A New Look at Pictures in Quantum Information

Arthur Jaffe and Zhengwei Liu

Workshop on Higher Category Approach to Certifiably Correct Quantum Information Processing Systems

College Park, February 4, 2019
Outline

• The Picture Language Project at Harvard
• Mathematical Background
• Some Problems Solved
• Some Interesting Directions for the Future
• Some Work in Progress (Zhengwei)
The Picture Language Project at Harvard
Mathematics, Physics, Quantum Information
Picture appears throughout mathematical history. The goal of the Mathematical Picture Language Project at Harvard is to reevaluate ways that one can use pictures, not only to gain mathematical insights, but also to prove mathematical theorems.

Arthur Jaffe and Zhengwei Liu began their collaboration by studying some problems in subfactor theory and quantum information. This led them to the discovery of the quon language, after which they realized that the quon language also sheds light on several other areas of mathematics. Here you can find links to this research or to articles about our project.

These events motivated the current project to use virtual and real mathematical concepts, simulated by pictures, as a tool to find new understanding ranging across operator algebras, subfactor theory, harmonic analysis, topology, representation theory, statistical physics, topological field theory, quantum field theory, and possibly other fields.

Upcoming Events

Seminar, Zhengwei Liu (Harvard), Quantum Error Correcting Codes

3:00pm
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Lots of Activity: 43 Recent Papers

Links to Recent Papers by Participants that are Related to the Project (Links to posters are at the bottom of this page.)


[42] Zhengwei Liu, William Norledge, and Adrian Ocneanu. The adjoint braid arrangement as a combinatorial Lie algebra via the Steinmann relations. arXiv:1901.03243

[41] Lu Li, Kaifeng Bu, and Zi-Wen Liu. Quantifying the resource content of quantum channels: An operational approach. arXiv:1812.02572


July 2, 2018, Shanghai
Institute of Jian-Wei Pan

Institute for Quantum Physics and Communication in Pudong
University of Science and Technology (USTC) and the Chinese Academy of Sciences (CAS)

US $12 Billion / 5 years in 1 Institute!!
Mathematical Background
2 Charged-String Language (2D)

Qubit Vectors

\[ |k\rangle = 2^{-\frac{1}{4}} \left( \begin{array}{c} k \end{array} \right) \]

\( k \) is a “charge” label in \( Z_d \). \( k = 0, \ldots, d-1 \)
Mathematical Background
2 Charged-String Language (2D)

Qudit Vectors

\[ |k\rangle = 2^{-\frac{1}{4}} \sum k \]

One-Qudit Transformations

\[ T = \begin{pmatrix} k \end{pmatrix} = \langle \ell | T |k\rangle \]

\( k \) is a “charge” label in \( \mathbb{Z}_d \). \( k = 0, ..., d-1 \)

Pauli I, X, Y, Z

\[ I = \begin{pmatrix} 1 \end{pmatrix} \quad X = \begin{pmatrix} 1 \end{pmatrix} \quad Y = \begin{pmatrix} -1 \end{pmatrix} \quad Z = \begin{pmatrix} 1 \end{pmatrix} \]

Vectors \( \sum k \) are an eigenbasis for Pauli Z.
Majorana Algebra (Case d=2)

Resolution of single strands (for Qubits)

\[ c = c^* = c^{-1} \]
\[ c = 1 = \begin{pmatrix} I = c^2 = 2 \end{pmatrix} = 0 = \]

Parasisotopy:
\[ = - \]
\[ YX = -XY \]

Twisted Tensor Product:
\[ (c \otimes I)(I \otimes c) = -(I \otimes c)(c \otimes I) \]

Interpolation:
\[ = -i \]
\[ YX = -iZ \]
\[ XYZ = i \]
Qubit Pauli Matrices

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
Fourier Transform (FT) on $Z_d$

$$(Ff)(k) = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} q^{k\ell} f(\ell)$$

Fourier transform on projections (neutral matrix units) is:

$$F_{-k} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} q^{k\ell} F_{-\ell}$$
String Fourier Transform (SFT)

\[ \mathcal{F}_s = \]

90° rotation of inputs and outputs on a 1-qubit transformation
Theorem. (SFT generalizes FT)

On neutral, one-qudit transformations,

\[ \mathcal{S}_s (-k) + k = F \]

One line proof of the theorem:

\[ \mathcal{S}_s (-k) + k = k \cdot -k = \frac{\zeta^2}{\sqrt{d}} \sum_{\alpha} (-k_{\alpha}) = \frac{q^2}{\sqrt{d}} \sum_{\alpha} q^{k_{\alpha}} = \frac{1}{\sqrt{d}} \sum_{\alpha} q^{k_{\alpha}} = F \]

Move \(-k\) up.  Move \(k\) across cap.  \(\alpha \rightarrow \alpha - k\)

Insert resolution of identity.
Some Results

• Translation between Algebraic and Diagrammatic Protocols
• Entanglement by SFT
• Topological Design of Protocols
Generalized Bell State: $\text{Max}$
Maximally Entangled $n$-qubit state

$2^{\frac{n}{4}} |\text{Max}\rangle_n = \cdots = \mathcal{F}_s = \cdots$

Kauffman entanglement by braid; SFT maximal entanglement.

$|\text{Max}\rangle_2 = |\text{GHZ}\rangle_2 = \frac{1}{2^{\frac{1}{2}}} (|0, 0\rangle + |1, 1\rangle)$

$|\text{Max}\rangle_n = F^{\otimes n} |\text{GHZ}\rangle_n$
One can use the dictionary to translate from the diagrammatic protocol to the algebraic circuit in (31).

The state $|\text{~}0\rangle$ denotes the $n$-qudit with charge 0 for each 1-qudit, and we call $|\text{~}0\rangle$ the ground state. It plays the role of ancilla qudits in our protocol. We mention the extremely interesting transformation $F_s$ that appears here, and that we call the string Fourier transform. We explore $F_s$ extensively in [16], and we show in [15] that $|\text{Max}\rangle_i = F_s |\text{~}0\rangle_i$, which replaces $|\text{Max}\rangle_i = (F_1 \cdots F_n) |\text{GHZ}\rangle_i$ given by (25).

One can simplify the protocol by the identity in Figure 2. Taking the conjugation of local transformations, we obtain the MCT protocol for other types of compressed transformations. In particular, taking the conjugate of the Fourier transform $F$, we obtain the MCT protocol for $Z$-compressed transformations or controlled transformations in (29).

Compressed = Though Line
Quon (4 strings, 3D)

\[ 2^{\frac{1}{2}} |k\rangle = k \bigcap -k \]

one cap replaced by neutral pair
2-dimensional neutral subspace in 4-dimensional space
Quon
Pauli I, X, Y, Z

\[ I = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c} \hline & & & & & & & & & & & & \hline \end{array} \]

\[ X = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} \hline & & & & & & & & & & & & \hline \end{array} \]

\[ Y = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} \hline & & & & & & & & & & & & \hline \end{array} \]

\[ Z = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} \hline & & & & & & & & & & & & \hline \end{array} \]

\[ X Y Z = \zeta \gamma , \quad \gamma = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} \hline & & & & & & & & & & & & \hline \end{array} \]

Must restrict to \( \gamma = 1 \) eigenspace.
Quon X, Y, and Z Bases

X Eigen Basis

Y Eigen Basis

Z Eigen Basis
GHZ\textsubscript{2} and Max\textsubscript{2} in Quon
3-Quon to 3-Quon Transformation

TQFT + Strings
String-Genus Relation

\[ = d^{-1/2} \]

\[ = d^{-1/2} \]
Some Results
Zhengwei Liu involved in all

• Topological Interpretation of Quantum CNOT
• Beautiful Teleportation Diagram
• Inequalities for Quantum Fourier Analysis
• Algebraic Identities for 6j Symbols of Quantum groups
• New proof of Verlinde fusion relations
• DeFinetti Theorem for Double Braiding
• SFT = S-Matrix
Topological CNOT
Teleportation Protocol in 3D Quon Language

3D Quon Language

Resource State

Bob

Recovery Maps X, Z

CNOT

Fourier

Measurement

Alice

Bennett et al. algebraic protocol

A. Jaffe & Z. Liu, Maryland, 2/4/2019
Two Multiplications

\[ xy \text{ multiplication} \quad x \ast y \text{ convolution} \]

\[ \mathcal{F}_S(xy) = \mathcal{F}_S(x) \ast \mathcal{F}_S(y) \]

Inequalities: Hausdorff-Young, Young, ..... 
Uncertainty Principles: Generalize classical Heisenberg, Hardy, Beckner, Tao, ,.....
6j Identities

- 6j symbol corresponds to a tetrahedron
- Duality of tetrahedron is a tetrahedron
- 6j identities for quantum groups from Max GHZ (Liu). \( \overline{X} \) dual to \( X \) in an MTC.

\[
\left| \begin{pmatrix} X_6 & X_5 & X_4 \\ X_3 & X_2 & X_1 \end{pmatrix} \right|^2 = \sum_{\overrightarrow{Y}} \left( \prod_{k=1}^{6} S_{Y_k}^{X_k} \right) \left| \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 \end{pmatrix} \right|^2.
\]
Open Questions

• Qtool for visualization of quon model
• Approach to quantum error correction
• New Algorithms
• Exactly soluble Statistical Mechanics Models
Youwei Zhou’s QTool
In the joint paper with Arthur Jaffe and Alex Wozniakowski (PNAS 2017), we introduce the quon language as a 3D picture language for quantum information. In this talk, we restrict the quon language on the 2D surfaces, which is a surface algebra.
The quon Language refines the tensor network approach to quantum information. A single string in tensor network is refined as four strings in a cylinder in the quon language. A Bosonic particle can be decomposed into a particle-antiparticle pair. In this way, we can see further internal symmetry and duality pictorially.
A new point of view can give new insight!
2019
Happy New Year
新年快乐