LECTURE 3

Reissner-Nördstrom Black Holes

3.1 The holes

Let us combine gravity with electromagnetism to find a charged black hole. The action is

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + \frac{1}{8\pi G}\int d^3x\sqrt{h}K - \frac{1}{16\pi}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$$

so there is now a source (electromagnetic field energy corresponds to mass). We have

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$

where (from electromagnetism)

$$T_{\mu\nu} = \frac{1}{4\pi}\left(g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

which has trace zero so there is no scale: no mass, no distance, no time, just photons. $R = 0$, just as in the Schwarzschild black hole, but we now have the Maxwell equations with no charges,

$$\nabla_\mu F^{\mu\nu} = 0$$

The most symmetric solution is the 4-vector with time component $A_0 = \frac{Q}{r}$; however, we want the potential to be zero at the horizon so we set

$$A_0 = \frac{Q}{r} - \frac{Q}{r_+}.$$

and

$$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$$

This is the Reissner-Nördstrom black hole. The horizon r_+ ($0 = f(r_+)$) is the solution to

$$r^2 - 2GMr + GQ^2 = 0$$

so

$$r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}$$

(r_- is inside r_+ so we cannot observe it). When $Q = 0$, we get $r_+ = 2GM$ as expected. The minimal r_+ occurs when $G^2M^2 - GQ^2 = 0$ so there is a minimal mass

$$M_{\min} = \frac{Q^2}{G}$$

Below this mass, we would have a naked singularity, but we stick with dressed singularities. The temperature is given by

$$\begin{aligned} T &= \frac{1}{4\pi} f'(r_+) \\ &= \frac{1}{4\pi}\left(\frac{2GM}{r_+^2} - \frac{2GQ^2}{r_+^3}\right) \\ &= \frac{1}{4\pi}\left(\frac{1}{r_+} - \frac{GQ^2}{r_+^3}\right) \end{aligned}$$

The entropy is

$$S = \frac{A_+}{4G} = \frac{\pi r_+^2}{G}.$$

The law of the thermodynamics reads

$$dM = TdS + \mu dQ$$

where $\mu = \frac{Q}{Gr_+}$ plays the role of a chemical potential and Q, in effect, counts the number of charges. Differentiating,

$$\begin{aligned} \left(\frac{\partial M}{\partial r_+}\right)_Q &= T\left(\frac{\partial S}{\partial r_+}\right)_Q \\ \left(\frac{\partial M}{\partial Q}\right)_{r_+} &= \frac{Q}{Gr_+} \end{aligned}$$

This seems to be nicer than a Schwarzschild black hole. In the extremal case, $r_+ = GM = \sqrt{G}Q$,

$$\begin{aligned} T &= \frac{1}{4\pi}\left(\frac{1}{r_+} - \frac{(GM)^2}{r_+^3}\right) = 0 \\ S &= \pi Q^2 \neq 0 \end{aligned}$$

violating the third law of thermodynamics.

Look at the classical action

$$S_{cl} = \frac{M}{2T} - \frac{Q^2}{2Tr_+^2}$$

and

$$\begin{aligned} \frac{F}{T} &= \frac{U}{T} - S \\ &= \frac{M}{T} - \frac{\pi r_+^2}{G} \end{aligned}$$

which is not equal to the classical action.

$$\begin{aligned} S_{cl} - \frac{F}{T} &= -\frac{M}{2T} + \frac{\pi r_+^2}{G} - \frac{Q^2}{2Tr_+^2} \\ &= -\frac{Q^2}{Tr_+} \\ &= -\frac{\mu Q}{T} \end{aligned}$$

so

$$TS_{cl} = F - \mu Q$$

is the Gibbs free energy. Two different calculations have given the same answer.

3.2 Extremal limit

Look at the extremal limit more closely

$$\begin{aligned} f(r) &= 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \\ &= 1 - \frac{2r_+}{r} + \frac{r_+^2}{r^2} \\ &= \left(1 - \frac{r_+}{r}\right)^2 \end{aligned}$$

If we change coordinates to $r' = r - r_+$,

$$f = \left(\frac{r'}{r' + r_+}\right)^2 = H^{-2}(r')$$

where $H(r') = 1 + r_+/r'$ is an harmonic function ($\nabla^2 H = 0$).

$$\begin{aligned} ds^2 &= -H^{-2}dt^2 + H^2 dr'^2 + (r' + r_+)^2 \, d\Omega_2^2 \\ &= -H^{-2}dt^2 + H^2 d\mathbf{x}^2 \, , \ \ \mathbf{x} \in \mathbb{R}^3 \end{aligned}$$

The Coulomb potential

$$A = \frac{Q}{r'} - \frac{Q}{r_+} = -\frac{Q}{Hr_+}$$

As we get near the horizon $r' \to 0$ and

$$ds^2 \approx -\left(\frac{r'}{r_+}\right)^2 dt^2 + \left(\frac{r_+}{r'}\right)^2 dr'^2 + r_+^2 d\Omega^2$$

so the black hole is the product of a 2-dimensional manifold and the 2-sphere; we shall show that the manifold is anti-de Sitter space (AdS).
Recall that in the case of a Schwarzschild black hole, after setting $\tau = it$, we obtained a cone. To get rid of the singularity, we required periodicity under $\tau \to \tau + \frac{1}{T}$ which determined the temperature T. Here, if we set $\tau = it/r_+^2$, we obtain

$$ds_2^2 = r_+^2 \left(r'^{\,2} d\tau^2 + \frac{1}{r'^{\,2}} dr'^{\,2}\right).$$

To see what manifold this is, in 3-dimensional Minkowski space with metric

$$ds_3^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2$$

consider a hyperboloid surface defined by

$$\left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 = r_+^2$$

If we change coordinates to (r_+, r', τ) with

$$\begin{aligned} x^0 &= r_+\left(\frac{1}{2r'} + \frac{r'}{2}\left(1+\tau^2\right)\right) \\ x^1 &= r_+\left(\frac{1}{2r'} - \frac{r'}{2}\left(1-\tau^2\right)\right) \\ x^2 &= r_+ r' \tau \end{aligned}$$

for a fixed r_+, we obtain the metric on the hyperboloid,

$$ds_2^2 = r_+^2 \left(r'^{\,2} d\tau^2 + \frac{1}{r'^{\,2}} dr'^{\,2}\right),$$

which is of the desired form (Euclidean anti-de Sitter space ($EAdS_2$)). There is no restriction on τ, which is consistent with the fact that $T = 0$. This surface has three isometries:

$$\begin{aligned} M_1 &= i\left(x^0\partial_1 - x^1\partial_0\right), \text{ a boost along the } x\text{-axis} \\ M_2 &= i\left(x^0\partial_2 - x^2\partial_0\right), \text{ a boost along the } x\text{-axis} \\ L &= i\left(x^1\partial_2 - x^2\partial_1\right), \text{ angular momentum.} \end{aligned}$$

satisfying the Lorentz algebra

$$\begin{aligned} [L, M_1] &= -iM_2 \\ [L, M_2] &= iM_1 \\ [M_1, M_2] &= iL. \end{aligned}$$

In terms of (r', τ),

$$\begin{aligned}
M_1 &= i\left(-r'\partial_{r'} + \tau\partial_\tau\right) \\
M_2 &= i\left(r'\tau\partial_{r'} + \frac{1}{2}\left(1 + \frac{1}{r'^{\,2}} - \tau^2\right)\partial_\tau\right) \\
L &= i\left(r'\tau\partial_{r'} + \frac{1}{2}\left(-1 + \frac{1}{r'^{\,2}} - \tau^2\right)\partial_\tau\right)
\end{aligned}$$

As we approach the boundary ($r\prime \to \infty$) staying on a $r' =$ const. slice (so that $\partial_{r'} \to 0$) we have

$$\begin{aligned}
M_1 &\to i\tau\partial_\tau \\
M_2 &\to \frac{i}{2}\left(1 - \tau^2\right)\partial_\tau \\
L &\to -\frac{i}{2}\left(1 + \tau^2\right)\partial_\tau
\end{aligned}$$

Define

$$D = M_1 = i\tau\partial_\tau\ , \quad H = M_2 - L = i\partial_\tau\ , \quad K = -M_2 - L = i\tau^2\partial_\tau$$

Evidently,

$$[D\ ,\ H] = -iH\ , \quad [D\ ,\ K] = iK\ , \quad [H\ ,\ K] = 2iD$$

which is the conformal algebra in one dimension. D generates dilations, H is the Hamiltonian (generates translations in τ) and K generates special conformal transformations.

Exercise: Show that for an operator $\mathcal{O}(\tau)$,

$$e^{-iaH}\mathcal{O}(\tau)e^{iaH} = \mathcal{O}(\tau + a)\ , \quad e^{-iaD}\mathcal{O}(\tau)e^{iaD} = \mathcal{O}(e^a\tau)$$

and find the corresponding transformation for K.

Thus the isometries of AdS space turned into the generators of the conformal group. Starting with a black hole, we have arrived at a conformal quantum mechanical system; this is the first hint at an AdS/CFT correspondence.

3.3 Generalizations

3.3.1 Multi-center solution

Recall that in the extremal limit,

$$ds^2 = -H^{-2}dt^2 + H^2 d\mathbf{x}^2 \quad (\mathbf{x} \in \mathbb{R}^3)\ , \quad A_t = -H^{-1}$$

and $H = 1 + \frac{r_+}{|\mathbf{x}|}$, an harmonic function. You may guess that the Einstein-Maxwell equations lead to $\nabla^2 H = 0$, so H can be any harmonic function, in particular a multi-center solution,

$$H = 1 + \frac{q_1}{|\mathbf{x} - \mathbf{x}_1|} + \frac{q_2}{|\mathbf{x} - \mathbf{x}_2|} + \dots$$

consisting of the Coulomb potentials due to charges q_i at positions $\mathbf{x}_i$ ($i = 1, 2, \dots$). The fist term equal to 1 is there for correct asymptotics. All these black holes are in equilibrium because the electric and gravitational forces cancel each other (due to $G^2 M_i M_j = G Q_i Q_j$, where M_i is the mass and $Q_i = q_i/\sqrt{G}$ is the charge of the ith black hole).

3.3.2 Arbitrary dimension

In dimensions $d \geq 4$, a Reissner-Nördstrom black hole has metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \; , \quad f(r) = 1 - \frac{2\mu}{r^{d-3}} + \frac{q}{r^{2(d-3)}}$$

where

$$\mu = \frac{8\pi G M}{(d-2)\Omega_{d-2}} \text{ and} \; , \quad q = \frac{8\pi G Q^2}{(d-3)(d-2)\Omega_{d-2}} \; , \quad \Omega_{d-2} = \frac{2\pi}{\Gamma\left(\frac{d-1}{2}\right)}$$

$\Omega + d - 2$ is the volume of a sphere in dimension $d-2$ (S^{d-2}).
The horizon is at r_+, where

$$r_\pm^{d-3} = \mu \pm \sqrt{\mu^2 - q}, \, \mu^2 \geq q$$

the temperature is

$$T = \frac{f\,'(r_+)}{4\pi} = \frac{d-3}{4\pi r_+}\left(1 - \left(\frac{r_-}{r_+}\right)^{d-3}\right) .$$

and the entropy is

$$S = \frac{A_+}{4G} = \frac{\Omega_{d-2} r_+^{d-2}}{4G}$$

At extremality, $\mu^2 = q$, so $r_+ = r_-$ and $T = 0$. The entropy is $S \propto Q^{\frac{d-2}{d-3}}$. For the metric, notice that

$$f(r) = \left[1 - \left(\frac{r_+}{r}\right)^{d-3}\right]^2$$

Changing coordinates to

$$r'^{d-3} = r^{d-3} - r_+^{d-3}$$

so that the horizon is at $r' = 0$, we obtain

$$\begin{aligned} ds^2 &= -H^{-2}dt^2 + H^{\frac{2}{d-3}} d\mathbf{x}^2 \; , \quad \mathbf{x} \in \mathbb{R}^{d-1} \\ H &= 1 + \left(\frac{r_+}{|\mathbf{x}|}\right)^{d-3} \end{aligned}$$

H is an harmonic function.

3.3.3 Kaluza-Klein reduction

Interestingly, the four-dimensional Reissner-Nördstrom black hole (in fact, all of electromagnetism) can also be derived from pure gravity in five dimensions. The action is

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g^{(5)}} R^{(5)}$$

The space is spanned by coordinates x^M ($M = 0, 1, 2, 3, 4$). If x^4 is a loop of length L (compactified) and there is no dependence on x^4 (which is true if we are interested in distances large compared to L), then

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g^{(5)}} R^{(5)} \ , \ \ \frac{1}{G} = \frac{L}{G_5}$$

where G is Newton's constant in our four-dimensional world.
Let us parametrize the metric as

$$ds^2 \equiv g^{(5)}_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + e^{2\Phi} \left(dx^4 + A_\mu dx^\mu \right)^2$$

Then the Ricci scalar reads

$$R^{(5)} = R - 2e^{-\Phi} \nabla^2 e^{\Phi} - \frac{1}{4} e^{2\Phi} F_{\mu\nu} F^{\mu\nu}$$

where R is the Ricci scalar in our four-dimensional world.
The action becomes

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{4} e^{2\Phi} F_{\mu\nu} F^{\mu\nu} \right) .$$

consisting of a gravitational field $g_{\mu\nu}$, the dilaton Φ and a photon A_μ. Notice that all three fields are massless.
Consider a particle of the dilaton field. SInce it is massless, $p^2 = p^M p_M = 0$. Because the fifth dimension has a finite length L, p_4 must be quantized: $e^{ip_4 x^4} = e^{ip_4 (x^4 + L)}$ so $p_4 L = 2\pi n$ for some integer n. Then in the four dimensions we can observe, the particle appears to have mass m with

$$m^2 = p_\mu p^\mu = p_4^2 = \left(\frac{2\pi}{L} \right)^2 n^2$$

so $m = 0, 2\pi/L, 4\pi/L, \dots$, the Kaluza-Klein tower. These masses are, in principle, observable, but we have not observed them. If $L \ll 1$, these masses would be very large and, therefore, may not be observable at present (or ever).
To get the Reissner-Nördstrom black hole, set $\Phi = 0$. Then the action reduces to one with gravity and electromagnetism which is exactly where we got the Reissner-Nördstrom black hole earlier.
Additionally, there exists a solution with $\Phi \neq 0$. We obtain

$$ds^2 = -\frac{f(r)}{H^{1/8}} dt^2 + H^{7/8} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \right) \ , \ \ f(r) = 1 - \frac{r_+}{r} \ , \ \ H = 1 + \frac{\rho}{r}$$

$e^{-2\Phi} = H^{3/2}$, and $A_t = H^{-1}$.
In the extremal limit $r_+ \to 0$

$$ds^2 = -H^{-1/8}dt^2 + H^{7/8}d\mathbf{x}^2 \ , \ \ \mathbf{x} \in \mathbb{R}^3 \ .$$

The singularity at $r = |\mathbf{x}| = 0$ has vanishing area, therefore the entropy $S = 0$.