LECTURE 2

Schwarzschild black hole

Spacetime is provided with a metric tensor $g_{\mu\nu}$ so that a line element has length

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

In flat spacetime, $ds^2 = -dt^2 + d\mathbf{x}^2$ ($\mathbf{x} \in \mathbb{R}^3$), so $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1\ 1\ 1\ 1)$ as a matrix. We denote the determinant of $g_{\mu\nu}$ by g. The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \begin{cases} 0 & , \quad \text{with just gravity not matter} \\ source & , \quad \text{in the presence of matter} \end{cases} .$$

$R_{\mu\nu}$ is the Ricci tensor (the contracted curvature tensor $R^\nu_{\mu\nu\rho}$) and $R = R^\mu_\mu$ (its trace) is the Ricci scalar. $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$ arises from the action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R.$$

If we take the trace of the Einstein equation in empty space, we get

$$R^\mu_\mu - \frac{1}{2}Rg^\mu_\mu = 0$$

which implies $R = 0$ so $R_{\mu\nu} = 0$. Schwarzschild found the solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the sphere S^2 and

$$f(r) = 1 - \frac{2GM}{r}$$

Where do G and M come from?
Compare to electromagnetism with Maxwell's equations without currents

$$\partial_\mu F^{\mu\nu} = 0.$$

Its simplest symmetric solution is $A_\mu = (A_0, \mathbf{0})$ with $\nabla^2 A_0 = 0$ so $A = \frac{Q}{r}$, i.e. the Coulomb potential. Q turns out to be the charge by Gauss's law.
In General Relativity, we can read off the mass from

$$g_{tt} \approx -1 + 2V(r)$$

if $V(r)$ is small (i.e., when $r \to \infty$), where $V(r)$ is the Newtonian potential ($V(r) = GM/r$).
In the Schwarzschild solution $f(r)$ diverges as $r \to 0$ which is a true singularity. There is also a singularity (which is an artifact of the coordinate system) at $r = 2GM \equiv r_+$ which is called the horizon. While not a coordinate singularity, the horizon is significant because inside ($r < r_+$) not even light can escape. Since $f(r_+) = 0$, proper time for an observer approaching the horizon ($\sqrt{-ds^2} = \sqrt{f(r)}dt$) passes quickly while to a distant observer it appears to take an infinite time to reach the horizon.
Now let us define $\tau = it$ so Minkowski space becomes Euclidean ($ds^2 = d\tau^2 + d\mathbf{x}^2$) and the Schwarzschild metric becomes

$$ds^2 \approx \left(1 - \frac{r_+}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)} + r^2 d\Omega_2^2.$$

When $r = r_+ + \varepsilon$ is outside but close to the horizon ($\varepsilon > 0$),

$$ds^2 \approx \frac{\varepsilon}{r_+} d\tau^2 + \frac{r_+}{\varepsilon} d\varepsilon^2 + r_+^2 d\Omega_2^2.$$

Note both ε and τ are completely independent of Ω so the space near the horizon neatly separates into a sphere S^2 of radius r_+ and a two-dimensional manifold of metric

$$ds_2^2 = \frac{\varepsilon}{r_+} d\tau^2 + \frac{r_+}{\varepsilon} d\varepsilon^2$$

To understand this manifold, change coordinates to $\rho = 2\sqrt{r_+ \varepsilon}$ and $\chi = \frac{\tau}{2r_+}$ to get

$$ds_2^2 = \rho^2 d\chi^2 + d\rho^2$$

which is similar to polar coordinates but χ is not restricted to be between 0 and 2π. The resulting spacetime is a cone (the circumference of a closed curve with constant ρ is not $2\pi\rho$). We want to eliminate the conical singularity. If it is to be a plane, χ must be between 0 and 2π so τ must be between 0 and $4\pi r_+$. Recall that periodic imaginary time is an attribute of statistical systems of temperature T which is the iverse period. Thus the black hole has temperature

$$T = \frac{1}{4\pi r_+} = \frac{1}{8\pi GM}$$

which is the Hawking temperature. It looks like we have a statistical system, but what are the states? In thermodynamics, $dU = TdS$ where U is the total energy; in this case, it must be the mass. Thus $dM = \frac{1}{8\pi GM} dS$ which means

$$S = \int 8\pi GM dM = 4\pi GM^2 = \frac{4\pi r_+^2}{4G} = \frac{A_+}{4G}$$

where A_+ is the area of the horizon; this is the Bekenstein-Hawking formula. It is remarkably universal.
Normally entropy is proportional to volume and therefore mass (in an ordinary star with N particles and n degrees of freedom, there are n^N possible states so entropy S is proportional to $N \ln n$ which is proportional to mass M (and volume) so we have two surprises: the entropy is proportional to surface area (the first hint of holography) and to the square of the mass. Also as $T \to 0$, the number of states usually goes to one so $S \to 0$, but, in this case, $T = \frac{1}{8\pi GM}$ so as $T \to 0$, $M \to \infty$: entropy is increasing. The relation between temperature and mass implies that the heat capacity is given by

$$C = \frac{dM}{dT} = -8\pi GM^2 < 0$$

which is an unstable thermodynamic system.
Introduce the partition function with the Euclidean action S_E

$$Z = \int [dg]\, e^{-S_E} \approx e^{-S_{cl}}.$$

which should be quantum gravity. If the Ricci scalar $R = 0$ then $S_{cl} = 0$ but we had better be careful. The Ricci tensor is a second derivative and the Lagrangian should be independent of second order and higher derivatives. We can integrate by parts but we must keep the surface terms at $R > r_+$ and let $R \to \infty$ at the end.

$$S_{surface} = -\frac{1}{8\pi G} \int_{surface} d^3x \sqrt{h} K$$

(York-Gibbons-Hawking action) where $K = trK_{\mu\nu}$ is the extrinsic curvature of the surface and

$$K_{\mu\nu} = \frac{1}{2} n_\alpha g^{\alpha\beta} \partial_\beta g_{\mu\nu}$$

($n^\mu_\alpha = \frac{1}{\sqrt{g_{rr}}} \delta^\mu_r$ is the unit vector perpendicular to the surface).

$$\begin{aligned}
K_{\tau\tau} &= -\frac{GM}{R^2}\sqrt{1 - \frac{2GM}{R}} \\
K_{\theta\theta} &= R\sqrt{1 - \frac{2GM}{R}} \\
K_{\varphi\varphi} &= R\sin^2\theta\sqrt{1 - \frac{2GM}{R}} \\
h_{\tau\tau} &= f(R)\; h_{\theta\theta} = R^2\; h_{\varphi\varphi} = R^2 \sin^2\theta
\end{aligned}$$

Then

$$\begin{aligned}
S_{surface} &= -\frac{1}{8\pi G}\int_0^{1/T} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sqrt{h} K \\
&= -\frac{2R - 3GM}{2GT}
\end{aligned}$$

which is the classical action.
It diverges as $R \to \infty$. But in order to properly define it, we need to introduce a reference point. We shall subtract the contribution of empty space. The latter is obtained by setting $M = 0$. However, there is a complication. We obtained the temperature in the case $M \neq 0$ by demanding that there be no conical singularity. There is no such constraint in empty space ($M = 0$). Instead, we shall match the boundaries of the two spaces (at $r = R$). The time direction at $r = R$ for the black hole has length $\frac{1}{T}\sqrt{g_{\tau\tau}} = \frac{1}{T}\sqrt{f(R)}$. If we make time periodic with period $1/T_0$ for $M = 0$, then that will be its length at any r. We need to match

$$\frac{1}{T_0} = \frac{1}{T}\sqrt{f(R)} = \frac{1}{T}\sqrt{1 - \frac{2GM}{R}}$$

i.e., choose the temperature of empty space to be T_0 (red-shifted). Then

$$\begin{aligned} S_{cl} &= -\frac{2R - 3GM}{2GT} + \frac{R}{GT_0} \\ &= -\frac{2R - 3GM}{2GT} + \frac{R}{GT}\sqrt{1 - \frac{2GM}{R}} \\ &= \frac{3M}{2T} - \frac{M}{T} + \mathcal{O}(1/R) \\ &= \frac{M}{2T} \end{aligned}$$

Now

$$\begin{aligned} F &= U - TS \\ &= M - TS \\ &= M - \frac{1}{8\pi GM}\frac{\pi r_+^2}{G} \\ &= M - \frac{1}{8\pi GM}\frac{4\pi G^2 M^2}{G} \\ &= \frac{1}{2}M \end{aligned}$$

since $r_+ = 2GM$. Therefore,

$$S_{cl} = \frac{F}{T}$$

as expected.
Can we understand the entropy microscopically by counting degrees of freedom? This is hard at finite temperature. It is easier to ask the question in the limit $T \to 0$ because the system then settles to its ground state. The problem is that in this limit, $M \to \infty$, so this limit is hard to understand. We need a better black hole.