is the propagator,

$$(-\nabla^2 + m^2) D_E(x, x') = \delta^4(x - x')$$

Using (1.4.1), we easily deduce

$$G(x_1, x_2) = D_E(x_1, x_2)$$

You are invited to check that this is still valid when interactions are included.

LECTURE 2

Schwarzschild Black Hole

Spacetime is provided with a metric tensor $g_{\mu\nu}$ so that a line element has length

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

In flat spacetime, $ds^2 = -dt^2 + d\mathbf{x}^2$ ($\mathbf{x} \in \mathbb{R}^3$), so $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1\ 1\ 1\ 1)$ as a matrix. We denote the determinant of $g_{\mu\nu}$ by g. The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \begin{cases} 0 & , \quad \text{with just gravity not matter} \\ source & , \quad \text{in the presence of matter} \end{cases} .$$

$R_{\mu\nu}$ is the Ricci tensor (the contracted curvature tensor $R^\nu_{\mu\nu\rho}$) and $R = R^\mu_\mu$ (its trace) is the Ricci scalar. $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$ arises from the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R.$$

If we take the trace of the Einstein equation in empty space, we get

$$R^\mu_\mu - \frac{1}{2} R g^\mu_\mu = 0$$

which implies $R = 0$ so $R_{\mu\nu} = 0$. Schwarzschild found the solution

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the sphere S^2 and

$$f(r) = 1 - \frac{2GM}{r}$$

Where do G and M come from?
Compare to electromagnetism with Maxwell's equations without currents

$$\partial_\mu F^{\mu\nu} = 0.$$

Its simplest symmetric solution is $A_\mu = (A_0, \mathbf{0})$ with $\nabla^2 A_0 = 0$ so $A = \frac{Q}{r}$, i.e. the Coulomb potential. Q turns out to be the charge by Gauss's law.
In General Relativity, we can read off the mass from

$$g_{tt} \approx -1 + 2V(r)$$

if $V(r)$ is small (i.e., when $r \to \infty$), where $V(r)$ is the Newtonian potential ($V(r) = GM/r$).
In the Schwarzschild solution $f(r)$ diverges as $r \to 0$ which is a true singularity. There is also a singularity (which is an artifact of the coordinate system) at $r = 2GM \equiv r_+$ which is called the horizon. While not a coordinate singularity, the horizon is significant because inside ($r < r_+$) not even light can escape. Since $f(r_+) = 0$, proper time for an observer approaching the horizon ($\sqrt{-ds^2} = \sqrt{f(r)}dt$) passes quickly while to a distant observer it appears to take an infinite time to reach the horizon.
Now let us define $\tau = it$ so Minkowski space becomes Euclidean ($ds^2 = d\tau^2 + d\mathbf{x}^2$) and the Schwarzschild metric becomes

$$ds^2 \approx \left(1 - \frac{r_+}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)} + r^2 d\Omega_2^2.$$

When $r = r_+ + \varepsilon$ is outside but close to the horizon ($\varepsilon > 0$),

$$ds^2 \approx \frac{\varepsilon}{r_+} d\tau^2 + \frac{r_+}{\varepsilon} d\varepsilon^2 + r_+^2 d\Omega_2^2.$$

Note both ε and τ are completely independent of Ω so the space near the horizon neatly separates into a sphere S^2 of radius r_+ and a two-dimensional manifold of metric

$$ds_2^2 = \frac{\varepsilon}{r_+} d\tau^2 + \frac{r_+}{\varepsilon} d\varepsilon^2$$

To understand this manifold, change coordinates to $\rho = 2\sqrt{r_+\varepsilon}$ and $\chi = \frac{\tau}{2r_+}$ to get

$$ds_2^2 = \rho^2 d\chi^2 + d\rho^2$$

which is similar to polar coordinates but χ is not restricted to be between 0 and 2π. The resulting spacetime is a cone (the circumference of a closed curve with constant ρ is not $2\pi\rho$). We want to eliminate the conical singularity. If it is to be a plane, χ must be between 0 and 2π so τ must be between 0 and $4\pi r_+$. Recall that periodic imaginary time is an attribute of statistical systems of temperature T which is the iverse period. Thus the black hole has temperature

$$T = \frac{1}{4\pi r_+} = \frac{1}{8\pi GM}$$

which is the Hawking temperature. It looks like we have a statistical system, but what are the states? In thermodynamics, $dU = TdS$ where U is the total energy; in this case, it must be the mass. Thus $dM = \frac{1}{8\pi GM} dS$ which means

$$S = \int 8\pi GM dM = 4\pi GM^2 = \frac{4\pi r_+^2}{4G} = \frac{A_+}{4G}$$

where A_+ is the area of the horizon; this is the Bekenstein-Hawking formula. It is remarkably universal.
Normally entropy is proportional to volume and therefore mass (in an ordinary star with N particles and n degrees of freedom, there are n^N possible states so entropy S is proportional to $N \ln n$ which is proportional to mass M (and volume) so we have two surprises: the entropy is proportional to surface area (the first hint of holography) and to the square of the mass. Also as $T \to 0$, the number of states usually goes to one so $S \to 0$, but, in this case, $T = \frac{1}{8\pi GM}$ so as $T \to 0$, $M \to \infty$: entropy is increasing. The relation between temperature and mass implies that the heat capacity is given by

$$C = \frac{dM}{dT} = -8\pi GM^2 < 0$$

which is an unstable thermodynamic system.
Introduce the partition function with the Euclidean action S_E

$$Z = \int [dg]\, e^{-S_E} \approx e^{-S_{cl}}.$$

which should be quantum gravity. If the Ricci scalar $R = 0$ then $S_{cl} = 0$ but we had better be careful. The Ricci tensor is a second derivative and the Lagrangian should be independent of second order and higher derivatives. We can integrate by parts but we must keep the surface terms at $R > r_+$ and let $R \to \infty$ at the end.

$$S_{surface} = -\frac{1}{8\pi G}\int_{surface} d^3x\sqrt{h}K$$

(York-Gibbons-Hawking action) where $K = trK_{\mu\nu}$ is the extrinsic curvature of the surface and

$$K_{\mu\nu} = \frac{1}{2}n_\alpha g^{\alpha\beta}\partial_\beta g_{\mu\nu}$$

($n^\mu_\alpha = \frac{1}{\sqrt{g_{rr}}}\delta^\mu_r$ is the unit vector perpendicular to the surface).

$$\begin{aligned}
K_{\tau\tau} &= -\frac{GM}{R^2}\sqrt{1-\frac{2GM}{R}} \\
K_{\theta\theta} &= R\sqrt{1-\frac{2GM}{R}} \\
K_{\varphi\varphi} &= R\sin^2\theta\sqrt{1-\frac{2GM}{R}} \\
h_{\tau\tau} &= f(R)\; h_{\theta\theta} = R^2\; h_{\varphi\varphi} = R^2\sin^2\theta
\end{aligned}$$

Then

$$\begin{aligned}
S_{surface} &= -\frac{1}{8\pi G}\int_0^{1/T} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\sqrt{h}K \\
&= -\frac{2R-3GM}{2GT}
\end{aligned}$$

which is the classical action.
It diverges as $R \to \infty$. But in order to properly define it, we need to introduce a reference point. We shall subtract the contribution of empty space. The latter is obtained by setting $M = 0$. However, there is a complication. We obtained the temperature in the case $M \neq 0$ by demanding that there be no conical singularity. There is no such constraint in empty space ($M = 0$). Instead, we shall match the boundaries of the two spaces (at $r = R$). The time direction at $r = R$ for the black hole has length $\frac{1}{T}\sqrt{g_{\tau\tau}} = \frac{1}{T}\sqrt{f(R)}$. If we make time periodic with period $1/T_0$ for $M = 0$, then that will be its length at any r. We need to match

$$\frac{1}{T_0} = \frac{1}{T}\sqrt{f(R)} = \frac{1}{T}\sqrt{1 - \frac{2GM}{R}}$$

i.e., choose the temperature of empty space to be T_0 (red-shifted). Then

$$\begin{aligned} S_{cl} &= -\frac{2R - 3GM}{2GT} + \frac{R}{GT_0} \\ &= -\frac{2R - 3GM}{2GT} + \frac{R}{GT}\sqrt{1 - \frac{2GM}{R}} \\ &= \frac{3M}{2T} - \frac{M}{T} + \mathcal{O}(1/R) \\ &= \frac{M}{2T} \end{aligned}$$

Now

$$\begin{aligned} F &= U - TS \\ &= M - TS \\ &= M - \frac{1}{8\pi GM}\frac{\pi r_+^2}{G} \\ &= M - \frac{1}{8\pi GM}\frac{4\pi G^2 M^2}{G} \\ &= \frac{1}{2}M \end{aligned}$$

since $r_+ = 2GM$. Therefore,

$$S_{cl} = \frac{F}{T}$$

as expected.
Can we understand the entropy microscopically by counting degrees of freedom? This is hard at finite temperature. It is easier to ask the question in the limit $T \to 0$ because the system then settles to its ground state. The problem is that in this limit, $M \to \infty$, so this limit is hard to understand. We need a better black hole.

LECTURE 3

Reissner-Nördstrom Black Holes

3.1 The holes

Let us combine gravity with electromagnetism to find a charged black hole. The action is

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + \frac{1}{8\pi G}\int d^3x\sqrt{h}K - \frac{1}{16\pi}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$$

so there is now a source (electromagnetic field energy corresponds to mass). We have

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$

where (from electromagnetism)

$$T_{\mu\nu} = \frac{1}{4\pi}\left(g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

which has trace zero so there is no scale: no mass, no distance, no time, just photons. $R = 0$, just as in the Schwarzschild black hole, but we now have the Maxwell equations with no charges,

$$\nabla_\mu F^{\mu\nu} = 0$$

The most symmetric solution is the 4-vector with time component $A_0 = \frac{Q}{r}$; however, we want the potential to be zero at the horizon so we set

$$A_0 = \frac{Q}{r} - \frac{Q}{r_+}.$$

and

$$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$$

This is the Reissner-Nördstrom black hole. The horizon r_+ $(0 = f(r_+))$ is the solution to

$$r^2 - 2GMr + GQ^2 = 0$$

so

$$r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}$$

(r_- is inside r_+ so we cannot observe it). When $Q = 0$, we get $r_+ = 2GM$ as expected. The minimal r_+ occurs when $G^2M^2 - GQ^2 = 0$ so there is a minimal mass

$$M_{\min} = \frac{Q^2}{G}$$

Below this mass, we would have a naked singularity, but we stick with dressed singularities. The temperature is given by

$$\begin{aligned} T &= \frac{1}{4\pi} f'(r_+) \\ &= \frac{1}{4\pi}\left(\frac{2GM}{r_+^2} - \frac{2GQ^2}{r_+^3}\right) \\ &= \frac{1}{4\pi}\left(\frac{1}{r_+} - \frac{GQ^2}{r_+^3}\right) \end{aligned}$$

The entropy is

$$S = \frac{A_+}{4G} = \frac{\pi r_+^2}{G}.$$

The law of the thermodynamics reads

$$dM = TdS + \mu dQ$$

where $\mu = \frac{Q}{Gr_+}$ plays the role of a chemical potential and Q, in effect, counts the number of charges. Differentiating,

$$\begin{aligned} \left(\frac{\partial M}{\partial r_+}\right)_Q &= T\left(\frac{\partial S}{\partial r_+}\right)_Q \\ \left(\frac{\partial M}{\partial Q}\right)_{r_+} &= \frac{Q}{Gr_+} \end{aligned}$$

This seems to be nicer than a Schwarzschild black hole. In the extremal case, $r_+ = GM = \sqrt{G}Q$,

$$\begin{aligned} T &= \frac{1}{4\pi}\left(\frac{1}{r_+} - \frac{(GM)^2}{r_+^3}\right) = 0 \\ S &= \pi Q^2 \neq 0 \end{aligned}$$

violating the third law of thermodynamics.

Look at the classical action

$$S_{cl} = \frac{M}{2T} - \frac{Q^2}{2Tr_+^2}$$

and

$$\begin{aligned} \frac{F}{T} &= \frac{U}{T} - S \\ &= \frac{M}{T} - \frac{\pi r_+^2}{G} \end{aligned}$$

which is not equal to the classical action.

$$\begin{aligned} S_{cl} - \frac{F}{T} &= -\frac{M}{2T} + \frac{\pi r_+^2}{G} - \frac{Q^2}{2Tr_+^2} \\ &= -\frac{Q^2}{Tr_+} \\ &= -\frac{\mu Q}{T} \end{aligned}$$

so

$$TS_{cl} = F - \mu Q$$

is the Gibbs free energy. Two different calculations have given the same answer.

3.2 Extremal limit

Look at the extremal limit more closely

$$\begin{aligned} f(r) &= 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \\ &= 1 - \frac{2r_+}{r} + \frac{r_+^2}{r^2} \\ &= \left(1 - \frac{r_+}{r}\right)^2 \end{aligned}$$

If we change coordinates to $r' = r - r_+$,

$$f = \left(\frac{r'}{r' + r_+}\right)^2 = H^{-2}(r')$$

where $H(r') = 1 + r_+/r'$ is an harmonic function ($\nabla^2 H = 0$).

$$\begin{aligned} ds^2 &= -H^{-2}dt^2 + H^2 dr'^2 + \left(r' + r_+\right)^2 d\Omega_2^2 \\ &= -H^{-2}dt^2 + H^2 d\mathbf{x}^2\ , \quad \mathbf{x} \in \mathbb{R}^3 \end{aligned}$$

The Coulomb potential

$$A = \frac{Q}{r'} - \frac{Q}{r_+} = -\frac{Q}{Hr_+}$$

As we get near the horizon $r' \to 0$ and

$$ds^2 \approx -\left(\frac{r'}{r_+}\right)^2 dt^2 + \left(\frac{r_+}{r'}\right)^2 dr'^2 + r_+^2 d\Omega^2$$

so the black hole is the product of a 2-dimensional manifold and the 2-sphere; we shall show that the manifold is anti-de Sitter space (AdS).
Recall that in the case of a Schwarzschild black hole, after setting $\tau = it$, we obtained a cone. To get rid of the singularity, we required periodicity under $\tau \to \tau + \frac{1}{T}$ which determined the temperature T. Here, if we set $\tau = it/r_+^2$, we obtain

$$ds_2^2 = r_+^2 \left(r'^{\,2} d\tau^2 + \frac{1}{r'^{\,2}} dr'^{\,2} \right).$$

To see what manifold this is, in 3-dimensional Minkowski space with metric

$$ds_3^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2$$

consider a hyperboloid surface defined by

$$\left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 = r_+^2$$

If we change coordinates to (r_+, r', τ) with

$$\begin{aligned} x^0 &= r_+ \left(\frac{1}{2r'} + \frac{r'}{2}\left(1+\tau^2\right) \right) \\ x^1 &= r_+ \left(\frac{1}{2r'} - \frac{r'}{2}\left(1-\tau^2\right) \right) \\ x^2 &= r_+ r' \tau \end{aligned}$$

for a fixed r_+, we obtain the metric on the hyperboloid,

$$ds_2^2 = r_+^2 \left(r'^{\,2} d\tau^2 + \frac{1}{r'^{\,2}} dr'^{\,2} \right),$$

which is of the desired form (Euclidean anti-de Sitter space ($EAdS_2$)). There is no restriction on τ, which is consistent with the fact that $T = 0$. This surface has three isometries:

$$\begin{aligned} M_1 &= i\left(x^0\partial_1 - x^1\partial_0\right), \text{ a boost along the } x\text{-axis} \\ M_2 &= i\left(x^0\partial_2 - x^2\partial_0\right), \text{ a boost along the } x\text{-axis} \\ L &= i\left(x^1\partial_2 - x^2\partial_1\right), \text{ angular momentum.} \end{aligned}$$

satisfying the Lorentz algebra

$$\begin{aligned} [L, M_1] &= -iM_2 \\ [L, M_2] &= iM_1 \\ [M_1, M_2] &= iL. \end{aligned}$$

In terms of (r', τ),

$$\begin{aligned} M_1 &= i\left(-r'\partial_{r'} + \tau\partial_\tau\right) \\ M_2 &= i\left(r'\tau\partial_{r'} + \frac{1}{2}\left(1 + \frac{1}{r'^{\,2}} - \tau^2\right)\partial_\tau\right) \\ L &= i\left(r'\tau\partial_{r'} + \frac{1}{2}\left(-1 + \frac{1}{r'^{\,2}} - \tau^2\right)\partial_\tau\right) \end{aligned}$$

As we approach the boundary ($r\prime \to \infty$) staying on a $r' =$ const. slice (so that $\partial_{r'} \to 0$) we have

$$\begin{aligned} M_1 &\to i\tau\partial_\tau \\ M_2 &\to \frac{i}{2}\left(1 - \tau^2\right)\partial_\tau \\ L &\to -\frac{i}{2}\left(1 + \tau^2\right)\partial_\tau \end{aligned}$$

Define

$$D = M_1 = i\tau\partial_\tau\ ,\ \ H = M_2 - L = i\partial_\tau\ ,\ \ K = -M_2 - L = i\tau^2\partial_\tau$$

Evidently,

$$[D\ ,\ H] = -iH\ ,\ \ [D\ ,\ K] = iK\ ,\ \ [H\ ,\ K] = 2iD$$

which is the conformal algebra in one dimension. D generates dilations, H is the Hamiltonian (generates translations in τ) and K generates special conformal transformations.

Exercise: Show that for an operator $\mathcal{O}(\tau)$,

$$e^{-iaH}\mathcal{O}(\tau)e^{iaH} = \mathcal{O}(\tau + a)\ ,\ \ e^{-iaD}\mathcal{O}(\tau)e^{iaD} = \mathcal{O}(e^a\tau)$$

and find the corresponding transformation for K.

Thus the isometries of AdS space turned into the generators of the conformal group. Starting with a black hole, we have arrived at a conformal quantum mechanical system; this is the first hint at an AdS/CFT correspondence.

3.3 Generalizations

3.3.1 Multi-center solution

Recall that in the extremal limit,

$$ds^2 = -H^{-2}dt^2 + H^2 d\mathbf{x}^2 \quad (\mathbf{x} \in \mathbb{R}^3)\ ,\ \ A_t = -H^{-1}$$

and $H = 1 + \frac{r_+}{|\mathbf{x}|}$, an harmonic function. You may guess that the Einstein-Maxwell equations lead to $\nabla^2 H = 0$, so H can be any harmonic function, in particular a multi-center solution,

$$H = 1 + \frac{q_1}{|\mathbf{x} - \mathbf{x}_1|} + \frac{q_2}{|\mathbf{x} - \mathbf{x}_2|} + \ldots$$

consisting of the Coulomb potentials due to charges q_i at positions $\mathbf{x}_i$ ($i = 1, 2, \ldots$). The fist term equal to 1 is there for correct asymptotics. All these black holes are in equilibrium because the electric and gravitational forces cancel each other (due to $G^2 M_i M_j = G Q_i Q_j$, where M_i is the mass and $Q_i = q_i/\sqrt{G}$ is the charge of the ith black hole).

3.3.2 Arbitrary dimension

In dimensions $d \geq 4$, a Reissner-Nördstrom black hole has metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \ , \ \ f(r) = 1 - \frac{2\mu}{r^{d-3}} + \frac{q}{r^{2(d-3)}}$$

where

$$\mu = \frac{8\pi GM}{(d-2)\Omega_{d-2}} \text{ and }, \ \ q = \frac{8\pi GQ^2}{(d-3)(d-2)\Omega_{d-2}} \ , \ \ \Omega_{d-2} = \frac{2\pi}{\Gamma\left(\frac{d-1}{2}\right)}$$

$\Omega + d - 2$ is the volume of a sphere in dimension $d-2$ (S^{d-2}).
The horizon is at r_+, where

$$r_\pm^{d-3} = \mu \pm \sqrt{\mu^2 - q}, \mu^2 \geq q$$

the temperature is

$$T = \frac{f\,'(r_+)}{4\pi} = \frac{d-3}{4\pi r_+}\left(1 - \left(\frac{r_-}{r_+}\right)^{d-3}\right).$$

and the entropy is

$$S = \frac{A_+}{4G} = \frac{\Omega_{d-2} r_+^{d-2}}{4G}$$

At extremality, $\mu^2 = q$, so $r_+ = r_-$ and $T = 0$. The entropy is $S \propto Q^{\frac{d-2}{d-3}}$. For the metric, notice that

$$f(r) = \left[1 - \left(\frac{r_+}{r}\right)^{d-3}\right]^2$$

Changing coordinates to

$$r'^{d-3} = r^{d-3} - r_+^{d-3}$$

so that the horizon is at $r' = 0$, we obtain

$$\begin{aligned} ds^2 &= -H^{-2}dt^2 + H^{\frac{2}{d-3}} d\mathbf{x}^2 \ , \ \ \mathbf{x} \in \mathbb{R}^{d-1} \\ H &= 1 + \left(\frac{r_+}{|\mathbf{x}|}\right)^{d-3} \end{aligned}$$

H is an harmonic function.

3.3.3 Kaluza-Klein reduction

Interestingly, the four-dimensional Reissner-Nördstrom black hole (in fact, all of electromagnetism) can also be derived from pure gravity in five dimensions. The action is

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g^{(5)}} R^{(5)}$$

The space is spanned by coordinates x^M ($M = 0, 1, 2, 3, 4$). If x^4 is a loop of length L (compactified) and there is no dependence on x^4 (which is true if we are interested in distances large compared to L), then

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g^{(5)}} R^{(5)} \ , \ \ \frac{1}{G} = \frac{L}{G_5}$$

where G is Newton's constant in our four-dimensional world.
Let us parametrize the metric as

$$ds^2 \equiv g^{(5)}_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + e^{2\Phi} \left(dx^4 + A_\mu dx^\mu \right)^2$$

Then the Ricci scalar reads

$$R^{(5)} = R - 2e^{-\Phi} \nabla^2 e^{\Phi} - \frac{1}{4} e^{2\Phi} F_{\mu\nu} F^{\mu\nu}$$

where R is the Ricci scalar in our four-dimensional world.
The action becomes

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{4} e^{2\Phi} F_{\mu\nu} F^{\mu\nu} \right) .$$

consisting of a gravitational field $g_{\mu\nu}$, the dilaton Φ and a photon A_μ. Notice that all three fields are massless.
Consider a particle of the dilaton field. SInce it is massless, $p^2 = p^M p_M = 0$. Because the fifth dimension has a finite length L, p_4 must be quantized: $e^{ip_4 x^4} = e^{ip_4(x^4+L)}$ so $p_4 L = 2\pi n$ for some integer n. Then in the four dimensions we can observe, the particle appears to have mass m with

$$m^2 = p_\mu p^\mu = p_4^2 = \left(\frac{2\pi}{L} \right)^2 n^2$$

so $m = 0, 2\pi/L, 4\pi/L, \ldots$, the Kaluza-Klein tower. These masses are, in principle, observable, but we have not observed them. If $L \ll 1$, these masses would be very large and, therefore, may not be observable at present (or ever).
To get the Reissner-Nördstrom black hole, set $\Phi = 0$. Then the action reduces to one with gravity and electromagnetism which is exactly where we got the Reissner-Nördstrom black hole earlier.
Additionally, there exists a solution with $\Phi \neq 0$. We obtain

$$ds^2 = -\frac{f(r)}{H^{1/8}} dt^2 + H^{7/8} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \right) \ , \ \ f(r) = 1 - \frac{r_+}{r} \ , \ \ H = 1 + \frac{\rho}{r}$$

$e^{-2\Phi} = H^{3/2}$, and $A_t = H^{-1}$.
In the extremal limit $r_+ \to 0$

$$ds^2 = -H^{-1/8}dt^2 + H^{7/8}d\mathbf{x}^2 \ , \ \ \mathbf{x} \in \mathbb{R}^3 \ .$$

The singularity at $r = |\mathbf{x}| = 0$ has vanishing area, therefore the entropy $S = 0$.

LECTURE 4

Black Branes from String Theory

The Kaluza-Klein picture we have presented above is classical and, just as we need quantum mechanics to understand the electron (which is a point particle, i.e. a singularity), we need quantum mechanics to understand a black hole. Unfortunately (because it is very complicated) the only quantum theory of gravity we possess is string theory. To avoid infinities in the construction of quantum gravity, string theory complicates things:

- Strings live in ten dimensions

- Strings vibrate. The greater the frequency, the higher the mass. Also an infinite number of vibrational modes should correspond to an infinite number of particles. The massless modes must be the particles we see. The electron's mass is $0.5\ MeV$, the proton's is $1\ GeV$, and the W particle has a mass of $80\ GeV$, but the vibrational modes have mass/energy on the order of the Planck mass/energy $10^{19}\ GeV$. Presumably these particles appeared only at the beginning of the universe.

- We need supersymmetry: every particle must have a super partner.

Example 1. Type IIA. Low energy (massless strings). In ten dimensions specify a metric $g_{\mu\nu}$, a scalar φ, an anti-symmetric tensor $B_{\mu\nu}$, a vector potential A_μ, and an anti-symmetric tensor $A_{\mu\nu\rho}$. As usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; we can do a similar thing for $B_{\mu\nu}$ and $A_{\mu\nu\rho}$ to get anti-symmetric tensors

$$\begin{aligned} H_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} \\ F_{\mu\nu\rho\sigma} &= \partial_\mu A_{\nu\rho\sigma}. \end{aligned}$$

Also we can anti-symmetrize $F_{\mu\nu\rho\sigma} + H_{\mu\nu\rho} A_\sigma$ to get a $G_{\mu\nu\rho\sigma}$. The action is

$$\begin{aligned} S_{10} &= \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \begin{pmatrix} R - \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{12} e^{\varphi/2} H_{\mu\nu\rho} H^{\mu\nu\rho} - \\ \frac{1}{4} e^{5\varphi/2} \left(F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} \right) \end{pmatrix} \\ & \quad - \frac{1}{32\pi G_{10}} \int \epsilon^{\mu_1 \cdots \mu_{10}} B_{\mu_1 \mu_2} F_{\mu_3 \dots \mu_6} F_{\mu_7 \dots \mu_{10}} \end{aligned}$$

where ϵ is the Levi-Civita tensor (zero if any indices are repeated, ± 1 if the permutation of the indices is even (odd)).
In eleven dimensions the situation is simpler. We only require an anti-symmetric tensor A_{ABC} with anti-symmetric F_{ABCD} as above. The action is

$$S_{11} = \frac{1}{16\pi G_{11}} \int d^{11}x \sqrt{-g^{(11)}} \left(R^{(11)} + \frac{1}{18} F_{ABCD} F^{ABCD} \right).$$

If the tenth dimension is tightly wrapped, we can partition the matrix $g_{AB}^{(11)}$ into a 9 by 9 matrix $g_{\mu\nu}^{(10)}$, a 9 by 1 column A_μ (and the same 1 by 9 row) and a scalar φ:

$$g_{AB}^{(11)} = \begin{pmatrix} g_{\mu\nu}^{(10)} & A_\mu \\ A_\mu & \varphi \end{pmatrix}.$$

If we restrict the indices of A_{ABC} to be from 0 to 9, we get a 10-dimensional tensor $A_{\mu\nu\rho}$ and setting $B_{\mu\nu} = A_{\mu\nu(10)}$, we obtain all the fields for the 10-dimensional case and $S^{(11)}$ gives $S^{(10)}$.
Example 2. Type IIB. p-brane solutions. We are given an anti-symmetric tensor $A_{\mu_1 \dots \mu_{p+1}}$ and a metric

$$ds^2 = -H^{(p-7)/8} \left(-f(r) dt^2 + d\mathbf{y}^2 \right) + H^{(p+1)/8} \left(\frac{1}{f(r)} dr^2 + r^2 d\Omega_{8-p}^2 \right)$$

where

$$H = 1 + \frac{\rho^{7-p}}{r^{7-p}},\ e^{2\varphi} = H^{(3-p)/2},\ f(r) = 1 - \frac{r_+^{7-p}}{r^{7-p}}, \text{ and } \mathbf{y} \in \mathbb{R}^p.$$

This would be a black hole except for the presence of $d\mathbf{y}$. It appears as if a black hole is stretched through other dimensions and is called a black brane around $\langle \mathbf{y} \rangle$.
In the extremal limit $r_+ \to 0$,

$$ds^2 = -H^{(p-7)/8} \left(-dt^2 + d\mathbf{y}^2 \right) + H^{(p+1)/8} d\mathbf{x}^2$$

where $\mathbf{x} \in \mathbb{R}^{9-p}$. As $r \to 0$, H has a singularity and

$$ds^2 \approx \left(\frac{\rho}{r} \right)^{-(7-p)^2/8} \left(-dt^2 + d\mathbf{y}^2 \right) + \left(\frac{\rho}{r} \right)^{(7-p)(p+1)/8} d\mathbf{x}^2.$$

When $p = 3$, we get something special, the dilaton φ is constant and

$$\begin{aligned} ds^2 &\approx \frac{r^2}{\rho^2} \left(-dt^2 + d\mathbf{y}^2 \right) + \frac{\rho^2}{r^2} d\mathbf{x}^2 \\ &= \frac{r^2}{\rho^2} \left(-dt^2 + d\mathbf{y}^2 \right) + \frac{\rho^2}{r^2} dr^2 + \rho^2 d\Omega_5^2. \end{aligned}$$

The last term is just the five sphere S^5 and r would not fall out if $p \neq 3$. The other terms are the anti-deSitter space AdS_5 (3 spatial and 1 time dimension). The situation is similar to two dimensional conformal theory on the boundary giving quantum mechanics.

LECTURE 5

Microscopic calculation of Entropy and Hawking Radiation

5.1 The hole

We want to understand why entropy is so large. Can it be explained from the action in this example? In five dimensions, the Reissner-Nördstrom black hole has the metric

$$ds^2 = H^{-2}dt^2 + H^2 d\mathbf{x}^2$$

where $H = 1 + Q^2/r^2$ ($r^2 = \mathbf{x}^2$). The 5-brane has metric

$$ds^2 = H^{-1/4}\left(dt^2\right) + d\mathbf{y}^2$$

where H is the same as in the Reissner-Nördstrom black hole and $e^{2\varphi} = H^{-1}$ is singular. Strominger and Vafa looked at

$$\begin{aligned} ds^2 &= H_1^{-3/4} H_5^{-1/4} - dt^2 + d\mathbf{y}_5^2 + (H_n - 1)(dt + d\mathbf{y})^2 \\ &\quad + \left(\frac{H_1}{H_5}\right)^{1/4} d\mathbf{y}^2 + H_1^{1/4} H_5^{3/4} d\mathbf{x}^2 \end{aligned}$$

where

$$H_n = 1 + \frac{\rho_n^2}{r^2} \text{ and } e^{-2\varphi} = \frac{H_5}{H_1},$$

$\mathbf{y}$ and $\mathbf{x}$ are in $\mathbb{R}^4$ and all the y-dimensions are very small.
Consider some special cases.
Case 1: When $\rho_1 = \rho_n = 0$, the metric becomes

$$\begin{aligned} ds^2 &= H_5^{-1/4}\left(-dt^2 + dy_5^2\right) + (H_5)^{-1/4} d\mathbf{y}^2 + H_5^{3/4} d\mathbf{x}^2 \\ &= H_5^{-1/4}\left(-dt^2 + dy_5^2 + d\mathbf{y}^2\right) + H_5^{3/4} d\mathbf{x}^2 \end{aligned}$$

where the y-terms form a 5-dimensional Euclidean space or, if small, a torus (5-brane).
Case 2: When $\rho_5 = \rho_n = 0$, the metric becomes

$$ds^2 = H_1^{-3/4}\left(-dt^2 + dy_5^2\right) + H_1^{1/4}\left(d\mathbf{y}^2 + d\mathbf{x}^2\right)$$

where the $\mathbf{x}$ and $\mathbf{y}$ terms contribute 8 dimensions and the y_5 term is a 1-brane.
Case 3: When $\rho_1 = \rho_5 = 0$, the metric becomes

$$ds^2 = -dt^2 + dy_5^2 + (H_n - 1)\left(dt + dy_5\right)^2 + d\mathbf{y}^2 + d\mathbf{x}^2$$

with no brane.
?? The complete solution is a bound state of a 5-brane from the survival of H_5 and a 1-brane from the survival of H_1. A 5-dimensional observer should perceive a black hole; the y_5-direction will have a momentum proportional to n/L which will be perceived as mass.

$$ds^2 = -H_n^{-1}dt^2 + H_n\left(dy_5 + \left(\frac{1}{H_n} - 1\right)dt\right)^2 + d\mathbf{y}^2 + d\mathbf{x}^2$$

looks like the Kaluza-Klein reduction where

$$ds^2 = ...stuff\left(dx^4 + A_\mu dx^\mu\right)^2$$

and, here we can set

$$A_t = \left(\frac{1}{H_n} - 1\right).$$

??
Now take the complete metric

$$ds^2 = H_1^{-3/4}H_5^{-1/4}H_n\left(dy_5 + A_t dt\right)^2 + \left(\frac{H_1}{H_5}\right)^{1/4}d\mathbf{y}^2 - H_1^{-3/4}H_5^{-1/4}H_n^{-1}dt^2 + H_1^{1/4}H_5^{3/4}d\mathbf{x}^2$$

where $\mathbf{y}$ (y_5) is a small, closed loop in $\mathbb{R}^5$ ($\mathbb{R}$). A 5-dimensional observer sees the last two terms which equal

$$H_1^{-3/4}H_5^{-1/4}H_n^{-1}\left(-dt^2 + H_1H_5H_n d\mathbf{x}^2\right)$$

which is similar to a Reissner-Nordström black hole:

$$ds^2 = -\frac{1}{H^2}dt^2 + Hd\mathbf{x}^2 = \frac{1}{H^2}\left(-dt^2 + H^3 d\mathbf{x}^2\right)$$

and, if $\rho_1 = \rho_5 = \rho_n$, it is exactly the same. We could set $H = (H_1H_5H_n)^{1/3}$ so the difference is that a RN black hole as one charge while the general case has three.

5.2 Entropy

To get the entropy, we need the area $A = 2\pi^2 r_+^3$ where, ?? in the RN case ($Hd\mathbf{x}^2 = Hr^2 d\Omega^2$), $r_+ = \lim_{r\to 0}\sqrt{Hr^2}$. ?? Therefore

$$A = 2\pi^2 \lim_{r\to 0}\left(\left(\frac{\rho_1^2}{r^2}\frac{\rho_5^2}{r^2}\frac{\rho_n^2}{r^2}\right)^{1/6} r\right)^3 = 2\pi^2\left(\rho_1\rho_5\rho_n\right)$$

Mass for a Schrödinger black hole, is deduced from $g_{tt} = -\left(1 - \frac{2GM}{r}\right)$. In the general case of d dimensions $g_{tt} \approx -1 + 2V$ as $r \to \infty$ where V is the potential energy; in fact

$$g_{tt} = -1 + \frac{\mu}{r^{d-3}},\ \mu = \frac{8\pi GM}{(d-2)\Omega_{d-2}},\ \text{and}\ \Omega_{d-1} = \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)}.$$

For 5 dimensions, $G_5 M = 3\pi\mu/4$ and $g_{tt} = -\left(H_1 H_5 H_n\right)^{-2/3}$; expanding in $1/r$, we get

$$g_{tt} = -1 + \frac{2}{3}\frac{\rho_1^2 + \rho_5^2 + \rho_n^2}{r^2} + \ldots$$

so

$$G_5 M = \frac{\pi}{2}\left(\rho_1^2 + \rho_5^2 + \rho_n^2\right).$$

Mass splits into three pieces (with no binding energy) due to the 1-brane, the 5-brane, and the momentum.
Let $\rho_n = 0$ and look at the metric near the horizon ?? $r \to 0$??

$$\begin{aligned} ds^2 &\approx \frac{r^2}{\sqrt{\rho_1^3\rho_5}}\left(-dt^2 + dy_5^2\right) + \sqrt{\frac{\rho_1}{\rho_5}}d\mathbf{y}^2 + \frac{\sqrt{\rho_1\rho_5^3}}{r^2}\left(dr^2 + r^2 d\Omega_3^2\right) \\ &= \left(\frac{r^2}{\sqrt{\rho_1^3\rho_5}}\left(-dt^2 + dy_5^2\right) + \sqrt{\rho_1\rho_5^3}\frac{dr^2}{r^2}\right) + \sqrt{\frac{\rho_1}{\rho_5}}d\mathbf{y}^2 + \sqrt{\rho_1\rho_5^3}d\Omega_3^2 \end{aligned}$$

where the terms in parenthesis represent AdS_3, the next term is a 4-torus, T^4, and the last term is just the 3-sphere, S^3.

?? Can we understand entropy in this model? The 1-brane can be taken to be wrapped tightly N_1 times, the 5-brane N_5 times, no momentum, and only one state so $S = 0$. This suggests entropy comes from momentum. String theory tells us that excitations of branes arise from the movement of strings attached to branes. Branes have no binding energy and they are points (no deeper structure). We can set the coupling constant to one. The only parameter that is available to determine energy is the length of the string ℓ_s; thus the unit of energy is $1/\ell_s$. If the 1-brane wraps N_1 times around a length L_1, it must have $N_1 L_1/\ell_s$ elementary units so its energy is $N_1 L_1/\ell_s^2$. $M_1 = N_1 L_5/2\pi\ell_s^2$. In 4-dimensions, $G \propto \ell^2$ so we expect $G_{10} \propto \ell_s^8$, in fact $G^{10} = 16\pi^6\ell_s^8$. The equation

$G_5 M = \frac{\pi}{2}\left(\rho_1^2 + \rho_5^2 + \rho_n^2\right)$ results from gravity alone, not string theory. Now

$$\begin{aligned} G_5 M_1 &= \frac{\pi}{2}\rho_1^2 \\ \frac{16\pi^6 \ell_s^8}{L_1 ... L_5}\frac{N_1 L_5}{2\pi \ell_s^2} &= \frac{\pi}{2}\rho_1^2 \\ \rho_1^2 &= \frac{(2\pi)^4}{L_1 ... L_4}\ell_s^6 N_1. \end{aligned}$$

Similarly the 5-brane

$$\begin{aligned} \frac{N_5 L_1 ... L_5}{\ell_s^2}\frac{1}{\ell_s} &= 16\pi^5 M_5 \\ \rho_5 &= N_5 \ell_s^2. \end{aligned}$$

The momentum in the y_5 direction $p_5 = 2\pi n / L_5$, $n = 0, 1, 2, ...$ is the last piece of the puzzle :

$$\rho_n^2 = \frac{(2\pi)^6 \ell_s^8}{L_1 ... L_4 L_5^2} n$$

and, putting it all together,

$$\begin{aligned} S &= 2\pi^2 \rho_1 \rho_5 \rho_n \\ &= 2\pi \sqrt{N_1 N_2 n}. \end{aligned}$$

This is one success of string theory. ??

We now intend to count the degrees of freedom of this system. Note that strings can begin and end on any brane, but only strings between the 1-brane and the 5-brane contribute. For instance, when $N_1 = 2$ and $N_5 = 2$, the string will return to its starting point after going around twice, but, if $N_1 = 2$ and $N_5 = 3$, the string will return to its starting point after going around six times. In general, if N_1 and N_5 are relatively prime, the string goes around $N_1 N_5$ times. If both numbers are very large, we can ignore the relatively prime requirement since they will be close to a relatively prime pair. Then $L_{eff} = N_1 N_5 L_5$. Then

$$p_5 = \frac{2\pi n}{L_5} = \frac{2\pi n}{L_{eff}}\frac{L_{eff}}{L_5} = N_1 N_5 n \frac{2\pi}{L_{eff}}$$

and $2\pi / L_{eff}$ can be considered the quantum of momentum. Given a momentum, we want to count all of the ways of getting that momentum to get the number of states. We do so in a physical way. Suppose the quantum of energy is E_0 and n_i particles have energy iE_0 at temperature T so the total energy is $E = n_1 E_0 + n_2 2E_0 + n_3 3E_0 + ...$

Define the partition function

$$\begin{aligned} Z &= \sum_E e^{-E/T} \\ &= \sum_{n_i} e^{-(n_1E_0+n_2 2E_0+n_3 3E_0+\ ...)/T} \\ &= \left(\sum_{n_1=0}^{\infty} e^{-n_1E_0/T}\right)\left(\sum_{n_2=0}^{\infty} e^{-n_2 2E_0/T}\right)... \\ &= \frac{1}{1-e^{-E_0/T}}\frac{1}{1-e^{-2E_0/T}}... \end{aligned}$$

The free energy is given by $F = -T\ln Z = T\sum_N \ln\left(1-e^{-NE_0/T}\right)$. Take the high temperature limit $T \to \infty$ and let $x = NE_0/T$ and $\Delta x = E_0/T \to 0$. Then

$$\begin{aligned} F &= E_0\frac{T}{E_0}\sum \ln\left(1-e^{-x}\right) \\ &= \frac{T^2}{E_0}\frac{E_0}{T}\sum \ln\left(1-e^{-x}\right) \\ &= \frac{T^2}{E_0}\Delta x\sum \ln\left(1-e^{-x}\right) \\ &\approx \frac{T^2}{E_0}\int \ln\left(1-e^{-x}\right)dx \\ &= \frac{T^2}{E_0}\frac{\pi^2}{6}. \end{aligned}$$

The entropy is $S = -\frac{\partial F}{\partial T} = \frac{\pi^2}{3}\frac{T}{E_0}$ and the internal energy is $E = -T^2\frac{\partial}{\partial T}\left(\frac{F}{T}\right) = \frac{\pi^2}{6}\frac{T^2}{E_0} = nE_0$ so $\frac{T^2}{E_0} = \frac{6n}{\pi^2}$ which we substitute into the entropy

$$S = \frac{\sqrt{6n}}{\pi}\frac{\pi^2}{3} = 2\pi\sqrt{\frac{n}{6}} = 2\pi\sqrt{\frac{N_1N_5n}{6}}.$$

This is not the whole story because string theory says there are four different strings so we should multiply by 4 under the radical; furthermore, supersymmetry gives fermions for which n_i must be either 0 or 1 so $S_{fermion} = 2\pi\sqrt{\frac{n}{12}}$. Therefore

$$S = 2\pi\sqrt{\frac{N_1N_5n}{6}\left(4+\frac{4}{2}\right)} = 2\pi\sqrt{N_1N_5n}.$$

We have come to the same conclusion independently of the calculation using the horizon area.

5.3 Finite temperature

Suppose T is slightly greater than zero. To heat the system, we could introduce an anti-brane or let the strings run in both directions. Let the number of left (right) moving strings be n_+ (n_-). A 5-dimensional observer would see a charge $Q \propto n_+ - n_-$ ($M_n \propto n_+ + n_-$). The metric is

$$ds^2 = H_1^{-3/4} H_5^{-1/4} \left(-dt^2 + dy_5^2 + \frac{1}{r^2} \left(\rho_+ dt + \rho_- dy_5 \right)^2 \right) + \left(\frac{H_1}{H_5} \right)^{1/4} d\mathbf{y}^2 + H_1^{1/4} H_5^{3/4} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \right)$$

where ρ splits into ρ_+ and ρ_+ and $f(r) = 1 - \frac{r_+^2}{r^2}$ so there are two horizons at 0 and r_+. $\rho_+^2 - \rho_-^2 = r_+^2$ so, when $\rho_+ = \rho_-$, $r_+ = 0$ and we are back to the case $T = 0$. Let us consider physical properties where

$$ds_5^2 = -\left(H_1 H_5 H_n \right)^{-2/3} f dt^2 + \left(H_1 H_5 H_n \right)^{1/3} \left(\frac{dr^2}{f} + r^2 d\Omega_3^2 \right)$$

with $H_n = 1 + \rho_-^2 / r^2$. The area of the horizon is

$$A_+ = 2\pi^2 r_+^3 \sqrt{H_1 H_5 H_n} \,\big|_{r=r_+} .$$

Assume $\rho_1,\ \rho_5 >> r_+$, then

$$A_+ = 2\pi^2 r_+^3 \sqrt{\frac{\rho_1^2 \rho_5^2}{r_+^4} \frac{r_+^2 + \rho_-^2}{r_+^2}} = 2\pi \rho_1 \rho_5 \rho_+ .$$

Expand the metric to get

$$g_{tt} = -\left(H_1 H_5 H_n \right)^{-2/3} f = -1 + \frac{2}{3} \frac{\rho_1^2 + \rho_5^2 + \rho_-^2}{r^2} + \frac{r_+^2}{r^2} + \ ....$$

The $1/r^2$ comprise μ / r^2 where in classical gravity $G_5 M = \frac{3\pi}{4} \mu$; therefore

$$\begin{aligned} G_5 M &= \frac{\pi}{2} \left(\rho_1^2 + \rho_5^2 + \rho_-^2 + \frac{3}{2} r_+^2 \right) \\ &= G_5 \left(M_1 + M_5 + M_n \right) . \end{aligned}$$

and

$$\begin{aligned} G_5 M_1 &= \frac{\pi}{2} \left(\rho_1^2 + \frac{1}{2} r_+^2 \right) \approx \frac{\pi}{2} \rho_1^2 \\ G_5 M_5 &= \frac{\pi}{2} \left(\rho_5^2 + \frac{1}{2} r_+^2 \right) \approx \frac{\pi}{2} \rho_5^2 \\ G_5 M_n &= \frac{\pi}{2} \left(\rho_-^2 + \frac{1}{2} r_+^2 \right) \approx \frac{\pi}{4} \left(\rho_+^2 + \rho_-^2 \right) . \end{aligned}$$

We now have

$$\begin{aligned} n_+ + n_- &\propto M_N \propto \rho_+^2 + \rho_-^2 \\ n_+ - n_- &\propto Q \propto 2 \rho_+ \rho_- \end{aligned}$$

so

$$\begin{aligned} n_+ &\propto \left(\rho_+ + \rho_-\right)^2 \\ n_- &\propto \left(\rho_+ - \rho_-\right)^2 . \end{aligned}$$

In terms of $n_\pm$, $S \propto \rho_+ \propto \sqrt{n_+} + \sqrt{n_-}$ or

$$S = 2\pi\sqrt{N_1 N_5}\left(\sqrt{n_+} + \sqrt{n_-}\right)$$

which is expected since the right and left moving modes are completely independent: $S_+ = 2\pi\sqrt{N_1 N_5 n_+}$, $S_- = 2\pi\sqrt{N_1 N_5 n_-}$, $S = S_+ + S_-$. With two independent quantum systems, our counting agrees with classical mechanics.
To count the degrees of freedom, we want to look near the horizon r_+, but carefully because r_+ is very small. When r is small the H functions can be approximated by ρ^2/r^2. As before the metric becomes

$$\begin{aligned} ds^2 &= \frac{r^2}{\sqrt{\rho_1^3\rho_5}}\left(-dt^2 + dy_5^2 + \frac{1}{r^2}\left(\rho_+ dt + \rho_- dy_5\right)^2\right) + \sqrt{\frac{\rho_1}{\rho_5}}d\mathbf{y}^2 + \frac{\sqrt{\rho_1\rho_5^3}}{r^2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_3^2\right) \\ &= \frac{r^2}{\sqrt{\rho_1^3\rho_5}}\left(-dt^2 + dy_5^2\right) + \frac{1}{\sqrt{\rho_1^3\rho_5}}\left(\rho_+ dt + \rho_- dy_5\right)^2 + \sqrt{\rho_1\rho_5^3}\frac{dr^2}{r^2 - r_+^2} + \sqrt{\rho_1\rho_5^3}d\Omega_3^2 + \sqrt{\frac{\rho_1}{\rho_5}}d\mathbf{y}^2 . \end{aligned}$$

The last two terms are the sphere S^3 and the torus T^4. The other terms form a three dimensional manifold M which was AdS_3 when T was zero. To study the metric ds_3^2 of M, set $r'^2 = r^2 + \rho_-^2$ to get ??

$$ds_3^2 = \frac{1}{\sqrt{\rho_1^3\rho_5}}\left(-\frac{\left(r'^2 - \rho_-^2\right)\left(r'^2 - \rho_+^2\right)}{r'^2}dt^2 + \rho_1^2\rho_5^2\frac{r'^2 dr'^2}{\left(r'^2 - \rho_-^2\right)\left(r'^2 - \rho_+^2\right)} + r'^2\left(dy_5 + \frac{\rho_+ + \rho_-}{r'^2}dt\right)^2\right).$$

?? This is Banados-Teitelboim-Zanelli or BTZ black hole. If $r' \to \infty$, the large parenthetical term reduces to $-r'^2 dt^2 + \rho_1^2\rho_5^2 dr'^2/r'^2 + r'^2 dy_5^2$ which is the metric of AdS_3. This is unusual because, far from the horizon, we do not get flat space; in this case going to infinity means going as far away as possible without leaving the horizon. Being near the horizon introduces a cosmological constant to get negative curvature.

5.4 Scattering and Hawking radiation

What do we mean by going to infinity and not leaving the horizon? Look at radiation thrown at the black hole. Introduce a scalar field φ which is massless and has minimal coupling (no term coupling with curvature); the action

$$S = \frac{1}{2}\int d^5\mathbf{x}\sqrt{-g}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}$$

give the Klein-Gordon field equation $\Box\varphi = 0$ which we want to solve for scattering. Use a spherically symmetric 5-dimensional wave not involving angles

$$\varphi(r,t) = \Phi_\omega(r)e^{-i\omega t}.$$

The Laplacian gives

$$\frac{1}{\sqrt{-g}}\left(g^{rr}\sqrt{-g}\Phi'_\omega\right)' + \omega^2\Phi_\omega = 0$$

which is

$$\frac{1}{H_1H_5H_n}\frac{f(r)}{r^3}\left(f(r)r^3\Phi'_\omega\right)' + \omega^2\Phi_\omega = 0. \quad (4.1)$$

In four or five dimensions behaves as $1/r^{3/2}$: $\Phi_\omega(r) = \psi(r)/r^{3/2}$ and we obtain

$$-f(r)\left(f(r)\psi'\right)' + \left(\omega^2 H_1H_5H_n - \frac{3}{4r^2}f(r)(4-3f(r))\right)\psi = 0 \quad (4.2)$$

which is actually a Schrödinger equation. Introduce tortoise coordinates $dr_* = dr/f(r)$ so

$$r_* = \int dr/f(r) = r + \frac{r_+}{2}\ln\left(\frac{r-r_+}{r+r_+}\right)$$

where $r > r_+$. Equation 4.2 becomes

$$-\frac{d^2\psi}{dr_*^2} + V\psi = 0$$

where V is the coefficient of ψ in Equation 4.2. As $r \to 0$, $V \to -\infty$ and, as $r \to \infty$, the H functions go to 1 and $V \to -\omega^2$. Calculating the transmission coefficient for a wave coming from $+\infty$ is a classical problem, not a quantum mechanical one, but we cannot possibly solve it exactly. With the assumption $\rho_1, \rho_5 >> r_+$, we will look for a solution near the horizon and one away from the horizon and patch them together. The intermediate region is given by the harmonic mean $\rho_m = \sqrt{r_+\rho_1}$:

$$\frac{r_+}{\rho_m} = \sqrt{\frac{r_+}{\rho_1}} << 1 \text{ and } \frac{\rho_m}{\rho_1} = \sqrt{\frac{r_+}{\rho_1}} << 1.$$

Far away, space is flat and $r \gtrsim \rho_m$, $\frac{r_+}{r} < \frac{r_+}{\rho_m} << 1$ so $f(r) = 1 - \frac{r_+^2}{r^2} \approx 1$ and r_* becomes r so

$$V = -\omega^2 + \frac{3}{4r^2}$$

which can be solved to give a Bessel function

$$\begin{aligned}\psi &= A\sqrt{r}J_1(\omega r) + B\sqrt{r}N_1(\omega r)\\ \Phi_\omega &= \frac{A}{r}J_1(\omega r) + \frac{B}{r}N_1(\omega r).\end{aligned}$$

If we now assume low frequency $\omega << 1/\rho_m$, then, as $r \to \rho_m$, ωr gets very small, $J_1(\omega r) \to \omega r/2$, and $N_1(\omega r) \to \infty$ so we must set $B = 0$ and we have

$$\Phi_\omega = \frac{A}{2}\omega.$$

Near the horizon, $r \lesssim \rho_m$, use equation 4.1 with $H_1 \approx \rho_1^2/r^2$, $H_5 \approx \rho_5^2/r^2$ to get

$$\frac{f}{r^3}\left(fr^3\Phi'_\omega\right)' + \omega^2\frac{\rho_1^2\rho_5^2}{r^4}\left(1+\frac{\rho_-^2}{r^2}\right)$$

which can be solved to yield

$$\Phi_\omega = A' f(r)^{-i(a+b)/2} F(-ia, -ib; 1-ia-ib; f(r))$$

where F is a hypergeometric function, $a = \omega\rho_1\rho_5/\left(\rho_+ + \rho_-\right)$, and $b = \omega\rho_1\rho_5/\left(\rho_+ - \rho_-\right)$. To match, let $r \to \rho_m$ so $f(r) \approx 1$ and

$$A'\frac{\Gamma(1-ia-ib)}{\Gamma(1-ia)\Gamma(1-ib)} \text{ must match } \frac{A}{2}\omega.$$

We now calculate the incoming flux and transmitted flux.

$$F_{in} = \frac{1}{2i}\left(f(r)r^3\Phi_\omega^*\Phi_\omega + c.c.\right)$$

and $\frac{dF_{in}}{dr} = 0$. Far away we have an incoming wave proportional to $e^{i\omega r}$ and a reflected wave proportional to $e^{-i\omega r}$; as $r \to \infty$,

$$\Phi_\omega = \frac{A}{\sqrt{2\pi\omega}}\left(e^{-3\pi i/4}\frac{e^{i\omega r}}{r^{3/2}} + c.c.\right)$$

so

$$F_{in} = \frac{1}{4\pi}\left|\frac{A}{2}\right|^2.$$

We now want the absorbed flux. Near the horizon ($r \to r_+$) $\Phi_\omega \to A' f(r)^{-i(a+b)/2}$ and

$$F_{abs} = -r_+^2(a+b)\left|A'\right|^2.$$

Then

$$\begin{aligned}
T &= \frac{F_{abs}}{F_{in}} \\
&= -r_+^2(a+b)16\pi\left|\frac{A'}{A}\right|^2 \\
&= -16\pi r_+^2(a+b)\frac{\omega^2}{4}\left|\frac{\Gamma(1-ia)\Gamma(1-ib)}{\Gamma(1-ia-ib)}\right|^2 \\
&\to -(2\pi\omega)^2\frac{abe^{2\pi(a+b)}}{(e^{2\pi a}-1)(e^{2\pi b}-1)}.
\end{aligned}$$

The factor

$$\frac{1}{e^{2\pi a}-1} = \left(e^{2\pi\omega\rho_1\rho_5/(\rho_+ + \rho_-)} - 1\right)^{-1}$$

looks like the Planck (black body) distribution

$$\left(e^{\hbar\omega/kT} - 1\right)^{-1}.$$