LECTURE 1

Path Integral

1.1 Quantum Mechanics

Set $\hbar = 1$. Given a particle described by coordinates (q, t) with initial and final coordinates of (q_i, t_i) and (q_f, t_f), the amplitude of the transition is

$$\langle q_i, t_i \mid q_f, t_f \rangle = \langle \psi \left| q_f, t_f \right\rangle = \psi^* \left(q_f, t_f \right) \ .$$

where ψ satisfies the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \frac{\partial^2 \psi}{\partial q^2} \ .$$

Feynman introduced the path integral formalism

$$\langle q_i, t_i \mid q_f, t_f \rangle = \int_{q(t_i)=q_i, q(t_f)=q_f} [dq] \, e^{-iS}$$

where $S = \int dt L$ is the action and $L = L(q, \dot{q})$ is the Lagrangian, e.g.,

$$L = \frac{1}{2} m \dot{q}^2 - V(q)$$

We convert to imaginary time via the Wick rotation $\tau = it$ so

$$L \rightarrow -\frac{1}{2} m \dot{q}^2 - V(q) \equiv -L_E$$

where the derivative is with respect to τ, L_E is the total energy (which is bounded below), and the subscript E stands for Euclidean. The action becomes $iS \rightarrow S_E = \int d\tau L_E$ so the path integral becomes

$$\int [dq] \, e^{-S_E} \ .$$

The major contribution to the path integral's value comes from minimum S_E, i.e.,

$$\delta S = 0$$

i.e., from the classical trajectory $q = q_{cl}(\tau)$.
Consider a perturbation of the classical trajectory

$$\begin{aligned} q(\tau) &= q_{cl}(\tau) + \delta q(\tau) \\ S &= S_{cl} + \,... \\ \int [dq]\, e^{-S_E} &= e^{-S_{cl}} \int [dq]\, e^{-\,\cdots} \\ \langle q_i, t_i \mid q_f, t_f \rangle &\approx e^{-S_{cl}} \end{aligned}$$

The correlators are

$$\langle q_i, \tau_i \,|T[q(\tau_1)\, q(\tau_2)\ \cdots]|\, q_f, \tau_f \rangle = \int_{q(\tau_i)=q_i, q(\tau_f)=q_f} [dq]\, q(\tau_1)\, q(\tau_2)\ \cdots e^{-S_E} \tag{1.1.1}$$

where T denotes a time ordered product. Notice that the path integral automatically takes care of time ordering. This can easily be seen by splitting the path integral into time intervals bounded by $\tau_1, \tau_2, \ldots$.
We are interested in the vacuum expectation values

$$G(\tau_1, \tau_2, \ldots) = \langle 0 \,|T\, [q(\tau_1)\, q(\tau_2)\ \cdots]|\, 0\rangle\ . \tag{1.1.2}$$

To write them in terms of a path integral, notice that by using the time evolution operator, we have

$$|q_f, \tau_f\rangle = e^{-\tau_f H}\, |q_f, 0\rangle \tag{1.1.3}$$

where H is the Hamiltonian with eigenstates $|n\rangle$; $H\,|n\rangle = E_n\,|n\rangle$ and we take $E_0 = 0$ (not necessary). Inserting the identity $\mathbb{I} = \sum_{states} |n\rangle\,\langle n|$ in equation (1.1.3),

$$|q_f, \tau_f\rangle = \sum_{states} e^{-\tau_f E_n}\, |n\rangle\, \langle n \mid q_f, 0\rangle$$

we see that in the limit $\tau_f \to \infty$, all terms but one vanish. Therefore,

$$|q_f, \tau_f\rangle \to |0\rangle\, \langle 0 \mid q_f, 0\rangle$$

Similarly, $\langle q_i, \tau_i| \to \langle q_i, 0 \mid 0\rangle\, \langle 0|$ as $\tau_i \to -\infty$. Substituting these limiting expressions into equation (1.1.1) gives

$$\begin{aligned} \langle 0 \,|T\, [q(\tau_1)\, q(\tau_2) \cdots]|\, 0\rangle\, \langle q_i, 0 \mid 0\rangle\, \langle 0 \mid q_f, 0\rangle &= \textstyle\int [dq]\, q(\tau_1)\, q(\tau_2)\ \ldots e^{-S_E} \\ \langle 0 \mid 0\rangle\, \langle q, 0 \mid 0\rangle\, \langle 0 \mid q, 0\rangle &= \textstyle\int [dq]\, e^{-S_E} \\ G(\tau_1, \tau_2, \ldots) &= \frac{\int [dq] q(\tau_1) q(\tau_2)\ \ldots\ e^{-S_E}}{\int [dq] e^{-S_E}} \end{aligned}$$

where we integrate over all trajectories $q(\tau)$ with $\tau \in (-\infty, +\infty)$ and no specified end points.

1.2 Statistical Mechanics

Set the Boltzmann constant $k_B = 1$. Denote temperature by $T \geq 0$; At $T = 0$ the system settles in the ground state. Vacuum expectation values correspond to $T = 0$. To study finite T, introduce the partition function

$$\begin{aligned}
Z &= \sum_{states} e^{-E_n/T} \\
&= \sum_{states} \left\langle n \left| e^{-H/T} \right| n \right\rangle \\
&= \sum_{states} \mathrm{Tr}\left(e^{-H/T} \left| n \right\rangle \left\langle n \right| \right) \\
&= \mathrm{Tr}\left(e^{-H/T} \right) \\
&= \int dq \left\langle q, 0 \left| e^{-H/T} \right| q, 0 \right\rangle \\
&= \int dq \left\langle q, 0 \mid q, 1/T \right\rangle \\
&= \int dq \int_{q'(0)=q'(1/T)=q} [dq'] \, e^{-S_E} \\
&= \int_{q(0)=q(1/T)} [dq] \, e^{-S_E}
\end{aligned}$$

Therefore, the partition function is given by a path integral over periodic orbits of period $1/T$.

Example 1 *Quantum mechanics: consider a free particle* $V = 0$ *and* $L = \frac{1}{2} m \dot{q}^2$. *Then using path integrals*

$$\langle q_i, t_i \mid q_f, t_f \rangle = \sqrt{\frac{m}{2\pi(\tau_f - \tau_i)}} e^{-m(q_f - q_i)^2/2(\tau_f - \tau_i)}$$

Example 2 *Statistical mechanics: Set* $\tau_i = 0$, $\tau_f = 1/T$, *and* $q_i = q_f = q$.

$$\begin{aligned}
Z &= \int dq \sqrt{\frac{mT}{2\pi}} \; which \; is \; infinite \; so \; put \; thesystem \; in \; a \; box \\
&= V \sqrt{\frac{mT}{2\pi}} \; where \; V \; is \; the \; volume \; (length).
\end{aligned}$$

which is the same as

$$\begin{aligned}
Z &= V \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-p^2/2m} \\
&= V \sqrt{\frac{mT}{2\pi}} \, .
\end{aligned}$$

1.3 Thermodynamics

1.3.1 Canonical ensemble

Expand the partition function around the classical trajectory to get

$$Z = e^{-S_{cl} + \cdots} = e^{-F/T}$$

where F is the Helmholtz free energy so

$$F = S_{cl} T \ .$$

The probability of a state $|n\rangle$ is $p_n = \frac{1}{Z} e^{-E_n/T}$. The internal energy is

$$\begin{aligned} U &= \langle E \rangle \\ &= \sum_{states} p_n E_n \\ &= T^2 \frac{\partial (\ln Z)}{\partial T} \end{aligned}$$

The entropy is

$$S = \sum_{states} p_n \ln p_n = T \frac{\partial (\ln Z)}{\partial T} - \ln Z = \frac{U}{T} - \frac{F}{T}$$

which means

$$F = U - TS \ .$$

1.3.2 Microcanonical ensemble

The microcanonical ensemble is isolated from its environment. The number of states with a given energy E is given by the multiplicity $g(E) = e^{S(E)}$, i.e., the entropy counts the number of different states.

$$\begin{aligned} Z &= \sum_{energy\ levels} e^{-E/T} g(E) \\ &\approx g(E_0) e^{-E_0/T} \\ &= e^{S(E_0) - E_0/T} \end{aligned}$$

where E_0 maximizes $S(E) - E/T$. We deduce $\frac{\partial S}{\partial E} = \frac{1}{T}$.

1.4 Field Theory

Set the speed of light $c = 1$. Let $x^\mu = (t, \mathbf{x})$ be the position 4-vector. Consider a real field $\varphi(x)$ with dynamics governed by the action $S = \int d^4x \mathcal{L}$ where $\mathcal{L}\left(\partial_\mu \varphi, \varphi\right)$ is the Lagrangian density; the Lagrangian is $L = \int d^3x \mathcal{L}$. Use the Wick rotation $\tau = it$ so

$ds^2 = -dt^2 + dx^2 = d\tau^2 + dx^2 = ds_E^2$ (Euclidean continuation). The correlation functions or Green functions are

$$\begin{aligned} G(x_1, x_2, \dots) &= \langle 0 \,| T\left(\varphi(x_1)\varphi(x_2)\cdots\right)|\, 0\rangle \\ &= \frac{\int [d\varphi]\, \varphi\left(x_1\right) \varphi\left(x_2\right) \cdots\; e^{-S_E}}{\int [d\varphi]\, e^{-S_E}} \end{aligned}$$

as before. Taking an arbitrary function J as a source current, the generating functional is given by

$$Z\left[J\right] = \int [d\varphi] \left(e^{-S_E + \int d^4x J\varphi} \right)$$

and

$$G(x_1, x_2 \dots) = \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots\; Z\left[J\right] \bigg|_{J=0} \tag{1.4.1}$$

Example 3 *For a free field of mass m,*

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + J\phi$$

The classical field equation is the Klein-Gordon equation

$$-\nabla^2 \phi_{cl} + m^2 \phi_{cl} = J$$

Writing a quantum field as

$$\phi = \phi_{cl} + \delta\phi$$

we obtain

$$S_E = S_{cl} + S^{(2)}(\delta\phi) \ , \quad S_{cl} = \int d^4 \mathcal{L}_E(\partial_{cl\mu}, \phi_{cl})$$

Notice that there is no linear terms, because it is proportional to the field equation. Then

$$Z[J] = e^{-S_{cl}}$$

where we omitted a constant which was independent of J. We have

$$S_{cl} = \frac{1}{2} \int d^4x d^4x' J(x) D_E(x, x') J(x')$$

where D_E is the propagator,

$$(-\nabla^2 + m^2) D_E(x, x') = \delta^4(x - x')$$

Using (1.4.1), we easily deduce

$$G(x_1, x_2) = D_E(x_1, x_2)$$

You are invited to check that this is still valid when interactions are included.