LECTURE 5

Microscopic calculation of Entropy and Hawking Radiation

5.1 The hole

We want to understand why entropy is so large. Can it be explained from the action in this example? In five dimensions, the Reissner-Nördstrom black hole has the metric

$$ds^2 = H^{-2}dt^2 + H^2 d\mathbf{x}^2$$

where $H = 1 + Q^2/r^2$ ($r^2 = \mathbf{x}^2$). The 5-brane has metric

$$ds^2 = H^{-1/4}\left(dt^2\right) + d\mathbf{y}^2$$

where H is the same as in the Reissner-Nördstrom black hole and $e^{2\varphi} = H^{-1}$ is singular. Strominger and Vafa looked at

$$\begin{aligned} ds^2 &= H_1^{-3/4} H_5^{-1/4} - dt^2 + d\mathbf{y}_5^2 + (H_n - 1)(dt + d\mathbf{y})^2 \\ &\quad + \left(\frac{H_1}{H_5}\right)^{1/4} d\mathbf{y}^2 + H_1^{1/4} H_5^{3/4} d\mathbf{x}^2 \end{aligned}$$

where

$$H_n = 1 + \frac{\rho_n^2}{r^2} \text{ and } e^{-2\varphi} = \frac{H_5}{H_1},$$

$\mathbf{y}$ and $\mathbf{x}$ are in $\mathbb{R}^4$ and all the y-dimensions are very small.
Consider some special cases.
Case 1: When $\rho_1 = \rho_n = 0$, the metric becomes

$$\begin{aligned} ds^2 &= H_5^{-1/4}\left(-dt^2 + dy_5^2\right) + (H_5)^{-1/4} d\mathbf{y}^2 + H_5^{3/4} d\mathbf{x}^2 \\ &= H_5^{-1/4}\left(-dt^2 + dy_5^2 + d\mathbf{y}^2\right) + H_5^{3/4} d\mathbf{x}^2 \end{aligned}$$

where the y-terms form a 5-dimensional Euclidean space or, if small, a torus (5-brane).
Case 2: When $\rho_5 = \rho_n = 0$, the metric becomes

$$ds^2 = H_1^{-3/4}\left(-dt^2 + dy_5^2\right) + H_1^{1/4}\left(d\mathbf{y}^2 + d\mathbf{x}^2\right)$$

where the $\mathbf{x}$ and $\mathbf{y}$ terms contribute 8 dimensions and the y_5 term is a 1-brane.
Case 3: When $\rho_1 = \rho_5 = 0$, the metric becomes

$$ds^2 = -dt^2 + dy_5^2 + (H_n - 1)\left(dt + dy_5\right)^2 + d\mathbf{y}^2 + d\mathbf{x}^2$$

with no brane.
?? The complete solution is a bound state of a 5-brane from the survival of H_5 and a 1-brane from the survival of H_1. A 5-dimensional observer should perceive a black hole; the y_5-direction will have a momentum proportional to n/L which will be perceived as mass.

$$ds^2 = -H_n^{-1}dt^2 + H_n\left(dy_5 + \left(\frac{1}{H_n} - 1\right)dt\right)^2 + d\mathbf{y}^2 + d\mathbf{x}^2$$

looks like the Kaluza-Klein reduction where

$$ds^2 = ...stuff\left(dx^4 + A_\mu dx^\mu\right)^2$$

and, here we can set

$$A_t = \left(\frac{1}{H_n} - 1\right).$$

??
Now take the complete metric

$$ds^2 = H_1^{-3/4}H_5^{-1/4}H_n\left(dy_5 + A_t dt\right)^2 + \left(\frac{H_1}{H_5}\right)^{1/4}d\mathbf{y}^2 - H_1^{-3/4}H_5^{-1/4}H_n^{-1}dt^2 + H_1^{1/4}H_5^{3/4}d\mathbf{x}^2$$

where $\mathbf{y}$ (y_5) is a small, closed loop in $\mathbb{R}^5$ ($\mathbb{R}$). A 5-dimensional observer sees the last two terms which equal

$$H_1^{-3/4}H_5^{-1/4}H_n^{-1}\left(-dt^2 + H_1 H_5 H_n d\mathbf{x}^2\right)$$

which is similar to a Reissner-Nordström black hole:

$$ds^2 = -\frac{1}{H^2}dt^2 + Hd\mathbf{x}^2 = \frac{1}{H^2}\left(-dt^2 + H^3 d\mathbf{x}^2\right)$$

and, if $\rho_1 = \rho_5 = \rho_n$, it is exactly the same. We could set $H = (H_1 H_5 H_n)^{1/3}$ so the difference is that a RN black hole as one charge while the general case has three.

5.2 Entropy

To get the entropy, we need the area $A = 2\pi^2 r_+^3$ where, ?? in the RN case ($Hd\mathbf{x}^2 = Hr^2 d\Omega^2$), $r_+ = \lim_{r\to 0}\sqrt{Hr^2}$. ?? Therefore

$$A = 2\pi^2 \lim_{r\to 0}\left(\left(\frac{\rho_1^2}{r^2}\frac{\rho_5^2}{r^2}\frac{\rho_n^2}{r^2}\right)^{1/6} r\right)^3 = 2\pi^2\left(\rho_1\rho_5\rho_n\right)$$

Mass for a Schrödinger black hole, is deduced from $g_{tt} = -\left(1 - \frac{2GM}{r}\right)$. In the general case of d dimensions $g_{tt} \approx -1 + 2V$ as $r \to \infty$ where V is the potential energy; in fact

$$g_{tt} = -1 + \frac{\mu}{r^{d-3}}, \mu = \frac{8\pi GM}{(d-2)\Omega_{d-2}}, \text{ and } \Omega_{d-1} = \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)}.$$

For 5 dimensions, $G_5 M = 3\pi\mu/4$ and $g_{tt} = -\left(H_1 H_5 H_n\right)^{-2/3}$; expanding in $1/r$, we get

$$g_{tt} = -1 + \frac{2}{3}\frac{\rho_1^2 + \rho_5^2 + \rho_n^2}{r^2} + ...$$

so

$$G_5 M = \frac{\pi}{2}\left(\rho_1^2 + \rho_5^2 + \rho_n^2\right).$$

Mass splits into three pieces (with no binding energy) due to the 1-brane, the 5-brane, and the momentum.
Let $\rho_n = 0$ and look at the metric near the horizon ?? $r \to 0$??

$$\begin{aligned} ds^2 &\approx \frac{r^2}{\sqrt{\rho_1^3\rho_5}}\left(-dt^2 + dy_5^2\right) + \sqrt{\frac{\rho_1}{\rho_5}}d\mathbf{y}^2 + \frac{\sqrt{\rho_1\rho_5^3}}{r^2}\left(dr^2 + r^2 d\Omega_3^2\right) \\ &= \left(\frac{r^2}{\sqrt{\rho_1^3\rho_5}}\left(-dt^2 + dy_5^2\right) + \sqrt{\rho_1\rho_5^3}\frac{dr^2}{r^2}\right) + \sqrt{\frac{\rho_1}{\rho_5}}d\mathbf{y}^2 + \sqrt{\rho_1\rho_5^3}d\Omega_3^2 \end{aligned}$$

where the terms in parenthesis represent AdS_3, the next term is a 4-torus, T^4, and the last term is just the 3-sphere, S^3.

?? Can we understand entropy in this model? The 1-brane can be taken to be wrapped tightly N_1 times, the 5-brane N_5 times, no momentum, and only one state so $S = 0$. This suggests entropy comes from momentum. String theory tells us that excitations of branes arise from the movement of strings attached to branes. Branes have no binding energy and they are points (no deeper structure). We can set the coupling constant to one. The only parameter that is available to determine energy is the length of the string ℓ_s; thus the unit of energy is $1/\ell_s$. If the 1-brane wraps N_1 times around a length L_1, it must have $N_1 L_1/\ell_s$ elementary units so its energy is $N_1 L_1/\ell_s^2$. $M_1 = N_1 L_5/2\pi\ell_s^2$. In 4-dimensions, $G \propto \ell^2$ so we expect $G_{10} \propto \ell_s^8$, in fact $G^{10} = 16\pi^6\ell_s^8$. The equation

$G_5 M = \frac{\pi}{2}\left(\rho_1^2 + \rho_5^2 + \rho_n^2\right)$ results from gravity alone, not string theory. Now

$$\begin{aligned} G_5 M_1 &= \frac{\pi}{2}\rho_1^2 \\ \frac{16\pi^6 \ell_s^8}{L_1 ... L_5}\frac{N_1 L_5}{2\pi \ell_s^2} &= \frac{\pi}{2}\rho_1^2 \\ \rho_1^2 &= \frac{(2\pi)^4}{L_1 ... L_4}\ell_s^6 N_1. \end{aligned}$$

Similarly the 5-brane

$$\begin{aligned} \frac{N_5 L_1 ... L_5}{\ell_s^2}\frac{1}{\ell_s} &= 16\pi^5 M_5 \\ \rho_5 &= N_5 \ell_s^2. \end{aligned}$$

The momentum in the y_5 direction $p_5 = 2\pi n / L_5$, $n = 0, 1, 2, ...$ is the last piece of the puzzle :

$$\rho_n^2 = \frac{(2\pi)^6 \ell_s^8}{L_1 ... L_4 L_5^2} n$$

and, putting it all together,

$$\begin{aligned} S &= 2\pi^2 \rho_1 \rho_5 \rho_n \\ &= 2\pi\sqrt{N_1 N_2 n}. \end{aligned}$$

This is one success of string theory. ??

We now intend to count the degrees of freedom of this system. Note that strings can begin and end on any brane, but only strings between the 1-brane and the 5-brane contribute. For instance, when $N_1 = 2$ and $N_5 = 2$, the string will return to its starting point after going around twice, but, if $N_1 = 2$ and $N_5 = 3$, the string will return to its starting point after going around six times. In general, if N_1 and N_5 are relatively prime, the string goes around $N_1 N_5$ times. If both numbers are very large, we can ignore the relatively prime requirement since they will be close to a relatively prime pair. Then $L_{eff} = N_1 N_5 L_5$. Then

$$p_5 = \frac{2\pi n}{L_5} = \frac{2\pi n}{L_{eff}}\frac{L_{eff}}{L_5} = N_1 N_5 n \frac{2\pi}{L_{eff}}$$

and $2\pi / L_{eff}$ can be considered the quantum of momentum. Given a momentum, we want to count all of the ways of getting that momentum to get the number of states. We do so in a physical way. Suppose the quantum of energy is E_0 and n_i particles have energy iE_0 at temperature T so the total energy is $E = n_1 E_0 + n_2 2E_0 + n_3 3E_0 + \ ....$

Define the partition function

$$
\begin{aligned}
Z &= \sum_E e^{-E/T} \\
&= \sum_{n_i} e^{-(n_1 E_0 + n_2 2E_0 + n_3 3E_0 + \ ...)/T} \\
&= \left(\sum_{n_1=0}^{\infty} e^{-n_1 E_0/T} \right) \left(\sum_{n_2=0}^{\infty} e^{-n_2 2E_0/T} \right) ... \\
&= \frac{1}{1-e^{-E_0/T}} \frac{1}{1-e^{-2E_0/T}} ...
\end{aligned}
$$

The free energy is given by $F = -T \ln Z = T \sum_N \ln\left(1 - e^{-NE_0/T}\right)$. Take the high temperature limit $T \to \infty$ and let $x = NE_0/T$ and $\Delta x = E_0/T \to 0$. Then

$$
\begin{aligned}
F &= E_0 \frac{T}{E_0} \sum \ln\left(1 - e^{-x}\right) \\
&= \frac{T^2}{E_0} \frac{E_0}{T} \sum \ln\left(1 - e^{-x}\right) \\
&= \frac{T^2}{E_0} \Delta x \sum \ln\left(1 - e^{-x}\right) \\
&\approx \frac{T^2}{E_0} \int \ln\left(1 - e^{-x}\right) dx \\
&= \frac{T^2}{E_0} \frac{\pi^2}{6}.
\end{aligned}
$$

The entropy is $S = -\frac{\partial F}{\partial T} = \frac{\pi^2}{3} \frac{T}{E_0}$ and the internal energy is $E = -T^2 \frac{\partial}{\partial T}\left(\frac{F}{T}\right) = \frac{\pi^2}{6} \frac{T^2}{E_0} = nE_0$ so $\frac{T^2}{E_0} = \frac{6n}{\pi^2}$ which we substitute into the entropy

$$
S = \frac{\sqrt{6n}}{\pi} \frac{\pi^2}{3} = 2\pi \sqrt{\frac{n}{6}} = 2\pi \sqrt{\frac{N_1 N_5 n}{6}}.
$$

This is not the whole story because string theory says there are four different strings so we should multiply by 4 under the radical; furthermore, supersymmetry gives fermions for which n_i must be either 0 or 1 so $S_{fermion} = 2\pi \sqrt{\frac{n}{12}}$. Therefore

$$
S = 2\pi \sqrt{\frac{N_1 N_5 n}{6} \left(4 + \frac{4}{2}\right)} = 2\pi \sqrt{N_1 N_5 n}.
$$

We have come to the same conclusion independently of the calculation using the horizon area.

5.3 Finite temperature

Suppose T is slightly greater than zero. To heat the system, we could introduce an anti-brane or let the strings run in both directions. Let the number of left (right) moving strings be n_+ (n_-). A 5-dimensional observer would see a charge $Q \propto n_+ - n_-$ ($M_n \propto n_+ + n_-$). The metric is

$$ds^2 = H_1^{-3/4} H_5^{-1/4} \left(-dt^2 + dy_5^2 + \frac{1}{r^2} \left(\rho_+ dt + \rho_- dy_5 \right)^2 \right) + \left(\frac{H_1}{H_5} \right)^{1/4} d\mathbf{y}^2 + H_1^{1/4} H_5^{3/4} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \right)$$

where ρ splits into ρ_+ and ρ_+ and $f(r) = 1 - \frac{r_+^2}{r^2}$ so there are two horizons at 0 and r_+. $\rho_+^2 - \rho_-^2 = r_+^2$ so, when $\rho_+ = \rho_-$, $r_+ = 0$ and we are back to the case $T = 0$. Let us consider physical properties where

$$ds_5^2 = - \left(H_1 H_5 H_n \right)^{-2/3} f dt^2 + \left(H_1 H_5 H_n \right)^{1/3} \left(\frac{dr^2}{f} + r^2 d\Omega_3^2 \right)$$

with $H_n = 1 + \rho_-^2 / r^2$. The area of the horizon is

$$A_+ = 2\pi^2 r_+^3 \sqrt{H_1 H_5 H_n} \,\Big|_{r=r_+} .$$

Assume $\rho_1, \ \rho_5 >> r_+$, then

$$A_+ = 2\pi^2 r_+^3 \sqrt{\frac{\rho_1^2 \rho_5^2}{r_+^4} \frac{r_+^2 + \rho_-^2}{r_+^2}} = 2\pi \rho_1 \rho_5 \rho_+ .$$

Expand the metric to get

$$g_{tt} = - \left(H_1 H_5 H_n \right)^{-2/3} f = -1 + \frac{2}{3} \frac{\rho_1^2 + \rho_5^2 + \rho_-^2}{r^2} + \frac{r_+^2}{r^2} + \ ....$$

The $1/r^2$ comprise μ / r^2 where in classical gravity $G_5 M = \frac{3\pi}{4} \mu$; therefore

$$\begin{aligned} G_5 M &= \frac{\pi}{2} \left(\rho_1^2 + \rho_5^2 + \rho_-^2 + \frac{3}{2} r_+^2 \right) \\ &= G_5 \left(M_1 + M_5 + M_n \right) . \end{aligned}$$

and

$$\begin{aligned} G_5 M_1 &= \frac{\pi}{2} \left(\rho_1^2 + \frac{1}{2} r_+^2 \right) \approx \frac{\pi}{2} \rho_1^2 \\ G_5 M_5 &= \frac{\pi}{2} \left(\rho_5^2 + \frac{1}{2} r_+^2 \right) \approx \frac{\pi}{2} \rho_5^2 \\ G_5 M_n &= \frac{\pi}{2} \left(\rho_-^2 + \frac{1}{2} r_+^2 \right) \approx \frac{\pi}{4} \left(\rho_+^2 + \rho_-^2 \right) . \end{aligned}$$

We now have

$$\begin{aligned} n_+ + n_- &\propto M_N \propto \rho_+^2 + \rho_-^2 \\ n_+ - n_- &\propto Q \propto 2 \rho_+ \rho_- \end{aligned}$$

so

$$\begin{aligned} n_+ &\propto (\rho_+ + \rho_-)^2 \\ n_- &\propto (\rho_+ - \rho_-)^2 . \end{aligned}$$

In terms of $n_\pm$, $S \propto \rho_+ \propto \sqrt{n_+} + \sqrt{n_-}$ or

$$S = 2\pi\sqrt{N_1 N_5}\left(\sqrt{n_+} + \sqrt{n_-}\right)$$

which is expected since the right and left moving modes are completely independent: $S_+ = 2\pi\sqrt{N_1 N_5 n_+}$, $S_- = 2\pi\sqrt{N_1 N_5 n_-}$, $S = S_+ + S_-$. With two independent quantum systems, our counting agrees with classical mechanics.
To count the degrees of freedom, we want to look near the horizon r_+, but carefully because r_+ is very small. When r is small the H functions can be approximated by ρ^2/r^2. As before the metric becomes

$$\begin{aligned} ds^2 &= \frac{r^2}{\sqrt{\rho_1^3\rho_5}}\left(-dt^2 + dy_5^2 + \frac{1}{r^2}\left(\rho_+ dt + \rho_- dy_5\right)^2\right) + \sqrt{\frac{\rho_1}{\rho_5}}d\mathbf{y}^2 + \frac{\sqrt{\rho_1\rho_5^3}}{r^2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_3^2\right) \\ &= \frac{r^2}{\sqrt{\rho_1^3\rho_5}}\left(-dt^2 + dy_5^2\right) + \frac{1}{\sqrt{\rho_1^3\rho_5}}\left(\rho_+ dt + \rho_- dy_5\right)^2 + \sqrt{\rho_1\rho_5^3}\frac{dr^2}{r^2 - r_+^2} + \sqrt{\rho_1\rho_5^3}d\Omega_3^2 + \sqrt{\frac{\rho_1}{\rho_5}}d\mathbf{y}^2 . \end{aligned}$$

The last two terms are the sphere S^3 and the torus T^4. The other terms form a three dimensional manifold M which was AdS_3 when T was zero. To study the metric ds_3^2 of M, set $r'^2 = r^2 + \rho_-^2$ to get ??

$$ds_3^2 = \frac{1}{\sqrt{\rho_1^3\rho_5}}\left(-\frac{\left(r'^2 - \rho_-^2\right)\left(r'^2 - \rho_+^2\right)}{r'^2}dt^2 + \rho_1^2\rho_5^2\frac{r'^2 dr'^2}{\left(r'^2 - \rho_-^2\right)\left(r'^2 - \rho_+^2\right)} + r'^2\left(dy_5 + \frac{\rho_+ + \rho_-}{r'^2}dt\right)^2\right).$$

?? This is Banados-Teitelboim-Zanelli or BTZ black hole. If $r' \to \infty$, the large parenthetical term reduces to $-r'^2 dt^2 + \rho_1^2\rho_5^2 dr'^2/r'^2 + r'^2 dy_5^2$ which is the metric of AdS_3. This is unusual because, far from the horizon, we do not get flat space; in this case going to infinity means going as far away as possible without leaving the horizon. Being near the horizon introduces a cosmological constant to get negative curvature.

5.4 Scattering and Hawking radiation

What do we mean by going to infinity and not leaving the horizon? Look at radiation thrown at the black hole. Introduce a scalar field φ which is massless and has minimal coupling (no term coupling with curvature); the action

$$S = \frac{1}{2}\int d^5\mathbf{x}\sqrt{-g}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}$$

give the Klein-Gordon field equation $\Box\varphi = 0$ which we want to solve for scattering. Use a spherically symmetric 5-dimensional wave not involving angles

$$\varphi(r,t) = \Phi_\omega(r)e^{-i\omega t}.$$

The Laplacian gives

$$\frac{1}{\sqrt{-g}}\left(g^{rr}\sqrt{-g}\Phi'_\omega\right)' + \omega^2\Phi_\omega = 0$$

which is

$$\frac{1}{H_1H_5H_n}\frac{f(r)}{r^3}\left(f(r)r^3\Phi'_\omega\right)' + \omega^2\Phi_\omega = 0. \tag{4.1}$$

In four or five dimensions behaves as $1/r^{3/2}$: $\Phi_\omega(r) = \psi(r)/r^{3/2}$ and we obtain

$$-f(r)\left(f(r)\psi'\right)' + \left(\omega^2H_1H_5H_n - \frac{3}{4r^2}f(r)(4-3f(r))\right)\psi = 0 \tag{4.2}$$

which is actually a Schrödinger equation. Introduce tortoise coordinates $dr_* = dr/f(r)$ so

$$r_* = \int dr/f(r) = r + \frac{r_+}{2}\ln\left(\frac{r-r_+}{r+r_+}\right)$$

where $r > r_+$. Equation 4.2 becomes

$$-\frac{d^2\psi}{dr_*^2} + V\psi = 0$$

where V is the coefficient of ψ in Equation 4.2. As $r \to 0$, $V \to -\infty$ and, as $r \to \infty$, the H functions go to 1 and $V \to -\omega^2$. Calculating the transmission coefficient for a wave coming from $+\infty$ is a classical problem, not a quantum mechanical one, but we cannot possibly solve it exactly. With the assumption $\rho_1, \rho_5 >> r_+$, we will look for a solution near the horizon and one away from the horizon and patch them together. The intermediate region is given by the harmonic mean $\rho_m = \sqrt{r_+\rho_1}$:

$$\frac{r_+}{\rho_m} = \sqrt{\frac{r_+}{\rho_1}} << 1 \text{ and } \frac{\rho_m}{\rho_1} = \sqrt{\frac{r_+}{\rho_1}} << 1.$$

Far away, space is flat and $r \gtrsim \rho_m$, $\frac{r_+}{r} < \frac{r_+}{\rho_m} << 1$ so $f(r) = 1 - \frac{r_+^2}{r^2} \approx 1$ and r_* becomes r so

$$V = -\omega^2 + \frac{3}{4r^2}$$

which can be solved to give a Bessel function

$$\begin{aligned}\psi &= A\sqrt{r}J_1(\omega r) + B\sqrt{r}N_1(\omega r)\\ \Phi_\omega &= \frac{A}{r}J_1(\omega r) + \frac{B}{r}N_1(\omega r).\end{aligned}$$

If we now assume low frequency $\omega << 1/\rho_m$, then, as $r \to \rho_m$, ωr gets very small, $J_1(\omega r) \to \omega r/2$, and $N_1(\omega r) \to \infty$ so we must set $B = 0$ and we have

$$\Phi_\omega = \frac{A}{2}\omega.$$

Near the horizon, $r \lesssim \rho_m$, use equation 4.1 with $H_1 \approx \rho_1^2/r^2$, $H_5 \approx \rho_5^2/r^2$ to get

$$\frac{f}{r^3}\left(fr^3\Phi_\omega'\right)' + \omega^2\frac{\rho_1^2\rho_5^2}{r^4}\left(1+\frac{\rho_-^2}{r^2}\right)$$

which can be solved to yield

$$\Phi_\omega = A' f(r)^{-i(a+b)/2}F(-ia,-ib;1-ia-ib;f(r))$$

where F is a hypergeometric function, $a = \omega\rho_1\rho_5/\left(\rho_+ + \rho_-\right)$, and $b = \omega\rho_1\rho_5/\left(\rho_+ - \rho_-\right)$. To match, let $r \to \rho_m$ so $f(r) \approx 1$ and

$$A'\frac{\Gamma(1-ia-ib)}{\Gamma(1-ia)\,\Gamma(1-ib)} \text{ must match } \frac{A}{2}\omega.$$

We now calculate the incoming flux and transmitted flux.

$$F_{in} = \frac{1}{2i}\left(f(r)r^3\Phi_\omega^*\Phi_\omega + c.c.\right)$$

and $\frac{dF_{in}}{dr} = 0$. Far away we have an incoming wave proportional to $e^{i\omega r}$ and a reflected wave proportional to $e^{-i\omega r}$; as $r \to \infty$,

$$\Phi_\omega = \frac{A}{\sqrt{2\pi\omega}}\left(e^{-3\pi i/4}\frac{e^{i\omega r}}{r^{3/2}} + c.c.\right)$$

so

$$F_{in} = \frac{1}{4\pi}\left|\frac{A}{2}\right|^2.$$

We now want the absorbed flux. Near the horizon ($r \to r_+$) $\Phi_\omega \to A' f(r)^{-i(a+b)/2}$ and

$$F_{abs} = -r_+^2(a+b)\left|A'\right|^2.$$

Then

$$\begin{aligned} T &= \frac{F_{abs}}{F_{in}} \\ &= -r_+^2(a+b)16\pi\left|\frac{A'}{A}\right|^2 \\ &= -16\pi r_+^2(a+b)\frac{\omega^2}{4}\left|\frac{\Gamma(1-ia)\,\Gamma(1-ib)}{\Gamma(1-ia-ib)}\right|^2 \\ &\to -(2\pi\omega)^2\frac{abe^{2\pi(a+b)}}{(e^{2\pi a}-1)(e^{2\pi b}-1)}. \end{aligned}$$

The factor

$$\frac{1}{e^{2\pi a}-1} = \left(e^{2\pi\omega\rho_1\rho_5/(\rho_+ + \rho_-)} - 1\right)^{-1}$$

looks like the Planck (black body) distribution

$$\left(e^{\hbar\omega/kT} - 1\right)^{-1}.$$