Quantum Field Theory I

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UNIT 1

Spin 0

1.1 Symmetry groups

1.1.1 Rotations

Let us first review rotations. Suppose $\vec{x} \in \mathbb{R}^3$ is a vector with components x^i ($i = 1, 2, 3$). We may also think of it as a single column matrix

$$x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{1.1.1}$$

Its norm is

$$||x||^2 = \vec{x} \cdot \vec{x} = x^i x_i = x^T x = (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{1.1.2}$$

For a long time (before Einstein) people thought that all observers agreed on norms of vectors (e.g., distance between two points). So, unlike coordinates, which were man-made, norms were numbers that belonged to Nature. It was then important to understand the transformations that preserved norms (isometries). A linear such transformation can be represented by a 3×3 matrix $\mathcal{O}$, so

$$x \to \mathcal{O} x \tag{1.1.3}$$

If the norm does not change,

$$x^T x = x^T \mathcal{O}^T \mathcal{O} x \quad \Rightarrow \quad \mathcal{O}^T \mathcal{O} = I \tag{1.1.4}$$

i.e., $\mathcal{O}$ is an orthogonal matrix. The group of all these matrices is called $O(3)$. Not all of them represent rotations, e.g.,

$$\mathcal{O} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{1.1.5}$$

represents a reflection ($x^1 \to -x^1$). Notice that $\det \mathcal{O} = -1$. In general, since $\det \mathcal{O}^T = \det \mathcal{O}$, we have

$$\mathcal{O}^T \mathcal{O} = I \;\Rightarrow\; (\det \mathcal{O})^2 = 1 \;\Rightarrow\; \det \mathcal{O} = \pm 1 \tag{1.1.6}$$

Rotations have $\det \mathcal{O} > 0$ and form the subgroup $SO(3)$ (S for special).
Example: Rotation around the z-axis:

$$\mathcal{O} = \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \tag{1.1.7}$$

1.1.2 Lorentz transformations

Einstein realized that norms were also man-made (two different observers could disagree on their value). To construct a number that truly belongs to Nature, we have to include time and consider four-vectors living in space-time, instead. Set $c = 1$ (the speed of light) and denote

$$x^\mu = (t, \vec{x}) \;\; \mu = 0, 1, 2, 3 \tag{1.1.8}$$

The invariant norm (on whose value all observers agree) is

$$x^\mu x_\mu = t^2 - \vec{x}^2 \tag{1.1.9}$$

It can be written in matrix form as

$$x^\mu x_\mu = x^T \eta x \;\;,\;\; \eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.1.10}$$

We need to understand the linear transformations that preserve the norm of a four-vector (Lorentz transformations). They are 4×4 matrices Λ ($x \to \Lambda x$). Since they preserve norms, we have

$$x^T \eta x = x^T \Lambda^T \eta \Lambda x \;\Rightarrow\; \Lambda^T \eta \Lambda = \eta \tag{1.1.11}$$

They form a group called $O(3,1)$. Notice that, similar to $O(3)$, $\det \Lambda = \pm 1$. Examples of transformations with negative determinant are reflections in space and time-inversions (for which $\Lambda = -\eta$). Restricting to $\det \Lambda > 0$, we obtain the subgroup $SO(3,1)$ which is the Lorentz group.
Example 1: Any rotation,

$$\Lambda = \begin{pmatrix} 1 & \\ & \mathcal{O} \end{pmatrix} \tag{1.1.12}$$

Example 2: A boost in the x-direction,

$$\Lambda = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & & \\ -\sinh\zeta & \cosh\zeta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.1.13}$$

where ζ is the rapidity related to the speed v by

$$v = \tanh \zeta \tag{1.1.14}$$

1.2 Canonical quantization

1.2.1 One particle

Consider a particle moving in one dimension whose trajectory is given by $q(t)$. Classically, its motion can be deduced from a Lagrangian

$$L(q, \dot{q}) \ , \ \dot{q} = \frac{dq}{dt} \tag{1.2.1}$$

by extremizing the action

$$S = \int dt \, L \tag{1.2.2}$$

$\delta S = 0$ implies the Euler-Lagrange equation

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = 0 \tag{1.2.3}$$

An often encountered case is one in which

$$L = T - V(q) \ , \ T = \tfrac{1}{2} m \dot{q}^2 \tag{1.2.4}$$

where T (V) is the kinetic (potential) energy (e.g., for a harmonic oscillator, $V = \frac{1}{2} m\omega^2 q^2$). Then the Euler-Lagrange equation implies Newton's Law,

$$m\ddot{q} = -\frac{dV}{dq} \tag{1.2.5}$$

Instead of the Lagrangian (a function of position q and velocity $\dot{q}$), we may use the Hamiltonian,

$$H = p\dot{q} - L \tag{1.2.6}$$

which is a function of the coordinate q and its conjugate momentum

$$p = \frac{\partial L}{\partial \dot{q}} \tag{1.2.7}$$

to define the system. In the case $L = T - V$, we have

$$H = \frac{p^2}{2m} + V(q) \tag{1.2.8}$$

which is the total energy of the particle. The equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} \ , \ \dot{p} = -\frac{\partial H}{\partial q} \tag{1.2.9}$$

In the special (important) case (1.2.8), we obtain

$$\dot{q} = \frac{p}{m} \quad , \quad \dot{p} = -\frac{dV}{dq} \tag{1.2.10}$$

For a general observable $\mathcal{O}(p,q)$, we obtain

$$\dot{\mathcal{O}} = \frac{\partial \mathcal{O}}{\partial q}\,\dot{q} + \frac{\partial \mathcal{O}}{\partial p}\,\dot{p} = \frac{\partial \mathcal{O}}{\partial q}\,\frac{\partial H}{\partial p} - \frac{\partial \mathcal{O}}{\partial p}\,\frac{\partial H}{\partial q} \tag{1.2.11}$$

which may be written as

$$\frac{d\mathcal{O}}{dt} = \{\mathcal{O}, H\} \tag{1.2.12}$$

in terms of the Poisson brackets

$$\{\mathcal{A}, \mathcal{B}\} = \frac{\partial \mathcal{A}}{\partial q}\,\frac{\partial \mathcal{B}}{\partial p} - \frac{\partial \mathcal{A}}{\partial p}\,\frac{\partial \mathcal{B}}{\partial q} \tag{1.2.13}$$

Notice that the general eq. (1.2.12) reduces to (1.2.9) for $\mathcal{O} = q, p$.
To quantize the system, we promote q and p to operators and replace the Poisson brackets by commutators as follows:

$$\{\mathcal{A}, \mathcal{B}\} \to -i[\mathcal{A}, \mathcal{B}] \tag{1.2.14}$$

where we set $\hbar = 1$. In particular, we have

$$[q, p] = i \tag{1.2.15}$$

and

$$\frac{d\mathcal{O}}{dt} = -i[\mathcal{O}, H] \tag{1.2.16}$$

which is the Heisenberg equation. Notice that, unlike in the Schrödinger picture, where the wave-function is time-dependent, in the Heisenberg picture the operators themselves are time dependent, so to be precise, we ought to write the basic commutation relation (1.2.15) as

$$[q(t), p(t)] = i \tag{1.2.17}$$

Notice that it is an equal-time commutation relation.
The general Heisenberg equation (1.2.16) can be solved if the Hamiltonian is constant in time. The solution is

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} \tag{1.2.18}$$

given in terms of the evolution operator $U(t) = e^{iHt}$.

1.2.2 N particles

The above discussion may be easily generalized to a system described by N coordinates,

$$q_a(t) \quad , \quad a = 1, \ldots N \tag{1.2.19}$$

The action is

$$S = \int dt\ L(q_a, \dot{q}_a) \tag{1.2.20}$$

from which we derive the Euler-Lagrange equations

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a}+\frac{\partial L}{\partial q_a}=0 \tag{1.2.21}$$

For example,

$$L=T-V(q_1,\ldots,q_N)\ ,\ T=\sum_{a=1}^{N}\tfrac{1}{2}m_a\dot{q}_a^2 \tag{1.2.22}$$

whose Euler-Lagrange equations are Newton's Law for the N coordinates,

$$m_a\ddot{q}_a=-\frac{\partial V}{\partial q_a} \tag{1.2.23}$$

The Hamiltonian is

$$H(q_a,p_a)=\sum_{a=1}^{N}p_a\dot{q}_a-L\ ,\ p_a=\frac{\partial L}{\partial \dot{q}_a} \tag{1.2.24}$$

Poisson brackets are defined by

$$\{\mathcal{A},\mathcal{B}\}=\sum_{a=1}^{N}\left(\frac{\partial \mathcal{A}}{\partial q_a}\frac{\partial \mathcal{B}}{\partial p_a}-\frac{\partial \mathcal{A}}{\partial p_a}\frac{\partial \mathcal{B}}{\partial q_a}\right) \tag{1.2.25}$$

Time evolution is given by

$$\frac{d\mathcal{O}}{dt}=\{\mathcal{O},H\} \tag{1.2.26}$$

in general, and in particular

$$\dot{q}_a=\frac{\partial H}{\partial p_a}\ ,\ \dot{p}_a=-\frac{\partial H}{\partial q_a} \tag{1.2.27}$$

The quantum system in the Heisenberg picture is based on the *equal-time* commutation relations

$$[q_a(t),p_b(t)]=i\delta_{ab} \tag{1.2.28}$$

with all other *equal-time* commutators between coordinates and momenta vanishing.

1.2.3 The Klein-Gordon field

Turning a into a *continuous* index yields a field theory. Then the coordinate $q_a(t)$ turns into a *function* $\phi(a,t)$ - a *field*. If a spans our three-dimensional world, it is a vector $\vec{x}\in\mathbb{R}$, and $\phi(\vec{x},t)$ is a function of *space-time*. The Lagrangian will be a function of ϕ as well as $\dot{\phi}=\partial_t\phi$. If we want to build a *relativistic* theory, the Lagrangian must be a function of the four-vector $\partial_\mu\phi=(\partial_t\phi,\vec{\nabla}\phi)$. If we want to build a *local* theory, the Lagrangian must be given in terms of a local Lagrangian density,

$$L=\int d^3x\mathcal{L}(\partial_\mu\phi,\phi) \tag{1.2.29}$$

Then the action is

$$S = \int dt L = \int d^4x \mathcal{L}(\partial_\mu \phi, \phi) \tag{1.2.30}$$

where $d^4x = dtd^3x$ is the Lorentz-invariant measure in space-time. Extremizing the action, we deduce the field equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{1.2.31}$$

in a manifestly Lorentz-invariant form.
The simplest Lagrangian density is a generalization of the harmonic oscillator,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 \tag{1.2.32}$$

where ϕ is real (a complex ϕ will be discussed later). It leads to the field equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{1.2.33}$$

which is the Klein-Gordon equation. It is a linear equation, so ϕ is a *non-interacting* (free) field. Its solutions can be easily obtained by Fourier transforming,

$$\phi = e^{-ik\cdot x} \ \ , \ \ k \cdot x = k_\mu x^\mu = k_0 t - \vec{k}\cdot\vec{x} \ \ , \ \ k_\mu k^\mu = k_0^2 - \vec{k}^2 = m^2 \tag{1.2.34}$$

Identifying k_0 with energy ($E = \hbar\omega = \hbar k_0$ and we have set $\hbar = 1$), we immediately realize that half of the solutions have negative energy, since

$$k_0 = \pm \omega_k \ \ , \ \ \omega_k = \sqrt{\vec{k}^2 + m^2} \tag{1.2.35}$$

Fixing this problem led to quantum field theory (as opposed to a simple relativistic generalization of quantum mechanics by simply replacing the Schrödinger equation with the Klein-Gordon equation). It was a triumph of the human mind and led to surprising (and important) conclusions.
To arrive at a quantum system, we proceed as before and derive the Hamiltonian. This is somewhat awkward in our relativistic system, because we will be singling out the time coordinate. However, it is a convenient procedure. At the end, we ought to check that we haven't inadvertently spoiled Lorentz invariance.
The Hamiltonian is given in terms of a Hamiltonian density,

$$H(\phi, \pi) = \int d^3x \mathcal{H} \ \ , \ \ \mathcal{H} = \pi \partial_t \phi - \mathcal{L} \ \ , \ \ \pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \tag{1.2.36}$$

For the Klein-Gordon field, we obtain

$$\mathcal{H} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\nabla}\phi)^2 + \tfrac{1}{2}m^2\phi^2 \tag{1.2.37}$$

a manifestly positive quantity and therefore worthy of being interpreted as energy density.
Quantization is based on the *equal-time* commutation relations

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}) \tag{1.2.38}$$

with all other *equal-time* commutators vanishing. To analyze this system, we expand the general field ϕ in terms of the complete set of solutions (1.2.34) of the Klein-Gordon equation. Notice that the solutions (1.2.34) are labeled by $\vec{k}$ and come in pairs $e^{\pm ik\cdot x}$, where $k_0 = \omega_k$, of negative and positive energy waves, respectively. Thus, we may write

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(e^{ik\cdot x} a^\dagger(\vec{k}) + e^{-ik\cdot x} a(\vec{k}) \right) \tag{1.2.39}$$

The factors in the measure are somewhat arbitrary and are there for convenience (you may recall similar factors arising in the harmonic oscillator case). The coefficients a and $a^\dagger$ are complex conjugate of each other in the classical theory (hermitian conjugate after quantization), since ϕ is real.
Differentiating with respect to time, we obtain the conjugate momentum in terms of a and $a^\dagger$,

$$\pi = \partial_t \phi = i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} \left(e^{ik\cdot x} a^\dagger(\vec{k}) - e^{-ik\cdot x} a(\vec{k}) \right) \tag{1.2.40}$$

The Hamiltonian is then found using the expression (1.2.37),

$$H = \tfrac{1}{2} \int \frac{d^3k}{(2\pi)^3}\, \omega_k \left(a^\dagger(\vec{k}) a(\vec{k}) + a(\vec{k}) a^\dagger(\vec{k}) \right) \tag{1.2.41}$$

Notice that H is manifestly independent of time.
To obtain the commutation relations between a and $a^\dagger$, first we Fourier transform (1.2.39) and (1.2.40),

$$\int d^3x e^{-i\vec{p}\cdot\vec{x}} \phi(\vec{x},t) = \frac{1}{\sqrt{2\omega_p}} \left(e^{i\omega_p t} a^\dagger(-\vec{p}) + e^{-i\omega_p t} a(\vec{p}) \right) \tag{1.2.42}$$

$$\int d^3x e^{-i\vec{p}\cdot\vec{x}} \pi(\vec{x},t) = i \sqrt{\frac{\omega_p}{2}} \left(e^{i\omega_p t} a^\dagger(-\vec{p}) - e^{-i\omega_p t} a(\vec{p}) \right) \tag{1.2.43}$$

Solving for a and $a^\dagger$, we obtain

$$e^{i\omega_p t} a^\dagger(\vec{p}) = \int d^3x e^{i\vec{p}\cdot\vec{x}} \left(\sqrt{\frac{\omega_p}{2}}\, \phi(\vec{x},t) - \frac{i}{\sqrt{2\omega_p}}\, \pi(\vec{x},t) \right) \tag{1.2.44}$$

$$e^{-i\omega_p t} a(\vec{p}) = \int d^3x e^{-i\vec{p}\cdot\vec{x}} \left(\sqrt{\frac{\omega_p}{2}}\, \phi(\vec{x},t) + \frac{i}{\sqrt{2\omega_p}}\, \pi(\vec{x},t) \right) \tag{1.2.45}$$

Notice that (1.2.44) and (1.2.45) are complex (hermitian) conjugate of each other. Using them, together with the commutation relations (1.2.38), we obtain

$$[a(\vec{p}), a^\dagger(\vec{p}\,')] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,') \tag{1.2.46}$$

and $[a(\vec{p}), a(\vec{p}\,')] = [a^\dagger(\vec{p}), a^\dagger(\vec{p}\,')] = 0$. Thus, $a^\dagger$ and a act as independent (for different momenta $\vec{p}$) creation and annihilation operators, respectively. We are now ready to build the Hilbert space on which they act (Fock space). It will be similar to the Hilbert space of a bunch of uncoupled harmonic oscillators.

1.3 The Fock space

1.3.1 The vacuum

The vacuum (ground state) is defined as the state annihilated by all annihilation operators,

$$a(\vec{p})|0\rangle = 0 \tag{1.3.1}$$

Acting with the Hamiltonian (1.2.41) on the vacuum, we obtain

$$H|0\rangle = E_0|0\rangle \tag{1.3.2}$$

where

$$E_0 = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega_k[a(\vec{k}), a^\dagger(\vec{k})] = \int \frac{d^3k}{(2\pi)^3}\,\tfrac{1}{2}\omega_k\,(2\pi)^3\delta^3(\vec{0}) \tag{1.3.3}$$

Therefore, the vacuum is an eigenstate of the Hamiltonian, but unfortunately the corresponding eigenvalue is infinite! On the other hand, E_0 represents the total energy of the system which should correspond to the total energy of the (empty!) Universe. This is not a quantity that is likely to affect our local world. A more relevant quantity would be the energy density

$$\rho = \frac{E_0}{V} \tag{1.3.4}$$

where V is the (infinite) volume of our system. To compute it, start with

$$\int d^3x e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{k}) \tag{1.3.5}$$

and set $\vec{k} = \vec{0}$. We deduce

$$V = \int d^3x = (2\pi)^3\delta^3(\vec{0}) \tag{1.3.6}$$

It follows that

$$\rho = \int \frac{d^3k}{(2\pi)^3}\,\tfrac{1}{2}\omega_k \tag{1.3.7}$$

This is the cosmological constant. The expression for ρ has a nice interpretation as the sum of zero-point energies of all oscillators (since the measure is the density of states and $\frac{1}{2}\omega$ is the ground-state energy of a harmonic oscillator). But alas! It is infinite.
In the absence of gravity, we may avoid this problem by shifting the Hamiltonian by its vacuum expectation value $E_0 = \langle 0|H|0\rangle$. This shift does not affect any measurements, because only differencies in energy can be measured. It leads to the normal-ordered Hamiltonian (all annihilation operators to the right of all creation operators)

$$:H: = H - E_0 = \int \frac{d^3k}{(2\pi)^3}\,\omega_k a^\dagger(\vec{k})a(\vec{k}) \tag{1.3.8}$$

We shall adopt this form of the Hamiltonian and write H instead of $: H :$ for simplicity. Then on the vacuum we have

$$H|0\rangle = 0 \tag{1.3.9}$$

If gravity is present, no type of energy can be easily subtracted away, because energy gravitates. Recent astronomical observations suggest that the vacuum energy is non-vanishing (positive). In fact, it makes up about 2/3 of our Universe! Its origin is still a mystery. We shall discuss these issues again later. For the moment we proceed by ignoring the effects of gravity which will allow us to understand all the other forces in our Universe.

1.3.2 One-particle states

We obtain a new set of states by acting with a creation operator $a^\dagger(\vec{p})$ on the vacuum. The resulting state is labeled by the momentum $\vec{p}$,

$$|\vec{p}\rangle = C_p a^\dagger(\vec{p})|0\rangle \tag{1.3.10}$$

where we included a normalization constant. To fix it, observe that

$$\langle \vec{k}'|\vec{k}\rangle = |C_k|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \tag{1.3.11}$$

Upon integration over all momenta $\vec{k}$, we should get 1. We ought to be careful and choose a measure which is Lorentz-invariant. To do this, recall that in $\mathbb{R}^3$, the way to get a rotationally-invariant measure is by restricting to a sphere,

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = R^2 \tag{1.3.12}$$

and defining the measure on it by

$$d^2\Sigma = \int d^3x \delta\left(\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} - R\right) \tag{1.3.13}$$

where we integrate over one variable (any choice is valid). If we choose to integrate over x^3, then we obtain the measure on the sphere

$$d^2\Sigma = R\,\frac{dx^1 dx^2}{x^3} \tag{1.3.14}$$

where x^3 is given in terms of x^1 and x^2 through (1.3.12) (restrict to the upper-half sphere, for definiteness). Switching to spherical polar coordinates,

$$x^1 = R\sin\theta\cos\phi \ , \ x^2 = R\sin\theta\sin\phi \tag{1.3.15}$$

brings the measure (1.3.14) into the more familiar form

$$d^2\Sigma = R^2 d\phi d\theta \sin\theta \tag{1.3.16}$$

Similarly, starting with the four-momentum k_μ and the measure d^4k, we restrict to the hyperboloid

$$k_\mu k^\mu = k_0^2 - \vec{k}^2 = m^2 \tag{1.3.17}$$

whose measure may be defined as

$$d^3\Sigma_k = \int \frac{d^4k}{(2\pi)^4}\,(2\pi)\delta(k_0^2 - \vec{k}^2 - m^2) \tag{1.3.18}$$

By not including a square root, as in (1.3.13), we are only introducing an (irrelevant) overall constant factor. Integrating over k_0, we obtain

$$d^3\Sigma_k = \frac{d^3k}{(2\pi)^3 2\omega_k} \tag{1.3.19}$$

as the Lorentz-invariant measure in the space of momenta $\vec{k}$. We may now fix the normalization constant C_p in (1.3.10) by demanding

$$\int d^3\Sigma_k \langle \vec{k}'|\vec{k}\rangle = 1 \tag{1.3.20}$$

Using (1.3.11) and (1.3.19), we deduce

$$C_p = \sqrt{2\omega_p} \tag{1.3.21}$$

We have omitted a phase which changes with time (see eq. (1.3.27) below). For definiteness, let us make the choice (1.3.21) at time $t = 0$. We shall evolve the state $|\vec{p}\rangle$ in time shortly.
We may also write a completeness relation,

$$\int d^3\Sigma_k |\vec{k}\rangle\langle\vec{k}| = \mathbb{I} \tag{1.3.22}$$

where $\mathbb{I}$ is the projection operator onto the subspace of one-particle states (in quantum mechanics, it would be the identity).
Next, we ought to figure out the energy of the state $|\vec{p}\rangle$. Using

$$[H, a^\dagger(\vec{p})] = \omega_p a^\dagger(\vec{p}) \tag{1.3.23}$$

we obtain the action of the (normal-ordered) Hamiltonian

$$H|\vec{p}\rangle = \omega_p|\vec{p}\rangle \tag{1.3.24}$$

showing that $|\vec{p}\rangle$ is an eigenstate with (finite!) eigenvalue ω_p. Thus, it represents a particle of definite momentum $\vec{p}$ (not localized in space) and (positive!) energy $\omega_p = \sqrt{\vec{p}^2 + m^2}$, as expected; $a^\dagger(\vec{p})$ creates a particle of momentum $\vec{p}$ from the vacuum.
A particle localized at position $\vec{x}$ at time $t = 0$ is described by the state

$$|\vec{x}\rangle = \phi(\vec{x}, 0)|0\rangle \tag{1.3.25}$$

Using the expansion (1.2.39), we obtain the inner-product

$$\langle\vec{x}|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}} \tag{1.3.26}$$

which is the position space representation of a momentum eigenstate in quantum mechanics. Thus, $\phi(\vec{x},0)$ creates a particle at position $\vec{x}$ from the vacuum.

TIME EVOLUTION

Since $|\vec{p}\rangle$ is an eigenstate of the Hamiltonian, its time evolution is easily deduced,

$$e^{iHt}|\vec{p}\rangle = e^{i\omega_p t}|\vec{p}\rangle \tag{1.3.27}$$

Correspondingly, the creation operators evolve as

$$e^{iHt}a^\dagger(\vec{p})e^{-iHt} = e^{i\omega_p t}a^\dagger(\vec{p}) \tag{1.3.28}$$

which is a direct consequence of eq. (1.3.23). The evolution of $a(\vec{p})$ is found by taking the hermitian conjugate of (1.3.28). Then using the expansion of ϕ (eq. (1.2.39)), we deduce

$$e^{iHt}\phi(\vec{x},0)e^{-iHt} = \phi(\vec{x},t) \tag{1.3.29}$$

as expected.

SPACETIME TRANSLATIONS

Since the Hamiltonian represents energy, it ought to be the time component of a four-vector P_μ representing total four-momentum. It is not hard to find an expression for P_μ. Simply replace $\omega_k = k_0$ in the expression (1.3.8) for the Hamiltonian with k_μ and define

$$P_\mu = (H,\vec{P}) = \int \frac{d^3k}{(2\pi)^3}\; k_\mu\; a^\dagger(\vec{k})a(\vec{k}) \tag{1.3.30}$$

Thus defined, P_μ is a conserved quantity (no time dependence). It can also be defined in terms of a *local* quantity (density). Explicitly,

$$P_\mu = \int d^3x \mathcal{P}_\mu \;\;,\;\; \mathcal{P}_\mu = (\mathcal{H},\vec{\mathcal{P}}) \;\;,\;\; \vec{\mathcal{P}} = -\pi\vec{\nabla}\phi \tag{1.3.31}$$

Local conserved quantities are rare and should be treasured.

We may now straightforwardly generalize the effects of time evolution to general spacetime translations. The states $|\vec{p}\rangle$ are eigenstates of P_μ,

$$P_\mu|\vec{p}\rangle = p_\mu|\vec{p}\rangle \;\;,\;\; p_\mu = (\omega_p,\vec{p}) \tag{1.3.32}$$

Translation by a four-vector λ^μ gives

$$e^{iP\cdot\lambda}|\vec{p}\rangle = e^{ip\cdot\lambda}|\vec{p}\rangle \tag{1.3.33}$$

Correspondingly, the creation operators are translated according to

$$e^{iP\cdot\lambda}a^\dagger(\vec{p})e^{-iP\cdot\lambda} = e^{ip\cdot\lambda}a^\dagger(\vec{p}) \tag{1.3.34}$$

and similarly for the annihilation operators $a(\vec{p})$. Finally, the field ϕ transforms as

$$e^{iP\cdot\lambda}\phi(x)e^{-iP\cdot\lambda} = \phi(x+\lambda) \tag{1.3.35}$$

as expected.

LORENTZ TRANSFORMATIONS
Under a Lorentz transformation,

$$p_\mu \to (\Lambda p)_\mu \ , \ \ \Lambda \in SO(3,1) \tag{1.3.36}$$

The state $|\vec{p}\rangle$ transforms as

$$|\vec{p}\rangle \to |\vec{\Lambda p}\rangle \equiv U(\Lambda)|\vec{p}\rangle \tag{1.3.37}$$

This defines the operator U implicitly. An explicit expression will be derived later. Let us now show that U is a unitary operator. We shall show this within the subspace of one-particle states in which $\mathbb{I}$ (eq. (1.3.22)) is the identity. We have

$$UU^\dagger = U\mathbb{I}U^\dagger = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \, |\vec{\Lambda k}\rangle\langle\vec{\Lambda k}| \tag{1.3.38}$$

Performing the inverse transformation $k_\mu \to (\Lambda^{-1}k)_\mu$, we have $|\vec{\Lambda k}\rangle \to |\vec{k}\rangle$ and the measure is invariant, hence

$$UU^\dagger = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \, |\vec{k}\rangle\langle\vec{k}| = \mathbb{I} \tag{1.3.39}$$

showing that U is unitary.
From the action of U on $|\vec{p}\rangle$ and the definition (1.3.10) we deduce the action of U on the creation operators,

$$U(\Lambda)C_p a^\dagger(\vec{p})U^\dagger(\Lambda) = C_{\Lambda p} a^\dagger(\vec{\Lambda p}) \tag{1.3.40}$$

where C_p is given by (1.3.21). The annihilation operators transform the same way (easily seen by taking the hermitian conjugate of (1.3.40)). Using (1.3.40) and the expansion (1.2.39), we deduce

$$U(\Lambda)\phi(x)U^\dagger(\Lambda) = \phi(\Lambda x) \tag{1.3.41}$$

which shows that ϕ is a scalar (0-spin) field under Lorentz transformations. This is a non-trivial result!

1.3.3 Two-particle states

By acting with two creation operators on the vacuum, we obtain a two-particle state labeled by the respective momenta of the two particles,

$$|\vec{p}_1, \vec{p}_2\rangle = Ca^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle \tag{1.3.42}$$

Notice that

$$|\vec{p}_1, \vec{p}_2\rangle = |\vec{p}_2, \vec{p}_1\rangle \tag{1.3.43}$$

since $a^\dagger(\vec{p}_1)$ commutes with $a^\dagger(\vec{p}_2)$. Thus, these are identical particles obeying Bose-Einstein statistics (another non-trivial result!).

The normalization constant is determined as in the one-particle case from the inner product

$$\langle \vec{p}_1', \vec{p}_2' | \vec{p}_1, \vec{p}_2 \rangle = (2\pi)^6 |C|^2 \left(\delta^3(p_1 - p_1')\delta^3(p_2 - p_2') + \delta^3(p_1 - p_2')\delta^3(p_2 - p_1') \right) \tag{1.3.44}$$

If $\vec{p}_1 \neq \vec{p}_2$, then we obtain

$$|C|^2 = (2\omega_{p_1})(2\omega_{p_2}) \tag{1.3.45}$$

However, when $\vec{p}_1 = \vec{p}_2$, then the two terms contributing to the inner product are equal to each other, which yields an additional factor of 2. Therefore, in this case

$$|C|^2 = \tfrac{1}{2}(2\omega_{p_1})^2 \tag{1.3.46}$$

1.3.4 Multi-particle states

Similarly, we obtain a multi-particle state of N identical particles with definite momenta,

$$|\vec{p}_1, \dots, \vec{p}_N\rangle = C a^\dagger(\vec{p}_1) \cdots a^\dagger(\vec{p}_N)|0\rangle \tag{1.3.47}$$

It is an eigenstate of the Hamiltonian,

$$H|\vec{p}_1, \dots, \vec{p}_N\rangle = E|\vec{p}_1, \dots, \vec{p}_N\rangle \tag{1.3.48}$$

with (positive) energy

$$E = \omega_{p_1} + \ldots + \omega_{p_N} \tag{1.3.49}$$

as expected. All these states span the Fock space.

1.4 Causality and propagation

1.4.1 Causality and measurement

Information cannot propagate faster than the speed of light. Equivalently, two events at x^μ and y^μ cannot influence each other if they are separated by a *spacelike* distance,

$$(x - y)^2 < 0 \tag{1.4.1}$$

Consequently, one should be able to make arbitrarily accurate measurements of observables $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$, therefore,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \;\; , \;\; (x - y)^2 < 0 \tag{1.4.2}$$

In particular, this should be true for $\mathcal{O}_1 = \mathcal{O}_2 = \phi$. Let us check it. Since the commutator is a c-number (*not* an operator), we have

$$[\phi(x), \phi(y)] = \langle 0|[\phi(x), \phi(y)]|0\rangle \tag{1.4.3}$$

Denote

$$D(x - y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{-ik\cdot(x-y)} \tag{1.4.4}$$

Notice that D is a function of the distance $(x-y)^\mu$ because of translational invariance of the vacuum. This can be seen in general by inserting translation operators,

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|e^{iP\cdot y}e^{-iP\cdot y}\phi(x)e^{iP\cdot y}e^{-iP\cdot y}\phi(y)e^{iP\cdot y}e^{-iP\cdot y}|0\rangle \\ &= \langle 0|\phi(x-y)\phi(0)|0\rangle \end{aligned} \quad (1.4.5)$$

where we used (1.3.35) and $e^{-iP\cdot y}|0\rangle = |0\rangle$. $D(x-y)$ is the amplitude for a particle created at x^μ to propagate to y^μ. It is non-vanishing outside the light-cone ($(x-y)^2 < 0$), but that's ok because it has nothing to do with a measurement. To calculate it, go to a frame in which $x^0 = y^0$ (coincident events) and use spherical polar coordinates in (1.4.4). After some algebra (involving contour deformation in the complex $|\vec{k}|$-plane), we obtain

$$D(x-y) = \frac{m}{4\pi^2\sqrt{-(x-y)^2}}\, K_1(m\sqrt{-(x-y)^2}) \quad (1.4.6)$$

where K_1 is a Bessel function. Notice that

$$D(x-y) = D(y-x) \quad (1.4.7)$$

therefore, using (1.4.3) and (1.4.4),

$$[\phi(x),\phi(y)] = D(x-y) - D(y-x) = 0 \quad (1.4.8)$$

(and also $[\partial_\mu\phi(x),\phi(y)] = 0$, etc), as required by causality (eq. (1.4.2)). This is a non-trivial result: it is the consequence of interference between the amplitudes of propagation $(x \to y)$ and $(y \to x)$ (in general, one forward and the other backward in time; the latter will be associated with the anti-particle later on). Both amplitudes must be present with a relative phase of π ($e^{i\pi} = -1$).

Inside the light-cone ($(x-y)^2 > 0$), go to a Lorentz frame in which $\vec{x} = \vec{y}$ and set $x^0 - y^0 = t$ (observer flies from $\vec{x}$ to $\vec{y}$ in time t). Then eq. (1.4.4) gives

$$D(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{-i\omega_k t} = \frac{1}{4\pi^2}\int_m^\infty d\omega_k\, \sqrt{\omega_k^2 - m^2}\, e^{-i\omega_k t} \quad (1.4.9)$$

therefore,

$$[\phi(x),\phi(y)] = D(x-y) - D(y-x) = -\frac{i}{2\pi^2}\int_m^\infty d\omega_k\, \sqrt{\omega_k^2-m^2}\, \sin\omega_k t \quad (1.4.10)$$

For large t, the major contribution comes from $\omega_k \approx m$, so

$$[\phi(x),\phi(y)] \sim \sin mt \ , \ \ t = \sqrt{(x-y)^2} \to \infty \quad (1.4.11)$$

which is non-vanishing.

1.4.2 The Feynman propagator

To gain further insight, it is advantageous to turn the integral (1.4.4) into a four-dimensional integral (instead of apparently fixing the energy $\omega_k = \sqrt{\vec{k}^2 + m^2}$ by hand). To this end, consider the integral

$$\int \frac{dk_0}{2\pi} \frac{i}{k^2 - m^2} e^{-ik_0 t} \;\; , \;\; k^2 = k_\mu k^\mu = k_0^2 - \vec{k}^2 \tag{1.4.12}$$

along the real axis. We shall calculate this integral as a contour integral in the complex k_0-plane. For $t > 0$ ($t < 0$), we ought to close the contour in the *lower-half (upper-half)* plane. The integrand has poles at $k_0 = \pm\omega_k$. We ought to get them off the real axis by giving them a small imaginary part. Let $\epsilon > 0$ be small ($\epsilon \to 0$ at the end). If we replace $m^2 \to m^2 - i\epsilon$, the pole at $k_0 = \omega_k$ ($-\omega_k$) will move below (above) the real axis. If we replace $k_0 \to k_0 + i\epsilon$, both poles go below the real axis (at $k_0 = \pm\omega_k - i\epsilon$). If, instead, $k_0 \to k_0 - i\epsilon$, both poles go above the real axis. Therefore, we obtain for $t = x^0 - y^0 > 0$ (similar expressions hold for $t < 0$),

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \tag{1.4.13}$$

and the commutator

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik\cdot(x-y)} \;\; , \;\; k^2 = (k_0 + i\epsilon)^2 - \vec{k}^2 \tag{1.4.14}$$

The above two expressions look very similar, but one should keep in mind that they represent different quantities: D is not a physical quantity; the commutator is. Notice that only positive energy contributes to D ($k_0 = \omega_k$) whereas both positive and negative energies contribute to the commutator. Negative energy states do not exist in the Hilbert space, but negative energy solutions to the wave equation have not disappeared. In fact, they provide essential contributions to physical quantities.
An important physical quantity was proposed by Feynman. It is defined by

$$D_F(x-y) = \begin{cases} D(x-y) & , \;\; x^0 - y^0 > 0 \\ D(y-x) & , \;\; x^0 - y^0 < 0 \end{cases} \tag{1.4.15}$$

and is called the Feynman propagator. Obviously, for $x^0 - y^0 > 0$, it is given by (1.4.13). For $x^0 - y^0 < 0$, it is also given by (1.4.13), as the substitution $k^\mu \to -k^\mu$ will easily convince you. Therefore,

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \tag{1.4.16}$$

always. For $t = x^0 - y^0 > 0$, only the positive energy pole contributes, because we close the contour in the lower-half plane. For $t < 0$, only the negative energy pole contributes. Thus, propagation backward in time is associated with negative energies (anti-particles, as we shall see later).
That D_F is a physical quantity follows from the fact that D_F is a Green function. Indeed,

$$(\partial_\mu \partial^\mu + m^2) D_F(x) = -i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} = -i\delta^4(x) \tag{1.4.17}$$

The Feynman propagator is also conveniently expressed in terms of the time-ordered product

$$T(\phi(x)\phi(y)) = \begin{cases} \phi(x)\phi(y) & , \quad x^0 > y^0 \\ \phi(y)\phi(x) & , \quad x^0 < y^0 \end{cases} \tag{1.4.18}$$

as

$$D_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle \tag{1.4.19}$$

Even though it is a physical quantity, it does not arise classically (e.g., you never saw it in electromagnetism; instead you saw advanced and retarded propagators) because it is *complex* (*not* real).

1.5 Symmetries and conservation laws

1.5.1 Noether currents

Consider a Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$ (no need to specify it further). Suppose that this system possesses a symmetry: under the variation of ϕ,

$$\phi \to \phi + \epsilon\delta\phi \tag{1.5.1}$$

where ϵ is small, the Lagrangian density changes by a total divergence,

$$\mathcal{L} \to \mathcal{L} + \epsilon\delta\mathcal{L} \ , \ \delta\mathcal{L} = \partial_\mu \mathcal{F}^\mu \tag{1.5.2}$$

It follows that the action $S = \int d^4x\mathcal{L}$ is invariant (δS only gets a boundary contribution which should vanish in infinite spacetime).
On the other hand, an arbitrary variation of $\mathcal{L}$ may be written as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu\delta\phi \tag{1.5.3}$$

After using the field equation (1.2.31), we may write this as a total divergence,

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right) \tag{1.5.4}$$

Comparing (1.5.2) and (1.5.4), we obtain the locally conserved current (Noether current)

$$\partial_\mu J^\mu = 0 \ , \ J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi - \mathcal{F}^\mu \tag{1.5.5}$$

and the corresponding globally conserved charge

$$\frac{dQ}{dt} = 0 \ , \ Q = \int d^3x J^0 \tag{1.5.6}$$

EXAMPLE 1: TIME TRANSLATION
Under time translation, $x^0 \to x^0 + \epsilon$, the change in ϕ may be found by Taylor expanding,

$$\phi \to \phi + \epsilon\delta\phi + \dots \ , \ \delta\phi = \partial_0\phi \tag{1.5.7}$$

The Lagrangian density also depends on time x^0 implicitly through ϕ and $\partial_\mu\phi$. Its variation is similarly given by

$$\delta\mathcal{L} = \partial_0\mathcal{L} \tag{1.5.8}$$

This is a total divergence if we define

$$\mathcal{F}^0 = \mathcal{L} \ , \ \ \vec{\mathcal{F}} = \vec{0} \tag{1.5.9}$$

Then the Noether current (1.5.5) may be written as

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_0\phi - \delta_0^\mu\mathcal{L} \tag{1.5.10}$$

The conserved charge is

$$Q = \int d^3x J^0 = \int d^3x \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\,\partial_0\phi - \mathcal{L}\right) = \int d^3x\,(\pi\dot{\phi} - \mathcal{L}) = H \tag{1.5.11}$$

the Hamiltonian of the system. Thus, time translation invariance implies that H is constant. This does not appear to be the case in our Universe, because it is expanding. We shall discuss this issue later. In the experiments we perform in our small world, this expansion is not an appreciable effect and may be ignored. We shall discuss this issue further later on.
The Hamiltonian generates translations in time,

$$\delta\phi = \partial_0\phi = i[H,\phi] \tag{1.5.12}$$

For *finite* (not small) ϵ, this exponentiates to (*cf.* eq. (1.3.29))

$$U\phi(t,\vec{x})U^\dagger = \phi(t+\epsilon,\vec{x}) \ , \ \ U = e^{i\epsilon H} \tag{1.5.13}$$

EXAMPLE 2: SPACETIME TRANSLATIONS
The above discussion may be generalized to an arbitrary spacetime translation

$$x^\mu \to x^\mu + \epsilon^\mu \tag{1.5.14}$$

Concentrating in the ν-direction, we have

$$\delta\phi = \partial_\nu\phi \ , \ \ \mathcal{F}_\nu^\mu = \delta_\nu^\mu\mathcal{L} \tag{1.5.15}$$

leading to the Noether current

$$T_\nu^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - \delta_\nu^\mu\mathcal{L} \tag{1.5.16}$$

These are the components of the stress-energy tensor. The conserved charges are

$$P_\nu = \int d^3x T_\nu^0 \tag{1.5.17}$$

(components of the total four-momentum). They generate translations,

$$\delta\phi = \partial_\nu \phi = -i[P_\nu, \phi] \tag{1.5.18}$$

and for finite ϵ^μ, we have (*cf.* eq. (1.3.35))

$$U\phi(x)U^\dagger = \phi(x+\epsilon) \ , \ U = e^{iP_\mu \epsilon^\mu} \tag{1.5.19}$$

EXAMPLE 3: ROTATIONS
A rotation is specified by a vector $\vec{\epsilon}$ pointing in the direction of the axis of rotation. Its magnitude is the angle of rotation. If ϵ is small, under a rotation,

$$\phi \to \phi + \vec{\epsilon} \cdot (\vec{x} \times \vec{\nabla})\phi \tag{1.5.20}$$

The corresponded charges form the angular momentum of the system,

$$\vec{L} = \int d^3x \, \vec{x} \times \vec{\mathcal{P}} \ , \ \mathcal{P}^i = T^{i0} = -\pi \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \tag{1.5.21}$$

where $\vec{\mathcal{P}}$ is the momentum density (consequently $\vec{x} \times \vec{\mathcal{P}}$ is the angular momentum density) and we used (1.5.16).
The components of the angular momentum satisfy the commutation relations

$$[L^i, L^j] = i\epsilon^{ijk} L_k \tag{1.5.22}$$

which is the algebra of the rotation group.
The angular momentum generates rotations:

$$\delta\phi = \vec{\epsilon} \cdot (\vec{x} \times \vec{\nabla})\phi = i[\vec{\epsilon} \cdot \vec{L}, \phi] \tag{1.5.23}$$

This exponentiates: for finite ϵ,

$$U\phi(\vec{x}, t)U^\dagger = \phi(\vec{Ox}, t) \ , \ U = e^{i\vec{\epsilon} \cdot \vec{L}} \tag{1.5.24}$$

where O is the corresponding 3×3 rotation matrix.
EXAMPLE 4: LORENTZ TRANSFORMATIONS
In addition to rotations, the group of Lorentz transformations contains boosts. Under a boost with (small) velocity $\vec{v}$,

$$\phi \to \phi + (t\vec{v} \cdot \vec{\nabla} - \vec{x} \cdot \vec{v}\partial_t)\phi \tag{1.5.25}$$

The corresponding charges are the components of the vector

$$\vec{M} = \vec{P}t - \int d^3x \, \vec{x}\mathcal{H} \tag{1.5.26}$$

They generate boosts: for infinitesimal velocities,

$$\delta\phi = (t\vec{v} \cdot \vec{\nabla} - \vec{x} \cdot \vec{v}\partial_t)\phi = i[\vec{v} \cdot \vec{M}, \phi] \tag{1.5.27}$$

and for finite $\vec{v}$, say $\vec{v} = v\hat{x}$ (boost in x-direction),

$$U\phi(x)U^{-1} = \phi(\Lambda x) \ , \ U = e^{i\zeta M_x} \tag{1.5.28}$$

where ζ is the rapidity (1.1.14) and Λ is the corresponding 4×4 Lorentz transformation matrix (1.1.13).
$\vec{M}$, together with $\vec{L}$ form the 6 generators of Lorentz transformations satisfying the algebra

$$[M^i, M^j] = -i\epsilon^{ijk} L_k \ , \ [L^i, M^j] = i\epsilon^{ijk} M_k \ , \ [L^i, L^j] = i\epsilon^{ijk} L_k \tag{1.5.29}$$

The last two relations show that $\vec{M}$ and $\vec{L}$ transform correctly as vectors in $\mathbb{R}^3$ under rotations.

1.5.2 Discrete spacetime symmetries

A Lorentz group matrix Λ satisfies $\Lambda^T \eta \Lambda = \eta$ (eq. (1.1.11)), where η is the matrix given in (1.1.10). In terms of indices,

$$\eta_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho = \eta_{\lambda\rho} \tag{1.5.30}$$

There are two important parameters that classify these matrices: $\det \Lambda$ and the component $\Lambda^0{}_0$. We have already seen that $\det \Lambda = \pm 1$. To constrain $\Lambda^0{}_0$, set $\nu = \rho = 0$ in (1.5.30),

$$(\Lambda^0{}_0)^2 - \vec{a}^2 = 1 \ , \ a^i = \Lambda^i{}_0 \tag{1.5.31}$$

It follows that $(\Lambda^0{}_0)^2 = 1 + \vec{a}^2 \geq 1$ and so we have two choices:

$$\Lambda^0{}_0 \geq 1 \ , \ \Lambda^0{}_0 \leq -1 \tag{1.5.32}$$

We obtain four disconnected components:

- $\det \Lambda = 1$, $\Lambda^0{}_0 \geq 1$, which includes the identity

$$I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.5.33}$$

  They form a subgroup of matrices that can be continuously reached from the identity. To see that they form a subgroup, consider two matrices Λ and Λ'. Their product has

$$\det(\Lambda\Lambda') = \det \Lambda \det \Lambda' = 1 \tag{1.5.34}$$

  and (from matrix multiplication)

$$(\Lambda\Lambda')^0{}_0 = \Lambda^0{}_0 \Lambda'^0{}_0 + \vec{a} \cdot \vec{b} \ , \ a^i = \Lambda^i{}_0 \ , \ b^i = \Lambda'^i{}_0 \tag{1.5.35}$$

  The Schwarz inequality implies

$$(\vec{a} \cdot \vec{b})^2 \leq \vec{a}^2 \vec{b}^2 = ((\Lambda^0{}_0)^2 - 1)((\Lambda'^0{}_0)^2 - 1) < (\Lambda^0{}_0 \Lambda'^0{}_0)^2 \tag{1.5.36}$$

  where we used (1.5.31). It follows that $|\vec{a} \cdot \vec{b}| < \Lambda^0{}_0 \Lambda'^0{}_0$ and so from (1.5.35) we deduce

$$(\Lambda\Lambda')^0{}_0 > 0 \tag{1.5.37}$$

  This excludes the second possibility in (1.5.32), therefore, $(\Lambda\Lambda')^0{}_0 \geq 1$.

- $\det \Lambda = -1$, $\Lambda^0{}_0 \geq 1$, which includes parity,

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.5.38}$$

- $\det \Lambda = -1$, $\Lambda^0{}_0 \leq -1$, which includes time reversal,

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.5.39}$$

- $\det \Lambda = 1$, $\Lambda^0{}_0 \leq -1$, which includes

$$PT = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.5.40}$$

Parity
P implements $\vec{x} \to -\vec{x}$, $t \to t$. Its action on an operator is through the unitary operator U_P,

$$U_P \phi(\vec{x}, t) U_P^\dagger = \phi(-\vec{x}, t) \tag{1.5.41}$$

satisfying $U_P^2 = I$ (consequently, $U_P^\dagger = U_P$). From the expansion of ϕ (1.2.39), we deduce

$$U_P a(\vec{k}) U_P^\dagger = a(-\vec{k}) \tag{1.5.42}$$

and same for $a^\dagger(\vec{k})$. It follows that U_P commutes with the Hamiltonian (1.3.24). It leaves the vacuum invariant (assuming the vacuum has no interesting structure)

$$U_P|0\rangle = |0\rangle \tag{1.5.43}$$

From (1.5.42) and (1.3.10), we deduce

$$U_P|\vec{p}\rangle = |-\vec{p}\rangle \tag{1.5.44}$$

(degenerate states, since U_P commutes with H).
Time reversal
Let U_T implement time reversal ($t \to -t$, $\vec{x} \to \vec{x}$). We ought to have

$$[U_T, H] = 0 \tag{1.5.45}$$

On the other hand

$$U_T e^{iHt} U_T^{-1} = e^{-iHt} \tag{1.5.46}$$

These two requirements seem to be in conflict with each other. To remedy this, we need to make U_T anti-linear,

$$U_T a|\Psi\rangle = a^* U_T|\Psi\rangle \tag{1.5.47}$$

and anti-unitary,

$$\langle U_T\Psi_1|U_T\Psi_2\rangle = \langle\Psi_2|\Psi_1\rangle = \langle\Psi_1|\Psi_2\rangle^* \tag{1.5.48}$$

Then, even though U_T commutes with H, it *anti-commutes* with iH. Heuristically, U_T anti-commutes with i.

1.5.3 Internal symmetry

Consider two scalar fields ϕ_1, ϕ_2 of the same mass. The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \tfrac{1}{2}m^2(\phi_1^2+\phi_2^2) \tag{1.5.49}$$

In addition to the symmetries studied earlier (Lorentz invariance, etc), this system possesses an internal symmetry which is a rotation in the abstract two-dimensional space of (ϕ_1, ϕ_2). Under the transformation

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to O\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{1.5.50}$$

where O is a 2×2 orthogonal matrix ($O \in O(2)$), the Lagrangian density is invariant ($\delta\mathcal{L} = 0$). To find the Noether current, consider an infinitesimal rotation,

$$O = I + \epsilon\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{1.5.51}$$

We obtain

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\,\delta\phi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\,\delta\phi_2 = \phi_2\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_2 \tag{1.5.52}$$

and the conserved charge

$$Q = \int d^3x(\phi_2\partial_0\phi_1 - \phi_1\partial_0\phi_2) = \int d^3x(\phi_2\pi_1 - \phi_1\pi_2) \tag{1.5.53}$$

It follows that

$$[Q, H] = 0 \tag{1.5.54}$$

so H and Q can be simultaneously diagonalized. To find the eigenstates of Q, define

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\ ,\ \ \psi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{1.5.55}$$

The equal-time commutation relations for ϕ_1 and ϕ_2 may be written in terms of ψ as

$$[\psi(\vec{x},t), \partial_0\psi^\dagger(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \tag{1.5.56}$$

Expanding in modes,

$$\psi = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(e^{-ik\cdot x} b(\vec{k}) + e^{ik\cdot x} c^\dagger(\vec{k})\right) \tag{1.5.57}$$

and its complex (hermitian) conjugate

$$\psi^\dagger = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(e^{-ik\cdot x} c(\vec{k}) + e^{ik\cdot x} b^\dagger(\vec{k})\right) \tag{1.5.58}$$

In terms of modes, the commutation rules read

$$[b(\vec{k}), b^\dagger(\vec{k}')] = [c(\vec{k}), c^\dagger(\vec{k}')] = \delta^3(\vec{k} - \vec{k}') \tag{1.5.59}$$

and all other commutators vanish. The normal-ordered Hamiltonian is

$$H = \int d^3k\omega_k \left(b^\dagger(\vec{k})b(\vec{k}) + c^\dagger(\vec{k})c(\vec{k})\right) \tag{1.5.60}$$

and the conserved charge is

$$Q = \int d^3k \left(b^\dagger(\vec{k})b(\vec{k}) - c^\dagger(\vec{k})c(\vec{k})\right) \tag{1.5.61}$$

Since $[Q, b^\dagger(\vec{k})] = b^\dagger(\vec{k})$ and $[Q, c^\dagger(\vec{k})] = -c^\dagger(\vec{k})$, states created with a $b^\dagger$ have charge $Q = +1$ whereas states created with a $c^\dagger$ have $Q = -1$. This is a system described by a single complex scalar field ψ with Lagrangian density

$$\mathcal{L} = \partial_\mu\psi\partial^\mu\psi^\dagger - m^2\psi^\dagger\psi \tag{1.5.62}$$

which contains particles ($Q = +1$) as well as anti-particles ($Q = -1$). The two have identical properties other than their charge.

1.5.4 Charge conjugation and the CPT theorem

The internal transformations include rotations

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.5.63}$$

which are continuously connected to the identity and in terms of ψ read

$$\psi \to e^{i\theta}\psi \tag{1.5.64}$$

and also ‘big’ transformations, e.g., the reflection

$$O = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{1.5.65}$$

which in terms of ψ reads

$$\psi \to \psi^\dagger \tag{1.5.66}$$

The latter is a discrete symmetry (in addition to P and T) called charge conjugation (C). Let U_C be the unitary operator implementing charge conjugation,

$$U_C \psi U_C^\dagger = \psi^\dagger \tag{1.5.67}$$

Evidently, $U_C^2 = I$ and $U_C^\dagger = U_C$ (as with parity). Using the expansion of ψ and $\psi^\dagger$, we deduce

$$U_C b(\vec{p}) U_C = c(\vec{p}) \ , \ U_C c(\vec{p}) U_C = b(\vec{p}) \tag{1.5.68}$$

Therefore, U_C interchanges patricles and anti-particles. U_C commutes with the Hamiltonian and anti-commutes with the charge,

$$[U_C, H] = 0 \ , \ \{U_C, Q\} = 0 \tag{1.5.69}$$

In a general quantum field theory, P, T and C are not separately symmetries. However, the product CPT is always a symmetry. This can be proved under very broad assumptions (the CPT theorem).

1.6 Symmetry breaking

1.6.1 Classical mechanics

Firstly, consider a ball of mass $m = 1$ moving in a one-dimensional harmonic oscillator potential

$$V(x) = \frac{1}{2}\omega^2 x^2 \tag{1.6.1}$$

It oscillates back and forth rolling up and down in the potential well as determined by its total (conserved) energy

$$E = \frac{p^2}{2} + V(x) \tag{1.6.2}$$

Both the Hamiltonian and the trajectory are symmetric under reflection,

$$x \to -x \tag{1.6.3}$$

The ground state has $E = 0$ (minimum energy) and a simple trajectory ($x = 0$) which is also symmetric under (1.6.3). Now perturb this harmonic oscillator, choosing

$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{1}{4}\lambda x^4 \ , \quad \lambda > 0 \tag{1.6.4}$$

None of the above will change, although we can no longer easily compute the trajectories.

However, if $\omega^2 < 0$, we may write (1.6.4) as

$$V(x) = \frac{1}{4}\lambda(x^2 - v^2)^2 + \text{const.} \ , \quad v = \sqrt{\frac{-\omega^2}{\lambda}} \tag{1.6.5}$$

Evidently, this potential has two minima at $x = \pm v$. The ground state (of minimum energy) may correspond to two different trajectories: $x = +v$ or $x = -v$. Neither one is invariant under (1.6.3). Instead, (1.6.3) maps one trajectory onto the other. For energies slightly above the minimum, we obtain trajectories around either $x = +v$ or $x = -v$. Thus the symmetry is *broken.* It should be noted that the underlying physical laws are still invariant under (1.6.3), however this symmetry is not at all evident to the ball rolling up and down the hill, unless it has sufficiently high energy to go past the middle point $x = 0$.

A low-energy ball rolling around $x = +v$ is oblivious to the existence of the other minimum of the potential (at $x = -v$) and will naturally describe its motion in terms of the coordinate

$$x' = x - v \tag{1.6.6}$$

and write the potential (1.6.5) as

$$V(x') = \lambda v^2 x'^2 + \lambda v x'^3 + \frac{1}{4}\lambda x'^4 \tag{1.6.7}$$

ignoring the irrelevant constant. The frequency of small oscillations is $\Omega = \sqrt{2\lambda v^2}$ and the potential has no apparent symmetry. An astute ball will realize that the potential is symmetric under

$$x' \to -x' \ , \quad v \to -v \tag{1.6.8}$$

but there is really no motivation for the ball to study this symmetry because v is a (God given) parameter of its world.

Let us now place the ball at $x = 0$. This is a point of *unstable* equilibrium. Let it correspond to energy $E = 0$ (as with the potential (1.6.7)). For $E \geq 0$, the trajectories are symmetric under (1.6.3). Now perturb the system by applying a small constant force changing the potential to

$$V(x) \to V(x) + hx \ , \quad h > 0 \tag{1.6.9}$$

We shall observe the response of the system and then let $h \to 0$. This is similar to a *ferromagnet:* if we switch on a uniform magnetic field field H, we break rotational symmetry and the material gets magnetized. When we turn H off, the magnetization does not go to zero; the symmetry has been (*spontaneously*) broken.

Back to our ball: since $V'(0) = h > 0$, the ball sitting at $x = 0$ will roll down to the left and perform oscillatory motion determined by its energy $E = 0$. As $h \to 0$, the original shape of V is restored together with its symmetry, but for the trajectory of the ball, the average position

$$\lim_{h \to 0^+} \langle x \rangle_h = -v \neq 0 \tag{1.6.10}$$

and the symmetry is spontaneously broken.

1.6.2 Quantum mechanics

The above classical picture changes drastically if one includes quantum effects. We may approximate the system by a harmonic oscillator around each minimum, however, the states corresponding to the two minima are not independent due to the possibility of *quantum tunneling*. There is a *unique* ground state which is symmetric under (1.6.3). Therefore, there is no symmetry breaking because $\langle x \rangle = 0$.

This conclusion does not change even when we perturb the potential as in (1.6.9). The effect of the perturbation is to shift the relative energies of the energy levels of the two wells, but the states can still mix! In the limit $h \to 0$, we thus recover $\langle x \rangle_h \to 0$, unlike in the classical system.

1.6.3 Quantum field theory

Consider a field ϕ described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) \tag{1.6.11}$$

where the potential is as above. For small fluctuations around each minimum of the potential, we have a Klein-Gordon field and we may build the Hilbert space acting with creation operators on the ground state. We have two ground states, call them $|\pm\rangle$, with

$$\langle + |\phi| + \rangle = +v \ , \quad \langle - |\phi| - \rangle = -v \tag{1.6.12}$$

Each of the two systems is equivalent to an infinite number of harmonic oscillators. If we perturb the system as in (1.6.9), then for each oscillator we obtain a small $O(h)$ difference in the energies of the two ground states. This vanishes as $h \to 0$, however, for the total energy of the ground state we need to include all oscillators. Sine we have an infinite number of them, the energy difference of the two ground states is infinite no matter how small h is. These states cannot mix and are orthogonal to each other,

$$\langle + | - \rangle = 0 \tag{1.6.13}$$

The corresponding Hilbert spaces obtained from $|\pm\rangle$ by acting with *local* operators are also orthogonal to each other by the same token. This does not change as $h \to 0$. Thus in quantum field theory, we may have

$$\lim_{h \to 0} \langle \phi \rangle_h \neq 0 \tag{1.6.14}$$

unlike in quantum mechanics, because we have an *infinite* number of degrees of freedom.

The above is true only in infinite volume. For a finite volume $\mathcal{V}$, we have a finite number of oscillators and the quantum mechanical result holds,

$$\lim_{h \to 0} \langle \phi \rangle_{h,\mathcal{V}} = 0 \tag{1.6.15}$$

Taking the infinite volume limit,

$$\lim_{\mathcal{V} \to \infty} \lim_{h \to 0} \langle \phi \rangle_{h,\mathcal{V}} = 0 \tag{1.6.16}$$

we do not recover quantum field theory. For the latter, we ought to take the infinite volume limit before we remove the perturbation. Then

$$\lim_{h \to 0} \lim_{\mathcal{V} \to \infty} \langle \phi \rangle_{h,\mathcal{V}} \neq 0 \tag{1.6.17}$$

The two limits are *not* interchangeable.

1.6.4 Goldstone bosons

Let us now consider a system with a continuous symmetry. The Lagrangian density for the complex field ϕ,

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \lambda \left(\phi^* \phi - \frac{v^2}{2} \right)^2 \tag{1.6.18}$$

has a global $U(1)$ symmetry,

$$\phi \to e^{i\theta} \phi \tag{1.6.19}$$

The potential is obtained from the one considered earlier by revoultion and looks like a Mexican hat. The minima are on the circle

$$|\phi| = \frac{v}{\sqrt{2}} \tag{1.6.20}$$

so the classical ground states are

$$\phi = \frac{v}{\sqrt{2}} e^{i\sigma} \quad , \quad \sigma \in [0, 2\pi) \tag{1.6.21}$$

None of these states is invariant under $U(1)$, instead the transformation (1.6.19) takes us from one ground state to another. Therefore, as before, the symmetry is spontaneously broken.
To study the system around one of these ground states, change variables to

$$\phi = \frac{1}{\sqrt{2}} \rho e^{i\sigma} \tag{1.6.22}$$

so that on the ground state

$$\langle \rho \rangle = v \tag{1.6.23}$$

and σ can take any value. The Lagrangian density may be written as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{\lambda}{4} (\rho^2 - v^2)^2 + \frac{1}{2} \rho^2 \partial_\mu \sigma \partial^\mu \sigma \tag{1.6.24}$$

The $U(1)$ transformation translates to $\sigma \to \sigma + \theta$, therefore the $U(1)$ symmetry implies that only *derivatives* of the field σ should appear. It follows that it is not important which ground state we choose - they all lead to the same physics ($\mathcal{L}$).
To see the particle content of this system, shift

$$\rho = v + \rho' \tag{1.6.25}$$

Then

$$\mathcal{L} = \frac{1}{2} \partial_\mu \rho' \partial^\mu \rho' - \lambda v^2 \rho'^2 + \frac{1}{2} \partial_\mu \tilde{\sigma} \partial^\mu \tilde{\sigma} + \text{interactions} \tag{1.6.26}$$

where we also introduced the canonically normalized field

$$\tilde{\sigma} = v\sigma \tag{1.6.27}$$

Thus, ρ' corresponds to a massive scalar particle of mass

$$m = \sqrt{2\lambda v^2} \tag{1.6.28}$$

and $\tilde{\sigma}$ is massless (Goldstone boson).
Notice also that by varying σ (or $\tilde{\sigma}$), we move along a *flat* direction of the potential. Since the potential does not change, σ generates the (spontaneously broken) $U(1)$ symmetry.
The above conclusions may be generalized to the Nambu-Goldstone theorem:
For each spontaneously broken symmetry one obtains a massless excitation (Goldstone boson). Moreover, these Goldstone bosons have derivative couplings.
PROOF: As we know, if the Lagrangian possesses a symmetry, there is a Noether current J_μ which is conserved,

$$\partial_\mu J^\mu = 0 \tag{1.6.29}$$

and the corresponding (conserved) charge generates the transformation,

$$\delta\phi = i[Q\ ,\ \phi]\ \ ,\quad Q = \int d^3x J^0(t, \vec{x}) \tag{1.6.30}$$

In the definition of Q, t is arbitrary because Q is conserved.
If the symmetry is spontaneously broken, then Q does not annihilate the vacuum,

$$Q|0\rangle \neq 0 \tag{1.6.31}$$

The vacuum is not invariant under this symmetry, instead the transformation relates different vacua. Since

$$\langle 0|\delta\phi|0\rangle = i\langle 0\ [Q\ ,\ \phi]|0\rangle \tag{1.6.32}$$

the symmetry is spontaneously broken only if

$$\langle 0|\delta\phi|0\rangle \neq 0 \tag{1.6.33}$$

Now consider the Green function

$$G^\mu(x) = \langle 0|T(J^\mu(x)\phi(0))|0\rangle = \langle 0|\theta(x^0)J^\mu(x)\phi(0) + \theta(-x^0)\phi(0)J^\mu(x)|0\rangle \tag{1.6.34}$$

Due to (1.6.29), its divergence is

$$\partial_\mu G^\mu(x) = \delta(x^0)\langle 0|[J^0(0, \vec{x})\ ,\ \phi(0)]|0\rangle \tag{1.6.35}$$

The right-hand side involves an equal-time commutator which must vanish unless $\vec{x} = \vec{0}$ due to causality. It follows that

$$\partial_\mu G^\mu(x) = \mathcal{C}\delta^4(x) \tag{1.6.36}$$

To find $\mathcal{C}$, we may integrate over spacetime,

$$\mathcal{C} = \int d^3x\langle 0|[J^0(0, \vec{x})\ ,\ \phi(0)]|0\rangle = \langle 0|[Q\ ,\ \phi(0)]|0\rangle = -i\langle 0|\delta\phi|0\rangle \neq 0 \tag{1.6.37}$$

Taking fourier transforms, we have

$$p_\mu \tilde{G}^\mu(p) = -i\mathcal{C} \neq 0 \tag{1.6.38}$$

By Lorentz invariance, we have

$$\tilde{G}^\mu(p) = p^\mu \mathcal{F}(p^2) \tag{1.6.39}$$

therefore

$$p^2 \mathcal{F}(p^2) = -i\mathcal{C} \neq 0 \tag{1.6.40}$$

It follows that

$$\tilde{G}^\mu(p) = -i\mathcal{C}\frac{p^\mu}{p^2} \tag{1.6.41}$$

showing that there exists a massless particle (due to the pole at $p^2 = 0$) which couples to both the current J^μ and the field ϕ. This completes the proof of the theorem.
The reverse is also true (the above proof also runs backwards):
If there exists a massless particle coupling to both J^μ and ϕ, then the symmetry associated with the current J^μ is spontaneously broken.
The above argument also shows that the Goldstone boson remains massless even when quantum effects are included.

1.7 The Schrödinger equation

At low energies, we may use the non-relativistic approximation. Quantum physics is then adequately described by the Schrödinger equation. If we are interested in many-body systems, it is convenient to introduce a field theoretic formulation, sometimes referred to as *second quantization*.
Consider charged spinless particles as in section 1. At low energies, only one kind of particles will be produced, say those with positive charge created by the creation operators $b^\dagger(\vec{k})$. Their anti-particles can be safely ignored. Moreover, if the system is in a cube of (large) size L, the momenta will be discreet,

$$\vec{k} = \left(\frac{2n_x\pi}{L} \ \frac{2n_y\pi}{L} \ \frac{2n_z\pi}{L} \right) \quad , \quad n_x, n_y, n_z \in \mathbb{Z} \tag{1.7.1}$$

The commutation relations turn into

$$[b(\vec{k}) \ , \ b^\dagger(\vec{k}')] = \delta_{\vec{k}\,\vec{k}'} \tag{1.7.2}$$

Instead of (1.5.57) and (1.5.58), we shall define respectively

$$\psi(\vec{x}) = \frac{1}{L^{3/2}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} b(\vec{k}) \ , \ \ \psi^\dagger(\vec{x}) = \frac{1}{L^{3/2}} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} b^\dagger(\vec{k}) \tag{1.7.3}$$

where we fixed time at $t = 0$ (to be considered separately from space) and normalized the fields so that they would obey the commutation relations

$$[\psi(\vec{x}) \ , \ \psi^\dagger(\vec{y})] = \delta^3(\vec{x} - \vec{y}) \tag{1.7.4}$$

replacing (1.5.56).
Define the Hamiltonian by

$$H = -\frac{1}{2m}\int d^3x\, \psi^\dagger \nabla^2 \psi \tag{1.7.5}$$

It can be expressed in terms of creation and annihilation operators as

$$H = \sum_{\vec{k}} E_k b^\dagger(\vec{k}) b(\vec{k}) \tag{1.7.6}$$

where

$$E_k = \frac{\vec{k}^2}{2m} \tag{1.7.7}$$

is the non-relativistic energy.
A basis of states (eigenstates of the Hamiltonian) is obtained by acting with a string of creation operators on the vacuum state $|0\rangle$ defined by

$$b(\vec{k})|0\rangle = 0 \ , \quad \forall \vec{k} \tag{1.7.8}$$

The state

$$|\vec{k}_1, \dots, \vec{k}_N\rangle \equiv b^\dagger(\vec{k}_1)\cdots b^\dagger(\vec{k}_N)|0\rangle \tag{1.7.9}$$

describes N particles with respective momenta $\vec{k}_1, \dots, \vec{k}_N$ and total energy (eigenvalue of H) $E_{k_1} + \dots + E_{k_N}$.
Another useful operator to introduce is the number operator

$$\mathcal{N} \equiv \int d^3x \psi^\dagger(\vec{x})\psi(\vec{x}) = \sum_{\vec{k}} b^\dagger(\vec{k}) b(\vec{k}) \tag{1.7.10}$$

which counts the total number of particles in the system. Indeed,

$$\mathcal{N}|\vec{k}_1, \dots, \vec{k}_N\rangle = N|\vec{k}_1, \dots, \vec{k}_N\rangle \tag{1.7.11}$$

Consider the general time-dependent N-particle state

$$|\Psi(t)\rangle = \sum_{\vec{k}_1,\dots,\vec{k}_N} A(\vec{k}_1, \dots, \vec{k}_N; t)|\vec{k}_1, \dots, \vec{k}_N\rangle \tag{1.7.12}$$

It is easy to see that it obeys the Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{1.7.13}$$

if and only if the N-particle wavefunction $A(\vec{k}_1, \dots, \vec{k}_N; t)$ obeys the Schrödinger equation in momentum space

$$i\frac{\partial}{\partial t}A = (E_{k_1} + \dots + E_{k_N})A \tag{1.7.14}$$

We may also act with a string of $\psi^\dagger$s to form the states

$$|\vec{x}_1, \ldots, \vec{x}_N\rangle \equiv \psi^\dagger(\vec{x}_1) \cdots \psi^\dagger(\vec{x}_N)|0\rangle \tag{1.7.15}$$

representing particles at positions $\vec{x}_1, \ldots, \vec{x}_N$. The general state (1.7.12) may also be expressed in this basis as

$$|\Psi(t)\rangle = \int d^3x_1 \cdots d^3x_N \widetilde{A}(\vec{x}_1, \ldots, \vec{x}_N; t)|\vec{x}_1, \ldots, \vec{x}_N\rangle \tag{1.7.16}$$

where $\widetilde{A}$ is the Fourier transform of A, obeying the Schrödinger equation in position space,

$$i\frac{\partial}{\partial t}\widetilde{A} = -\frac{1}{2m}\left(\nabla^2_{x_1} + \cdots + \nabla^2_{x_N}\right)\widetilde{A} \tag{1.7.17}$$

The above discussion may be generalized to include interactions. Suppose the particles interact via a symmetric two-body potential $V(\vec{x}, \vec{x}')$ (with $V(\vec{x}, \vec{x}') = V(\vec{x}', \vec{x})$).
The Hamiltonian is modified to

$$H = -\frac{1}{2m}\int d^3x\, \psi^\dagger \nabla^2 \psi + V \tag{1.7.18}$$

where the new term

$$V = \frac{1}{2}\int d^3x d^3x' \psi^\dagger(\vec{x})\psi^\dagger(\vec{x}')V(\vec{x}, \vec{x}')\psi(\vec{x}')\psi(\vec{x}) \tag{1.7.19}$$

describes the interactions.
To see that this is the right form, let us act on the state (1.7.15). We obtain

$$V|\vec{x}_1, \ldots, \vec{x}_N\rangle = \frac{1}{2}\sum_{m,n\neq m} V(\vec{x}_m, \vec{x}_n)|\vec{x}_1, \ldots, \vec{x}_N\rangle \tag{1.7.20}$$

which is the right form of the potential energy.
Moreover, the general state (1.7.16) obeys the Schrödinger equation (1.7.13) with H given by (1.7.18) if and only if the wavefunction $\widetilde{A}$ obeys the Schrödinger equation

$$i\frac{\partial}{\partial t}\widetilde{A} = -\frac{1}{2m}\left(\nabla^2_{x_1} + \cdots + \nabla^2_{x_N}\right)\widetilde{A} + \frac{1}{2}\sum_{m,n\neq m} V(\vec{x}_m, \vec{x}_n)\widetilde{A} \tag{1.7.21}$$

1.8 Finite temperature

1.8.1 Statistical mechanics

To describe phenomena at a finite temperature, we need statistical mechanics. Suppose that the system is at temperature T and held at chemical potential μ. Define

$$\beta = \frac{1}{k_B T} \tag{1.8.1}$$

The grand partition function is

$$\mathcal{Z} = \mathrm{Tr} e^{-\beta(H-\mu\mathcal{N})} = \sum_{\vec{k}_1,\vec{k}_2,\dots} \langle \vec{k}_1, \vec{k}_2, \dots | e^{-\beta(H-\mu\mathcal{N})} | \vec{k}_1, \vec{k}_2, \dots \rangle \tag{1.8.2}$$

Ensemble averages of operators $\mathcal{A}$ are given by

$$\langle \mathcal{A} \rangle = \frac{1}{\mathcal{Z}} \mathrm{Tr} \left(e^{-\beta(H-\mu\mathcal{N})} \mathcal{A} \right) \tag{1.8.3}$$

This expression suggests that we treat $H - \mu\mathcal{N}$ as an effective (*grand canonical*) Hamiltonian and $e^{-\tau(H-\mu\mathcal{N})}$ as an evolution operator in imaginary time τ. Then, e.g.,

$$\psi(\vec{x}, \tau) \equiv e^{\tau(H-\mu\mathcal{N})} \psi(\vec{x}) e^{-\tau(H-\mu\mathcal{N})} \ , \ \ \psi^\dagger(\vec{x}, \tau) \equiv e^{\tau(H-\mu\mathcal{N})} \psi^\dagger(\vec{x}) e^{-\tau(H-\mu\mathcal{N})} \tag{1.8.4}$$

Notice that these two fields are conjugate of each other only if τ is imaginary. They obey the Heisenberg equation,

$$\partial_\tau \mathcal{A} = [H - \mu\mathcal{N} \ , \ \mathcal{A}] \tag{1.8.5}$$

We obtain

$$\begin{aligned} \partial_\tau \psi(\vec{x}, \tau) &= \left[\frac{1}{2m} \nabla^2 + \mu - \int d^3x' \psi^\dagger(\vec{x}', \tau) \psi(\vec{x}', \tau) V(\vec{x}, \vec{x}') \right] \psi(\vec{x}, \tau) \\ \partial_\tau \psi^\dagger(\vec{x}, \tau) &= - \left[\frac{1}{2m} \nabla^2 + \mu - \int d^3x' \psi^\dagger(\vec{x}', \tau) \psi(\vec{x}', \tau) V(\vec{x}, \vec{x}') \right] \psi^\dagger(\vec{x}, \tau) \end{aligned} \tag{1.8.6}$$

Define the propagator as the ensemble average

$$D(\vec{x}, \tau; \vec{x}', \tau') \equiv -\langle T[\psi(\vec{x}, \tau) \psi^\dagger(\vec{x}', \tau')] \rangle \tag{1.8.7}$$

It possesses a remarkable property: it is periodic in the time variables with period β. To see this, set $\tau = 0$ and let $0 < \tau' < \beta$. Then

$$\begin{aligned} D(\vec{x}, 0; \vec{x}', \tau') &= -\frac{1}{\mathcal{Z}} \mathrm{Tr} \left(\psi^\dagger(\vec{x}', \tau') \psi(\vec{x}, 0) e^{-\beta(H-\mu\mathcal{N})} \right) \\ &= -\frac{1}{\mathcal{Z}} \mathrm{Tr} \left(\psi^\dagger(\vec{x}', \tau') e^{-\beta(H-\mu\mathcal{N})} \psi(\vec{x}, \beta) \right) \\ &= D(\vec{x}, \beta; \vec{x}', \tau') \end{aligned} \tag{1.8.8}$$

The above argument applies to general τ and also τ' because in most cases of interest the propagator is a function of the difference $\tau - \tau'$. It has the Fourier representation

$$D(\vec{x}, \tau; \vec{x}', \tau') = \frac{1}{\beta} \sum_n e^{-i\omega_n(\tau-\tau')} D(\vec{x}, \vec{x}'; \omega_n) \ , \quad \omega_n = \frac{2\pi n}{\beta} \tag{1.8.9}$$

in terms of the *Matsubara* frequencies ω_n. Inversely,

$$D(\vec{x}, \vec{x}'; \omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} D(\vec{x}, \tau; \vec{x}', 0) \tag{1.8.10}$$

1.8.2 Bose-Einstein condensation

Let us ignore interactions (ideal gas). Then we are summing over common eigenstates of H and $\mathcal{N}$, therefore,

$$\begin{aligned}\mathcal{Z} &= \sum_{N=0}^{\infty} \sum_{\vec{k}_1,\dots,\vec{k}_N} e^{-\beta(E_{k_1}+\dots+E_{k_N}-\mu N)} = \prod_{\vec{k}} \sum_{N=0}^{\infty} \left(e^{-\beta(E_k-\mu)}\right)^N \\ &= \prod_{\vec{k}} \left(1-e^{-\beta(E_k-\mu)}\right)^{-1} \end{aligned} \tag{1.8.11}$$

We deduce the thermodynamic potential

$$\Omega = -pV = E - TS - \mu N = -\frac{1}{\beta}\ln\mathcal{Z} = \frac{1}{\beta}\sum_{\vec{k}} \ln\left(1-e^{-\beta(E_k-\mu)}\right) \tag{1.8.12}$$

where $V = L^3$ is the volume.
The mean number (ensemble average) of particles is

$$\langle\mathcal{N}\rangle = \frac{1}{\mathcal{Z}}\mathrm{Tr}\left(e^{-\beta(H-\mu\mathcal{N})}\mathcal{N}\right) = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} \tag{1.8.13}$$

Using the explicit form of Ω, we obtain

$$\langle\mathcal{N}\rangle = \sum_{\vec{k}} \frac{1}{e^{\beta(E_k-\mu)}-1} \tag{1.8.14}$$

This may also be obtained from the definition (1.7.10). We have

$$\langle\mathcal{N}\rangle = -\int d^3x D(\vec{x},\tau;\vec{x},\tau-\epsilon) \tag{1.8.15}$$

where $\epsilon \to 0^+$.
From the equation satisfied by ψ,

$$\partial_\tau\psi = \frac{1}{2m}\nabla^2\psi + \mu\psi \tag{1.8.16}$$

we easily deduce the Fourier transform of the propagator,

$$\tilde{D}(\vec{k},\omega_n) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} D(\vec{x},\vec{0};\omega_n) = -\frac{1}{i\omega_n - E_k + \mu} \tag{1.8.17}$$

It follows that

$$\langle\mathcal{N}\rangle = -\frac{1}{\beta}\sum_{n,\vec{k}} e^{i\omega_n\epsilon}\tilde{D}(\vec{k},\omega_n) = \frac{1}{\beta}\sum_{n,\vec{k}} e^{i\omega_n\epsilon}\frac{1}{i\omega_n - E_k+\mu} \tag{1.8.18}$$

With the aid of a contour integral,

$$\oint_C \frac{dz}{2\pi i}\,\frac{e^{\epsilon z}}{(e^{\beta z}-1)(z-E_k+\mu)} \tag{1.8.19}$$

we can show that

$$\lim_{\epsilon\to 0^+}\sum_n \frac{e^{i\omega_n\epsilon}}{i\omega_n - E_k+\mu} = -\frac{\beta}{e^{\beta(E_k-\mu)}-1} \tag{1.8.20}$$

showing that (1.8.18) agrees with the earlier result (1.8.14).
In the large volume limit ($L\to\infty$), we may approximate

$$\sum_{\vec{k}} \approx L^3\int\frac{d^3k}{(2\pi)^3} \tag{1.8.21}$$

We deduce

$$pV = -\Omega = \frac{L^3}{4\pi^2}(2m)^{3/2}\frac{2}{3}\int_0^\infty dE_k \frac{E_k^{3/2}}{e^{\beta(E_k-\mu)}-1} \tag{1.8.22}$$

Also using

$$E = TS - pV + \mu N\ ,\ \ S = -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\mu}\ ,\ \ p = -\left(\frac{\partial\Omega}{\partial V}\right)_{T,\mu} \tag{1.8.23}$$

we deduce

$$pV = \frac{2}{3}E \tag{1.8.24}$$

which is the equation of state of an ideal Bose gas.
Moreover,

$$\frac{N}{V} = \frac{1}{4\pi^2}(2m)^{3/2}\int_0^\infty dE_k\frac{E_k^{1/2}}{e^{\beta(E_k-\mu)}-1} \tag{1.8.25}$$

showing that $\mu \le E_k$ (otherwise we would obtain negative contributions to the mean number for certain energy levels). Since the spectrum of energies extends down to zero, we must have

$$\mu\le 0 \tag{1.8.26}$$

At high temperatures ($\beta\to 0$),

$$\mu\approx\frac{1}{\beta}\ln\left(\frac{N}{V}\left(\frac{2\pi\beta}{m}\right)^{3/2}\right) \tag{1.8.27}$$

which is the classical value of the chemical potential.
As we lower the temperature, we hit a critical temperature T_c at which $\mu = 0$, given through

$$\frac{N}{V} = \frac{1}{4\pi^2}(2m)^{3/2}\int_0^\infty dE_k\frac{E_k^{1/2}}{e^{\beta E_k}-1} \tag{1.8.28}$$

We have

$$\frac{N}{V} = \frac{1}{4\pi^2}\left(\frac{2m}{\beta}\right)^{3/2}\int_0^\infty dx\frac{x^{1/2}}{e^{\beta x}-1} = \frac{1}{4\pi^2}\left(\frac{2m}{\beta}\right)^{3/2}\Gamma(\frac{3}{2})\zeta(\frac{3}{2}) \tag{1.8.29}$$

We obtain

$$k_BT_c = \frac{1}{\beta} = \frac{1}{2m}\left(\frac{4\pi^2}{\Gamma(\frac{3}{2})\zeta(\frac{3}{2})}\frac{N}{V}\right)^{2/3} = \frac{3.31}{m}\left(\frac{N}{V}\right)^{2/3} \tag{1.8.30}$$

Below T_c, μ is negligible and the above results are still valid, except N represents the total number N_1 of particles with energies $E_k > 0$,

$$\frac{N_1}{V} = \frac{1}{4\pi^2}\,(2mk_BT_c)^{3/2}\,\Gamma(\frac{3}{2})\zeta(\frac{3}{2}) = \frac{N}{V}\left(\frac{T}{T_c}\right)^{3/2} \tag{1.8.31}$$

The rest of the particles, N_0, are all in the ground state,

$$\frac{N_0}{V} = \frac{N-N_1}{V} = \frac{N}{V}\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right] \tag{1.8.32}$$

The total energy is given by (1.8.22) and (1.8.24) with $\mu = 0$,

$$E = \frac{L^3}{4\pi^2}(2m)^{3/2}\frac{2}{3}\int_0^\infty dE_k\,\frac{E_k^{3/2}}{e^{\beta E_k}-1} \tag{1.8.33}$$

We obtain

$$E = \frac{\zeta(\frac{5}{2})\Gamma(\frac{5}{2})}{\zeta(\frac{3}{2})\Gamma(\frac{3}{2})}NK_BT\left(\frac{T}{T_c}\right)^{3/2} \tag{1.8.34}$$

and the constant volume heat capacity below the critical temperature is

$$C_V = \frac{\partial E}{\partial T} = \frac{5}{2}\frac{\zeta(\frac{5}{2})\Gamma(\frac{5}{2})}{\zeta(\frac{3}{2})\Gamma(\frac{3}{2})}NK_B\left(\frac{T}{T_c}\right)^{3/2} = 1.925Nk_B\left(\frac{T}{T_c}\right)^{3/2} \tag{1.8.35}$$

Near the critical temperature we can calculate the chemical potential as an expansion,

$$\mu(T) = \mu(T_c) + \mu'(T_c)(T-T_c) + \frac{1}{2}\mu''(T_c)(T-T_c)^2 + \dots \tag{1.8.36}$$

We already know $\mu(T_c) = 0$. Right above T_c, we obtain from (1.8.25)

$$\mu'(T_c) = 0\;,\quad \mu''(T_c) = -\frac{9}{2}\left(\frac{\zeta(\frac{3}{2})\Gamma(\frac{3}{2})}{\pi}\right)^2\frac{k_B}{T_c} \tag{1.8.37}$$

showing that μ has a discontinuous second derivative at $T = T_c$. This discontinuity gives rise to a discontinuity in the first derivative of the heat capacity. Indeed, the energy near the critical temperature is (viewed as a function of μ)

$$E = E(\mu=0) + \left.\frac{\partial E}{\partial \mu}\right|_{\mu=0}\mu + \dots \tag{1.8.38}$$

where $E(\mu = 0)$ is the energy at $T = T_c$ and

$$\left.\frac{\partial E}{\partial \mu}\right|_{\mu=0} = -\frac{3}{2}\left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V} = \frac{3}{2}N \tag{1.8.39}$$

where we used the equation of state and $pV = -\Omega$.
We deduce the discontinuity

$$\Delta\frac{\partial C_V}{\partial T} = \Delta\frac{\partial^2 E}{\partial T^2} = \frac{3}{2}N\mu''(T_c) = -\frac{27}{4}\left(\frac{\zeta(\frac{3}{2})\Gamma(\frac{3}{2})}{\pi}\right)^2\frac{Nk_B}{T_c} = -3.66\frac{Nk_B}{T_c} \tag{1.8.40}$$

showing that the gas undergoes a second-order phase transition at $T = T_c$ (Bose-Einstein condensation).
To understand this phase transition better, recall the expansion of ψ in modes (1.7.3).
We have

$$N_0 = \langle b^\dagger(\vec{0})b(\vec{0})\rangle \tag{1.8.41}$$

If the expectation value is calculated in the ground state $|0\rangle$, then $N_0 = 0$, which is the case for $T > T_c$. Below T_c, it appears that $N_0 \neq 0$. In fact $N_0 \sim \mathcal{O}(N)$ even though $[b(\vec{0})\ ,\ b^\dagger(\vec{0})] = 1$. This shows that the system has switched to a different ground state.
To see why, let us concentrate on the contribution of the zero modes,

$$H - \mu\mathcal{N} = -\mu b^\dagger(\vec{0})b(\vec{0}) + \frac{g}{2L^3}(b^\dagger(\vec{0})b(\vec{0}))^2 + \dots \tag{1.8.42}$$

where we used the Hamiltonian (1.7.18) including interactions (1.7.19) and defined

$$gL^3 = \int d^3x d^3x' V(\vec{x}, \vec{x}') \tag{1.8.43}$$

with $g > 0$ (repulsive interaction, e.g., a contact interaction $V(\vec{x}, \vec{x}') = g\delta^3(\vec{x} - \vec{x}')$).
Above the critical temperature ($\mu < 0$), the minimum of (1.8.42) is attained for $b^\dagger(\vec{0})b(\vec{0}) = 0$, showing that $N_0 = \langle b^\dagger(\vec{0})b(\vec{0})\rangle = 0$.
Below the critical temperature ($\mu > 0$, no longer interpreted as a chemical potential), the minimum of (1.8.42) is at

$$b^\dagger(\vec{0})b(\vec{0}) = \frac{\mu L^3}{g} \tag{1.8.44}$$

leading to spontaneous symmetry breaking. The zero mode condenses,

$$N_0 = \langle b^\dagger(\vec{0})b(\vec{0})\rangle = \frac{\mu L^3}{g} \neq 0 \tag{1.8.45}$$

breaking the $U(1)$ symmetry of the system. This gives rise to a Goldstone mode (massless excitation) which is intimately related to the phenomenon of superfluidity.

1.8.3 Superfluidity

Parametrize ψ in terms of real fields,

$$\psi = \sqrt{\rho}e^{i\sigma} \tag{1.8.46}$$

The vacuum expectation value of ρ below the critical temperature is

$$v = \langle\rho\rangle = \frac{N_0}{L^3} = \frac{\mu}{g} \tag{1.8.47}$$

For definiteness, consider a contact potential $V(\vec{x},\vec{x}') = g\delta^3(\vec{x}-\vec{x}')$. Then the field equations (1.8.6) become (Gross-Pitaevskii eq.)

$$\partial_\tau \psi = \left[\frac{1}{2m}\nabla^2 + \mu - g\rho\right]\psi \ , \ \ \partial_\tau \psi^\dagger = -\left[\frac{1}{2m}\nabla^2 + \mu - g\rho\right]\psi^\dagger \tag{1.8.48}$$

or in terms of ρ and σ,

$$i\partial_\tau \sigma = \mu - g\rho - \frac{1}{2m}(\vec{\nabla}\sigma)^2 \ , \quad \partial_\tau \rho = i\frac{\rho}{m}\nabla^2\sigma \tag{1.8.49}$$

where we ignored gradients of ρ.
These two field equations yield

$$\partial_\tau^2 \sigma + \frac{gv}{m}\nabla^2\sigma = -\frac{g}{m}\rho'\nabla^2\sigma + \frac{i}{2m}\partial_\tau(\vec{\nabla}\sigma)^2 \tag{1.8.50}$$

where $\rho = v + \rho'$. The right-hand side represents interactions. Ignoring them, the left-hand side yields a Klein-Gordon equation (in Euclidean space) for σ with dispersion relation $\omega = c|\vec{k}|$, where $c = \sqrt{gv/m}$, showing that σ is indeed a massless (Goldstone) mode.
Moreover, we obtain a steady-state solution (setting $\partial_\tau\rho = 0$),

$$\vec{\nabla}\cdot\vec{J} = 0 \ , \quad \vec{J} = \frac{i}{2m}\left[\psi\vec{\nabla}\psi^\dagger - \psi^\dagger\vec{\nabla}\psi\right] \approx \frac{v}{m}\vec{\nabla}\sigma \tag{1.8.51}$$

showing that a configuration with a uniform density ρ can support a divergenceless supercurrent $\vec{J}$. This would not be possible if σ were massive. A normal (dissipative) current also exists corresponding to the massive fluctuations ρ'.
It should be emphasized that even though σ looks like a Klein-Gordon field, it is only defined modulus 2π. This is gives rise to interesting excitations, vortices; as one moves around the vortex center, ϕ changes by a multiple of 2π. (E.g., if $\langle\sigma\rangle = \theta + f(r)$ in polar coordinates (r,θ), σ will change by 2π if we move around the vortex at $r = 0$. The function $f(r)$ can be found by solving (1.8.48).)

UNIT 2

Spin $1/2$

As we saw, the Klein-Gordon equation led to a quantum field theory of scalar (spin-0) particles. What equation leads to a quantum field theory of spin-1/2 particles? The answer was given by Dirac before the question about quantum field theory was asked; Dirac had to resort to a "hole theory" to interpret his equation. Quantum field theory has no need for a "hole theory" or any other weird interpretation.

2.1 Symmetry groups again

2.1.1 Rotations

To motivate Dirac's suggestion, we start with a review of the familiar non-relativistic spin-1/2 particle and the group of rotations. Under an infinitesimal rotation, a scalar field ϕ changes by

$$\phi \to \phi + \vec{\theta} \cdot (\vec{x} \times \vec{\nabla})\phi \tag{2.1.1}$$

Thus, the angular momentum

$$\vec{L} = -i\vec{x} \times \vec{\nabla} \tag{2.1.2}$$

generates rotations in the sense that

$$\delta\phi = i[\vec{\theta} \cdot \vec{L}, \phi] \tag{2.1.3}$$

This operation exponentiates for finite rotations,

$$\phi \to U\phi U^\dagger \quad , \quad U = e^{-i\vec{\theta}\cdot\vec{L}} \tag{2.1.4}$$

U is a unitary operator, since $\vec{L}$ is Hermitian (an observable). The components of $\vec{L}$ obey the algebra

$$[L^i, L^j] = i\epsilon^{ijk} L_k \tag{2.1.5}$$

All objects are classified by their transformation properties under rotations (scalars, vectors, etc). The simplest non-trivial object is a spinor. It transforms under the spinor (two-dimensional) representation of the algebra realized by the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1.6}$$

Indeed, we have

$$[S^i, S^j] = i\epsilon^{ijk} S_k \ , \ \vec{S} = \tfrac{1}{2}\vec{\sigma} \tag{2.1.7}$$

Finite rotations are implemented with the 2×2 unitary matrix

$$U = e^{-i\vec{\theta}\cdot\vec{\sigma}/2} \tag{2.1.8}$$

The next irreducible representation is the adjoint representation acting on vectors $\vec{V}$. It is realized by the 3×3 matrices

$$(\hat{L}^i)_{jk} = i\epsilon^i{}_{jk} \tag{2.1.9}$$

It can easily be checked that they satisfy the algebra (2.1.5). They rotate vectors: an infinitesimal rotation acts as

$$\delta V^i = -i(\vec{\theta}\cdot\vec{\hat{L}})^i_j V^j = \epsilon^{ijk}\theta_j V_k \tag{2.1.10}$$

which may also be written in the more familiar form from Euclidean geometry,

$$\delta\vec{V} = \vec{\theta} \times \vec{V} \tag{2.1.11}$$

Notice that $\vec{\sigma}$ is a vector: using (2.1.3) to define the "change" in $\vec{\sigma}$ under an infinitesimal rotation, where $\vec{L} = \vec{S}$ in the spinor representation (eq. (2.1.7)), we obtain

$$\delta\sigma^i \equiv i[\vec{\theta}\cdot\vec{S}, \sigma^i] = \epsilon^{ijk}\theta_j\sigma_k \tag{2.1.12}$$

For the same reason, $\vec{L}$ is a vector, as well as any three-component object V^i ($i = 1,2,3$) satisfying

$$[L^i, V^j] = (\hat{L}^i)^{jk} V_k = i\epsilon^{ijk} V_k \tag{2.1.13}$$

Other irreducible representations are $(2s+1)$-dimensional, where s is a half-integer (the spin).

2.1.2 Lorentz transformations

In four dimensional spacetime, in addition to rotations, we have boosts. Under a boost with velocity $\vec{v}$, a scalar field transforms as

$$\phi \to \phi + (t\vec{v}\cdot\vec{\nabla} - \vec{x}\cdot\vec{v}\partial_t)\phi \tag{2.1.14}$$

This is generated by

$$\vec{M} = -i(t\vec{\nabla} - \vec{x}\partial_t) \tag{2.1.15}$$

in the sense that $\delta\phi = i[\vec{v}\cdot\vec{M}, \phi]$. Thus the Lorentz group has six generators ($\vec{L}$ and $\vec{M}$). They satisfy the algebra

$$[M^i, M^j] = -i\epsilon^{ijk} L_k \ , \ [L^i, M^j] = i\epsilon^{ijk} M_k \ , \ [L^i, L^j] = i\epsilon^{ijk} L_k \tag{2.1.16}$$

A finite Lorentz transformation is implemented with the unitary operator

$$U = e^{-i\vec{\theta}\cdot\vec{L}+i\vec{v}\cdot\vec{M}} \tag{2.1.17}$$

The above may be written in a more compact form if we introduce

$$J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2.1.18}$$

There are six independent components of $J_{\mu\nu}$. They are identified as

$$J_{ij} = \epsilon_{ijk} L^k \ \ , \ \ J_{0i} = M_i \tag{2.1.19}$$

The Lorentz algebra (2.1.16) can be written as

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho}) \tag{2.1.20}$$

as is easily verified.
We would like to find the representations of this algebra as we did with angular momentum. To this end, define

$$\vec{J}_R = \tfrac{1}{2}(\vec{L} + i\vec{M}) \ \ , \ \ \vec{J}_L = \tfrac{1}{2}(\vec{L} - i\vec{M}) \tag{2.1.21}$$

where the subscript R (L) stands for “right” (“left”) - to be justified later. The Lorentz algebra reads

$$[J_L^i, J_L^j] = i\epsilon^{ijk} J_{Lk} \ \ , \ \ [J_R^i, J_R^j] = i\epsilon^{ijk} J_{Rk} \ \ , \ \ [J_L^i, J_R^j] = 0 \tag{2.1.22}$$

i.e., we obtain two angular momentum algebras that don’t talk to (commute with) each other. Thus, an irreducible representation of the Lorentz group is a product of two irreducible representations of the rotation group. It is labeled by the pair (s_R, s_L) of the respective spins.
The simplest representation is $(0,0)$ - the scalar.
An important representation is the *adjoint* acting on four-vectors. It is realized by the 4×4 matrices

$$(\hat{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) \tag{2.1.23}$$

It is easily checked that $\hat{J}^{\mu\nu}$ satisfies the algebra (2.1.20). A four-vector V^μ satisfies

$$[V^\mu, J^{\rho\sigma}] = (\hat{J}^{\rho\sigma})^\mu{}_\nu V^\nu = i(\eta^{\mu\rho} V^\sigma - \eta^{\mu\sigma} V^\rho) \tag{2.1.24}$$

generalizing eq. (2.1.13) and reducing to it in the case of rotations.

2.2 Weyl spinors

The simplest non-trivial representations are $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. The corresponding spinors are ψ_R and ψ_L (right- and left-handed Weyl spinors), respectively. By definition, the action of $\vec{J}_{R,L}$ on ψ_R is given by

$$\vec{J}_R \psi_R = \tfrac{1}{2}\vec{\sigma}\psi_R \ \ , \ \ J_L \psi_R = 0 \tag{2.2.1}$$

This leads to the action of rotations and boosts, respectively,

$$\vec{L}\psi_R = (\vec{J}_R + \vec{J}_L)\psi_R = \tfrac{1}{2}\vec{\sigma}\psi_R \ , \ \vec{M}\psi_R = -i(\vec{J}_R - \vec{J}_L)\psi_R = -\tfrac{i}{2}\vec{\sigma}\psi_R \quad (2.2.2)$$

Thus, under a finite Lorentz transformation,

$$\psi_R \to U_R\psi_R \ , \ U_R = e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}++\frac{1}{2}\vec{v}\cdot\vec{\sigma}} \quad (2.2.3)$$

where we used eq. (2.1.17) - see also (2.1.8).
Working similarly, we find the effect of a general Lorentz transformation on ψ_L,

$$\psi_L \to U_L\psi_L \ , \ U_L = e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}-\frac{1}{2}\vec{v}\cdot\vec{\sigma}} \quad (2.2.4)$$

It is useful to construct quantities which transform nicely, such as vectors and scalars. These will represent physical quantities. One such quantity is

$$V^\mu = \psi_R^\dagger\sigma^\mu\psi_R \ , \ \sigma^\mu = (I, \vec{\sigma}) \quad (2.2.5)$$

Let us show that V^μ is a four-vector. To this end, we consider infinitesimal rotations and boosts. Under a rotation, it is clear from (2.2.3) that $V^0 = \psi_R^\dagger\psi_R$ doesn't change. The spatial components of V^μ transform as

$$\begin{aligned} \vec{V} &\to \psi_R^\dagger(1 + \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + o(\theta^2))\vec{\sigma}(1 - \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + o(\theta^2))\psi_R \\ &= \vec{V} + \tfrac{i}{2}\theta_i\psi_R^\dagger[\sigma^i, \vec{\sigma}]\psi_R + o(\theta^2) \end{aligned} \quad (2.2.6)$$

Using (2.1.7), we deduce

$$\delta\vec{V} = \vec{\theta}\times\vec{V} \quad (2.2.7)$$

showing that $\vec{V}$ is a vector in three dimensions (*cf.* eq. (2.1.11)). Notice that this property was a direct consequence of the fact that $\vec{\sigma}$ transforms as a vector, which is the content of eq. (2.1.7).
Under an infinitesimal boost,

$$V^0 \to \psi_R^\dagger(1 + \tfrac{1}{2}\vec{v}\cdot\vec{\sigma} + o(v^2))^2\psi_R = V^0 + \vec{v}\cdot\vec{V} + o(v^2) \quad (2.2.8)$$

and

$$\begin{aligned} \vec{V} &\to \psi_R^\dagger(1 + \tfrac{1}{2}\vec{v}\cdot\vec{\sigma} + o(v^2))\vec{\sigma}(1 + \tfrac{1}{2}\vec{v}\cdot\vec{\sigma} + o(v^2))\psi_R \\ &= \vec{V} + \vec{v}V^0 + o(v^2) \end{aligned} \quad (2.2.9)$$

where we used $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$. Thus, V^μ transforms correctly under boosts as well.[1]
It follows that V^μ is a four-vector.
To construct a Lagrangian density, we need to turn ψ_R into a *field* and then find a scalar. Then a Lorentz transformation will not only act on the spinor indices of ψ_R but also on its argument. It is easy to show that $\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R$ is a scalar. For the Lagrangian density we need a *real* quantity, so we define

$$\mathcal{L} = i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R \quad (2.2.10)$$

[1] Note that the factor $\gamma = (1 - v^2)^{-1/2}$ is missing because $\gamma = 1 + o(v^2)$.

By treating ψ_R and $\psi_R^\dagger$ as independent variablem the field equation is simple,

$$\sigma^\mu \partial_\mu \psi_R = 0 \tag{2.2.11}$$

This is the Weyl equation. Notice that

$$\det \sigma^\mu \partial_\mu = \begin{vmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{vmatrix} = \partial_\mu \partial^\mu \tag{2.2.12}$$

It follows that a Weyl spinor also satisfies the *massless* Klein-Gordon equation

$$\partial_\mu \partial^\mu \psi_R = 0 \tag{2.2.13}$$

which admits plane-wave solutions

$$\psi_R = u\, e^{-ip\cdot x} \tag{2.2.14}$$

where u is a spinor and $p^2 = 0$ (massless particle). Suppose the energy is positive. We are free to choose axes so that the momentum is along the z-axis. Then $p^\mu = (p, 0, 0, p)$. Plugging into the Weyl equation, we obtain

$$p_\mu \sigma^\mu u = p(I - \sigma^3)u = 0 \tag{2.2.15}$$

Therefore, $\sigma^3 u = u$, whose solution is $u = \binom{1}{0}$. It follows that the spin is along the momentum, i.e., the helicity is positive, i.e., this is a right-handed particle, which justifies our use of the subscript R in ψ_R.
The above discussion may be repeated for the left-handed spinors ψ_L. Under Lorentz tranformations,

$$\vec{L}\psi_L = \tfrac{1}{2}\vec{\sigma}\psi_L \ \ , \ \ \vec{M}\psi_L = \tfrac{i}{2}\vec{\sigma}\psi_L \tag{2.2.16}$$

It follows that

$$V^\mu = \psi_L^\dagger \bar{\sigma}^\mu \psi_L \ \ , \ \ \bar{\sigma}^\mu = (I, -\vec{\sigma}) \tag{2.2.17}$$

is a vector. The Lagrangian density (a scalar) is

$$L = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \tag{2.2.18}$$

leading to the field equation

$$\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \tag{2.2.19}$$

The solutions also solve the massless Klein-Gordon equation.[2] A plane-wave solution $\psi_L = u\, e^{-ip\cdot x}$ with momentum along the z-axis satisfies the Weyl equation if $u = \binom{0}{1}$. Thus, ψ_L is a left-handed spinor (of negative helicity).
Finally, two important observations:

- The Lagrangian density for a Weyl spinor is linear in the derivatives, unlike in the Klein-Gordon case, where it is quadratic. This is because there is no Lorentz-invariant operator linear in the derivatives that acts non-trivially on a scalar field.
- ψ_R and ψ_L are related to each other by parity ($P : \psi_R \leftrightarrow \psi_L$). This follows from the action of P on Lorentz generators, $P : \vec{L} \to \vec{L}$, $\vec{M} \to -\vec{M}$ and so, on account of (2.1.21), $P : \vec{J}_R \leftrightarrow \vec{J}_L$.

[2] This is a consequence of $\det \sigma \cdot \partial = \partial^2$; notice also that $(\sigma \cdot \partial)(\bar{\sigma} \cdot \partial) = \partial^2$.

2.3 The Dirac equation

2.3.1 The equation

Let us combine ψ_R and ψ_L by adding their respective Lagrangian densities,

$$\mathcal{L} = i\psi_R^\dagger \sigma \cdot \partial \psi_R + i\psi_L^\dagger \bar{\sigma} \cdot \partial \psi_L \qquad (2.3.1)$$

We may add a mixed quadratic term, $\psi_R^\dagger \psi_L$, which is Lorentz-invariant (this follows easily from (2.2.3) and (2.2.4)).[3] Actually, only the real part contributes to the Lagrangian. This is implemented by adding the complex conjugate of $\psi_R^\dagger \psi_L$. We obtain the more general Lagrangian density

$$\mathcal{L} = i\psi_R^\dagger \sigma \cdot \partial \psi_R + i\psi_L^\dagger \bar{\sigma} \cdot \partial \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R) \qquad (2.3.2)$$

where m is an arbitrary constant whose physical significance is yet to be determined. The field equations are easily deduced,

$$\begin{aligned} i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla})\psi_R - m\psi_L &= 0 \\ i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla})\psi_L - m\psi_R &= 0 \end{aligned} \qquad (2.3.3)$$

They reduce to the respective Weyl equations (2.2.11) and (2.2.19) in the limit $m = 0$. They can be collectively written in terms of a four-component spinor as

$$i\gamma^\mu \partial_\mu \psi = m\psi \ , \ \ \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \ , \ \ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad (2.3.4)$$

This is the Dirac equation and the 4×4 matrices γ^μ are the Dirac γ-matrices. The Lagrangian density (2.3.2) reads

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \ , \ \ \bar{\psi} = \psi^\dagger \gamma^0 = (\psi_R^\dagger \ \ \psi_L^\dagger) \qquad (2.3.5)$$

They obey the anti-commutation rules

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{I}_{4\times 4} \qquad (2.3.6)$$

which may be viewed as the defining equation for γ^μ. Also,

$$(\gamma \cdot \partial)^2 = \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = \partial^2 \qquad (2.3.7)$$

and so the Dirac equation implies the Klein-Gordon equation,

$$\partial^2 \psi = (\gamma \cdot \partial)^2 \psi = -m^2 \psi \qquad (2.3.8)$$

showing that the parameter m represents the mass of the particle.

2.3.2 Lorentz transformations

[3] Note that neither $\psi_R^\dagger \psi_R$ nor $\psi_L^\dagger \psi_L$ is Lorentz-invariant which is why we did not consider adding them to the Lagrangian density in our discussion of Weyl spinors.

The action of Lorentz transformations can be deduced from corresponding properties of Weyl spinors. Eqs. (2.2.2) and (2.2.16) can be written collectively as

$$\vec{L}\psi = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \ , \ \vec{M}\psi = \frac{i}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \tag{2.3.9}$$

We may summarize these equations neatly by using $J^{\mu\nu}$ (eq. (2.1.19)),

$$J^{\mu\nu}\psi = S^{\mu\nu}\psi \ , \ S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{2.3.10}$$

Thus the matrices $S^{\mu\nu}$ form a representation of the Lorentz group and obey the algebra (2.1.20) as can be easily checked. This generalizes the three-dimensional result that $\vec{S} = \frac{1}{2}\vec{\sigma}$ is a representation of the rotation group whose components obey the angular-momentum algebra (2.1.7). A finite Lorentz transformation on a Dirac spinor acts as the 4×4 matrix

$$U = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \tag{2.3.11}$$

generalizing eq. (2.1.8) for rotations on spinors. Generalizing the statement that $\vec{\sigma}$ is a vector (eq. (2.1.12)), we shall show that γ^μ is a four-vector, i.e., it satisfies eq. (2.1.24) with $J^{\mu\nu} = S^{\mu\nu}$. Indeed,

$$\begin{aligned} [\gamma^\mu, S^{\rho\sigma}] &= \frac{i}{4}[\gamma^\mu, [\gamma^\rho, \gamma^\sigma]] \\ &= \frac{i}{4}\Big(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu - (\rho \leftrightarrow \sigma)\Big) \\ &= \frac{i}{4}\Big(\{\gamma^\mu, \gamma^\rho\}\gamma^\sigma - \gamma^\rho\{\gamma^\sigma, \gamma^\mu\} - (\rho \leftrightarrow \sigma)\Big) \\ &= i(\eta^{\mu\rho}\gamma^\sigma - \eta^{\mu\sigma}\gamma^\rho) \end{aligned} \tag{2.3.12}$$

where we used (2.3.6) in the last step.

Let us also check that ∂^μ is a four-vector. Using $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ (eq. (2.1.18)) and $[\partial^\mu, x^\nu] = \eta^{\mu\nu}$ (a consequence of $\frac{\partial x^\nu}{\partial x^\mu} = \delta_\mu^\nu$), it is easy to check that ∂^μ satisfies eq. (2.1.24).

It follows that $\gamma^\mu\partial_\mu$ which appears in the Lagrangian is a scalar. It is easy to check that so is the Lagrangian density (2.3.5).

2.3.3 Bilinears

Bilinears are important physical quantities. There are 16 of them

$$\psi_\alpha^\dagger\psi_\beta \ , \ \alpha, \beta = 0, 1, 2, 3 \tag{2.3.13}$$

We would like to group them according to their properties under Lorentz transformations. We have already seen that $\psi_R^\dagger\psi_L$ and its complex conjugate, $\psi_L^\dagger\psi_R$, are scalars. Under parity, $\psi_R \leftrightarrow \psi_L$. The eigenstates of parity are

$$S = \psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R \ , \ S' = \psi_R^\dagger\psi_L - \psi_L^\dagger\psi_R \tag{2.3.14}$$

where S (S') is even (odd) under parity - a scalar (pseudoscalar). In terms of the Dirac spinor,

$$S = \bar{\psi}\psi \ , \ S' = \bar{\psi}\gamma_5\psi \tag{2.3.15}$$

where we introduced the matrix

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \ , \ \{\gamma_5, \gamma^\mu\} = 0 \tag{2.3.16}$$

Vectors:

$$V^\mu = \bar{\psi}\gamma^\mu\psi = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L \tag{2.3.17}$$

(sum of two vectors, (2.2.5) and (2.2.17)) is evidently even under parity; and

$$A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi = \psi_R^\dagger\sigma^\mu\psi_R - \psi_L^\dagger\bar{\sigma}^\mu\psi_L \tag{2.3.18}$$

which is odd under parity (axial (pseudo) vector). V^μ and A^μ form $(\frac{1}{2}, \frac{1}{2})$ representations of the Lorentz group. Also, V^μ is the conserved Noether current under the symmetry

$$\psi \to e^{i\theta}\psi \tag{2.3.19}$$

It represents the fermion number. A^μ is the Noether current under the transformation

$$\psi \to e^{i\theta\gamma_5}\psi \tag{2.3.20}$$

This is a symmetry of the theory (chiral symmetry) only if $m = 0$.
Another set of bilinears is

$$T^{\mu\nu} = \bar{\psi}S^{\mu\nu}\psi \tag{2.3.21}$$

where $S^{\mu\nu}$ is given by (2.3.10). It is an anti-symmetric tensor with 6 independent components. Unlike the rest, it forms a reducible representation of the Lorentz group. It can be split into its self-dual and anti-self-dual pieces, respectively,

$$T = T_S + T_A \ , \ T_S^{\mu\nu} = \bar{\psi}S^{\mu\nu}\mathcal{P}_R\psi \ , \ T_A^{\mu\nu} = \bar{\psi}S^{\mu\nu}\mathcal{P}_L\psi \tag{2.3.22}$$

of 3 components each, where $\mathcal{P}_{R,L} = \frac{1}{2}(1 \pm \gamma_5)$ are projection operators. T_S and T_A form $(1, 0)$ and $(0, 1)$ representations of the Lorentz group, respectively.
We now have all the bilinears, as can be seen by counting $(1+1+4+4+3+3 = 16)$.

2.3.4 Solutions

We need to solve the Dirac equation (2.3.4) in order to quantize it. Plugging in the plane-wave solution

$$\psi = u(\vec{p})e^{-ip\cdot x} \tag{2.3.23}$$

we obtain

$$(\gamma^\mu p_\mu - m)u(\vec{p}) = 0 \tag{2.3.24}$$

This implies $p^\mu p_\mu = m^2$, therefore

$$p_0 = E = \pm\omega_p \ , \ \omega_p = \sqrt{\vec{p}^2 + m^2} \tag{2.3.25}$$

We expect 4 linearly independent solutions, 2 of positive and 2 of negative energy.

First let $p_0 = +\omega_p > 0$. Go to the rest frame of the particle, so that $\vec{p} = \vec{0}$. Then $p_0 = m$ and (2.3.24) reads

$$(m\gamma^0 - m)u(\vec{p}) = 0 \;\Rightarrow\; \gamma^0 u = u \tag{2.3.26}$$

Writing $u = \binom{u_R}{u_L}$ and using the explicit form of γ^0 (2.3.4), we deduce $u_L = u_R$. Therefore, we are only free to choose u_R, say. Two independent choices are $u_R = \binom{1}{0}$, $\binom{0}{1}$ yielding two linearly independent solutions of (2.3.26),

$$u^{(1)}(\vec{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \;,\; u^{(2)}(\vec{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \tag{2.3.27}$$

They are eigenstates of the z-component of spin,

$$L^3 u^{(1)} = \tfrac{1}{2}u^{(1)} \;,\; L^3 u^{(2)} = -\tfrac{1}{2}u^{(2)} \;,\; L^3 = \begin{pmatrix}\frac{1}{2}\sigma^3 & 0\\ 0 & \frac{1}{2}\sigma^3\end{pmatrix} \tag{2.3.28}$$

on account of (2.2.2) and (2.2.16). The normalization constant $\sqrt{m}$ was inserted for convenience. It may seem odd, since it vanishes in the massless limit; however, this is an illusion, as we shall see shortly.
Now let us generalize to $\vec{p} \neq \vec{0}$. Choose axes so that $\vec{p}$ is in the z-direction and insist on eigenstates of L^3 (the latter is an invariant statement: we want eigenvalues of the helicity operator, $h = \frac{\vec{p}\cdot\vec{L}}{|\vec{p}|}$, i.e., the component of $\vec{L}$ along $\vec{p}$). We have $p^\mu = (\omega_p\, 0\, 0\, p)$, so

$$\gamma^\mu p_\mu = \begin{pmatrix}0 & \omega_p - p\sigma^3\\ \omega_p + p\sigma^3 & 0\end{pmatrix} \tag{2.3.29}$$

Assuming a solution of the Dirac eq. (2.3.24) of the form

$$u^{(1)} = \begin{pmatrix}\alpha\\0\\\beta\\0\end{pmatrix} \tag{2.3.30}$$

we deduce

$$(\omega_p - p)\beta = m\alpha \;\Rightarrow\; \frac{\alpha}{\beta} = \sqrt{\frac{\omega_p - p}{\omega_p + p}} \tag{2.3.31}$$

We shall fix the normalization so that

$$u^{(1)} = \begin{pmatrix}\sqrt{\omega_p - p}\\0\\\sqrt{\omega_p + p}\\0\end{pmatrix} \tag{2.3.32}$$

In the limit $p \to 0$, $\omega_p \to m$, we recover our earlier result (2.3.27) in the rest frame. Notice that in the limit $m \to 0$, eq. (2.3.27) cannot be used because a massless particle has no rest frame (always travels at the speed of light). Eq. (2.3.32) yields

$$u^{(1)} \to \begin{pmatrix} 0 \\ 0 \\ \sqrt{2\omega_p} \\ 0 \end{pmatrix} \tag{2.3.33}$$

which is a right-handed spinor. The other solution of eq. (2.3.24) is similarly obtained,

$$u^{(2)} = \begin{pmatrix} 0 \\ \sqrt{\omega_p + p} \\ 0 \\ \sqrt{\omega_p - p} \end{pmatrix} \tag{2.3.34}$$

In the rest frame it reduces to expression (2.3.27) and in the massless limit it becomes a left-handed spinor.
The above formulas are kind of useless. We will need bilinears. They can be obtained from the simple expressions (2.3.27) in the rest frame.

(i) By an explicit calculation using (2.3.27), it is easy to see that

$$\bar{u}^{(r)} u^{(s)} = 2m\delta^{rs} \quad , \quad r, s = 1, 2 \tag{2.3.35}$$

Since these are Lorentz-invariant quantities, this result is valid in any frame.

(ii) From (2.3.27), we have

$$\bar{u}^{(r)}(\vec{0})\gamma^\mu u^{(s)}(\vec{0}) = 2 \begin{pmatrix} m \\ \vec{0} \end{pmatrix} \delta^{rs} \tag{2.3.36}$$

In an arbitrary frame ($\vec{p} \neq \vec{0}$), this becomes

$$\bar{u}^{(r)}(\vec{p})\gamma^\mu u^{(s)}(\vec{p}) = 2p^\mu \delta^{rs} \tag{2.3.37}$$

(iii) Define the 4×4 matrix

$$M = \frac{1}{2m} \sum_{r=1}^{2} u^{(r)} \bar{u}^{(r)} \tag{2.3.38}$$

Using (2.3.35), we deduce

$$M u^{(r)} = u^{(r)} \quad , \quad M^2 = I \tag{2.3.39}$$

Thus, M is a projection operator (onto positive-energy states, as we shall see). In the rest frame, we obtain

$$M = \tfrac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \tfrac{1}{2}(I + \gamma^0) \tag{2.3.40}$$

In an arbitrary frame, this generalizes to

$$M = \frac{1}{2m}(\gamma^\mu p_\mu + m) \tag{2.3.41}$$

We may check that $M^2 = I$ by using this explicit expression and $(\gamma \cdot p)^2 = p^2 = m^2$.

Turning to negative energy solutions, let

$$\psi = v(\vec{p})e^{+ip\cdot x} \tag{2.3.42}$$

The Dirac equation becomes

$$(\gamma^\mu p_\mu + m)v(\vec{p}) = 0 \tag{2.3.43}$$

Working as before, we obtain two independent solutions in the rest frame,

$$v^{(1)}(\vec{0}) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad , \quad v^{(2)}(\vec{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.3.44}$$

normalized so that

$$\bar{v}^{(r)}v^{(s)} = -2m\delta^{rs} \tag{2.3.45}$$

(valid in *any* Lorentz frame). We also obtain the vector bilinear

$$\bar{v}^{(r)}(\vec{p})\gamma^\mu v^{(s)} = 2p^\mu\delta^{rs} \tag{2.3.46}$$

Notice that positive and negative energy spinors are orthogonal to each other in the sense

$$\bar{u}^{(r)}v^{(s)} = \bar{v}^{(r)}u^{(s)} = 0 \tag{2.3.47}$$

It follows from (2.3.38) that

$$Mv^{(r)} = 0 \tag{2.3.48}$$

This also follows from the explicit form of M (2.3.41) and the Dirac eq. (2.3.43). Together with (2.3.39), they show that M is a projection operator onto the positive-energy solutions of the Dirac equation.
Similarly, it is shown that

$$N = -\frac{1}{2m}\sum_{r=1}^{2} v^{(r)}\bar{v}^{(r)} = \frac{1}{2m}(m - \gamma^\mu p_\mu) \tag{2.3.49}$$

is a projection operator onto negative-energy states ($Nu^{(r)} = 0$, $Nv^{(r)} = v^{(r)}$, $N^2 = I$).

2.4 Canonical quantization

2.4.1 Quantization

Expand ψ in a complete set of solutions to the Dirac equation (eqs. (2.3.23) and (2.3.42)),

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{r=1}^{2} \left\{ b^{(r)}(\vec{k})u^{(r)}(\vec{k})e^{-ik\cdot x} + c^{(r)\dagger}(\vec{k})v^{(r)}(\vec{k})e^{ik\cdot x} \right\} \quad (2.4.1)$$

The coefficients $b^{(r)}$ and $c^{(r)}$ are not related to each other (*cf.* eq. (1.5.57) for a complex scalar field). Taking hermitian conjugates, we obtain

$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{r=1}^{2} \left\{ c^{(r)}(\vec{k})\bar{v}^{(r)}(\vec{k})e^{-ik\cdot x} + b^{(r)\dagger}(\vec{k})\bar{u}^{(r)}(\vec{k})e^{ik\cdot x} \right\} \quad (2.4.2)$$

From the Lagrangian density (2.3.5), we deduce the momentum conjugate to ψ_α,

$$\pi^\alpha \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0\psi_\alpha)} = i(\bar{\psi}\gamma^0)^\alpha = i\psi^{\dagger\alpha} \quad (2.4.3)$$

The Hamiltonian density is

$$\mathcal{H} = \pi^\alpha\partial_0\psi_\alpha - \mathcal{L} = i\bar{\psi}\gamma^0\partial_0\psi - \mathcal{L} \quad (2.4.4)$$

If ψ satisfies the Dirac equation, then $\mathcal{L} = 0$. Therefore, the Hamiltonian is

$$H \equiv \int d^3x\mathcal{H} = \int d^3x \, i\bar{\psi}\gamma^0\partial_0\psi \quad (2.4.5)$$

Plugging in the expansions (2.4.1) and (2.4.2), we obtain

$$\begin{aligned} H = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3} \sum_{r,s} \quad & \Big\{ b^{(s)\dagger}(\vec{k})b^{(r)}(\vec{k})\bar{u}^{(s)}(\vec{k})\gamma^0u^{(r)}(\vec{k}) \\ - \quad & c^{(s)}(\vec{k})c^{(r)\dagger}(\vec{k})\bar{v}^{(s)}(\vec{k})\gamma^0v^{(r)}(\vec{k}) \\ - \quad & b^{(s)\dagger}(-\vec{k})c^{(r)\dagger}(\vec{k})\bar{u}^{(s)}(-\vec{k})\gamma^0v^{(r)}(\vec{k})e^{2i\omega_k t} \\ + \quad & c^{(s)}(-\vec{k})b^{(r)}(\vec{k})\bar{v}^{(s)}(-\vec{k})\gamma^0u^{(r)}(\vec{k})e^{-2i\omega_k t}\Big\} \end{aligned} \quad (2.4.6)$$

where I have been careful with the ordering of the coefficients b and c. To simplify this expression, observe that if $u(\vec{p})$ satisfies the Dirac eq. (2.3.24), then so does $\gamma^0u(\vec{p})$ but with $\vec{p}\to-\vec{p}$. Therefore,

$$\gamma^0u^{(r)}(\vec{p}) = u^{(r)}(-\vec{p}) \quad (2.4.7)$$

where the constant of proportionality is fixed by (2.3.26). Similarly, we reach the same conclusion for negative-energy spinors: $v^{(r)}(\vec{p}) = \gamma^0v^{(r)}(-\vec{p})$. It follows that the last two terms in the expression for H (2.4.5) are proportional to $\bar{u}^{(s)}(-\vec{k})v^{(r)}(-\vec{k})$ and $\bar{v}^{(s)}(-\vec{k})u^{(r)}(-\vec{k})$, respectively, which vanish by the orthogonality relation (2.3.47).

This shows that H is time-independent. Using the vector bilinears (2.3.37) and (2.3.46) to simplify the remaining terms in H, we finally obtain

$$H = \int \frac{d^3k}{(2\pi)^3}\,\omega_k \sum_{r=1}^{2} \left\{ b^{(r)\dagger}(\vec{k})b^{(r)}(\vec{k}) - c^{(r)}(\vec{k})c^{(r)\dagger}(\vec{k}) \right\} \tag{2.4.8}$$

If we impose commutation relations similar to the ones for a and $a^\dagger$ in the expansion of the scalar field ψ (eq. (1.2.46)), the minus sign in the $cc^\dagger$ term will lead to negative-energy eigenstates. In fact, the spectrum of the Hamiltonian will be unbounded from below. To avoid this garbage, we will impose *anti*-commutation relations,

$$\{b^{(r)}(\vec{p}), b^{(s)\dagger}(\vec{p}')\} = \{c^{(r)}(\vec{p}), c^{(s)\dagger}(\vec{p}')\} = \delta^{rs}(2\pi)^3\delta^3(\vec{p}-\vec{p}') \tag{2.4.9}$$

All other *anti*-commutators will be assumed to vanish. They trivially imply

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(y)\} = 0 \tag{2.4.10}$$

The only non-trivial anti-commutator is between ψ and $\bar{\psi}$. We obtain the *equal-time* anti-commutator (suppressing spinor indices)

$$\{\psi(\vec{x},t), \bar{\psi}(\vec{y},t)\} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \sum_r \left\{ u^{(r)}(\vec{k})\bar{u}^{(r)}(\vec{k}) + v^{(r)}(-\vec{k})\bar{v}^{(r)}(-\vec{k}) \right\} \tag{2.4.11}$$

The spinorial sum is easily evaluated using the projection operators M (eqs. (2.3.38) and (2.3.41)) and N (eq. (2.3.49)). Finally,

$$\{\psi(\vec{x},t), \bar{\psi}(\vec{y},t)\} = \gamma^0\delta^3(\vec{x}-\vec{y}) \tag{2.4.12}$$

2.4.2 The vacuum

b and c will be annihilation operators,

$$b^{(r)}(\vec{p})|0\rangle = c^{(r)}(\vec{p})|0\rangle = 0 \tag{2.4.13}$$

We could call $b^\dagger$ and $c^\dagger$ creation operators, instead (notice that b and $b^\dagger$ are treated symmetrically in the anti-commutation relations, so their role is ambiguous, unlike a and $a^\dagger$ in the scalar case where we had no choice but to call a annihilation operator). However, this would lead to an equivalent formulation. Another option is to call b and $c^\dagger$ annihilation operators. That would be foolish, for then ψ would annihilate the vacuum leading to a trivial theory.
Acting on the vacuum with the Hamiltonian, we obtain

$$H|0\rangle = E_0|0\rangle \;\;,\;\; E_0 = -\int \frac{d^3k}{(2\pi)^3}\, 2\omega_k\delta^3(\vec{0}) \tag{2.4.14}$$

This leads to a vacuum energy density

$$\rho = \frac{E_0}{V} = -4\int \frac{d^3k}{(2\pi)^3}\, \tfrac{1}{2}\omega_k \tag{2.4.15}$$

to be compared with the scalar field expression (1.3.7). The factor of 4 accounts for the four degrees of freedom of the Dirac spinor, but unlike in the scalar case, the fermions yield a *negative* contribution to the vacuum energy (cosmological constant). In a supersymmetric theory, there are equal numbers of bosonic and fermionic degrees of freedom, so the cosmological constant should vanish in a supersymmetric Universe. Ours is obviously not supersymmetric, but people thought that if it started its life with supersymmetry (which was mysteriously broken later on), that would provide a neat explanation for a vanishing cosmological constant. Unfortunately, Nature did not play along and chose a *positive* cosmological constant instead, challenging us to find the fundamental reason behind Her choice. It is a challenge because any answer which is not zero seems to be infinity. At any rate, our immediate humble goal is to build a quantum theory of fermions in a flat spacetime ignoring gravity. In this case, we may simply shift H by its vacuum expectation value E_0 and define the *normal-ordered* Hamiltonian

$$:H := H - E_0 = \int \frac{d^3k}{(2\pi)^3}\,\omega_k \sum_{r=1}^{2}\left\{b^{(r)\dagger}(\vec{k})b^{(r)}(\vec{k}) + c^{(r)\dagger}(\vec{k})c^{(r)}(\vec{k})\right\} \quad (2.4.16)$$

Then the vacuum corresponds to zero eigenvalue and any other state has positive energy. From now on, we shall call the normal-ordered Hamiltonian simply H.

2.4.3 One-particle states and the charge

Acting with creation operators on the vacuum, we produce states

$$|\vec{p}, r, b\rangle = b^{(r)\dagger}(\vec{p})|0\rangle \ , \ |\vec{p}, r, c\rangle = c^{(r)\dagger}(\vec{p})|0\rangle \quad (2.4.17)$$

which are eigenstates of H with eigenvalue ω_p. Thus, they represent single particle states with momentum $\vec{p}$ and spin specified by the quantum number r. There are two types of each such state, one created by a b oscillator and one created by a c oscillator. How do they differ? Recall that the system possesses an internal symmetry, $\psi \to e^{i\theta}\psi$ (eq. (2.3.19)) whose Noether current is given by (2.3.17). The corresponding conserved charge (fermion number) is

$$Q = \int d^3x V^0 = \int d^3x\, \bar{\psi}\gamma^0\psi \quad (2.4.18)$$

Expanding ψ and $\bar{\psi}$ in modes and working as we did with H, we obtain

$$Q = \int \frac{d^3k}{(2\pi)^3} \sum_{r=1}^{2}\left\{b^{(r)\dagger}(\vec{k})b^{(r)}(\vec{k}) + c^{(r)}(\vec{k})c^{(r)\dagger}(\vec{k})\right\} \quad (2.4.19)$$

showing explicitly that Q is independent of time. We need to normal-order this expression. We obtain

$$:Q: \equiv Q - \langle 0|Q|0\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_{r=1}^{2}\left\{b^{(r)\dagger}(\vec{k})b^{(r)}(\vec{k}) - c^{(r)\dagger}(\vec{k})c^{(r)}(\vec{k})\right\} \quad (2.4.20)$$

We shall call the normal-ordered charge simply Q from now on. Then $Q|0\rangle = 0$. Also

$$Q|\vec{p}, r, b\rangle = |\vec{p}, r, b\rangle \ , \ \ Q|\vec{p}, r, c\rangle = -|\vec{p}, r, c\rangle \tag{2.4.21}$$

Thus, they are both eigenstates of Q but with opposite charges. It follows that the b oscillator creates a particle whereas the c-oscillator creates an anti-particle. The total fermion number is the sum of particles *minus* the sum of anti-particles.

2.4.4 Two-particle states and statistics

Acting with two b-oscillators, we produce a two-particle state,

$$|\vec{p}_1, r_1, b; \vec{p}_2, r_2, b\rangle = b^{(r_1)\dagger}(\vec{p}_1) b^{(r_2)\dagger}(\vec{p}_2)|0\rangle \tag{2.4.22}$$

Similarly, acting with two c-oscillators, we produce a two-antiparticle state. Since the oscillators anti-commute, we have

$$|\vec{p}_1, r_1, b; \vec{p}_2, r_2, b\rangle = -|\vec{p}_2, r_2, b; \vec{p}_1, r_1, b\rangle \tag{2.4.23}$$

showing that these particles obey Fermi statistics; they are *fermions*. We cannot have two particles with the same quantum numbers, since $[b^{(r)\dagger}(\vec{p})]^2 = 0$. Thus the Pauli exclusion principle is a consequence of this theory. The connection between spin and statistics (the spin-statistics theorem: bosons have integer spin, fermions do not) can be *proved* in general under very broad assumptions (such as *relativity* and *causality*).

2.4.5 On the classical limit

Some philosophical remarks on the classical limit of our quantum field theory of fermions are in order here. In the classical limit in general, uncertainties are small compared with mean values of observables. This can only be true if the occupation numbers are large. Then we get a classical macroscopic system. The occupation numbers of a fermion, such as the Dirac field, can only take two values: 0 or 1. Therefore, there is no classical limit! So the Dirac equation is not really a classical equation that the quantum theory we just developed approaches in the classical limit. This was the case with the Klein-Gordon field we discussed earlier. We followed the same approach with the Dirac field, but our motivation was not the same as in the Klein-Gordon case. We simply wanted a relativistic quantum theory for a spin-1/2 particle that would work. We were not motivated by classical considerations.

2.5 Causality and propagation

2.5.1 Causality and measurement

Causality implies that fermionic field *anti-commute* outside the light-cone (at spacelike separation). This is trivially true for two ψ's or two $\bar{\psi}$'s, since their anti-commutators vanish everywhere (eq. (2.4.10)) and not just outside the light-cone. We need to verify (suppressing indices)

$$\{\psi(x), \bar{\psi}(y)\} = 0 \ , \ \ (x-y)^2 < 0 \tag{2.5.1}$$

To this end, define the propagator

$$S_+(x-y) = \langle 0|\psi(x)\bar{\psi}(y)|0\rangle \tag{2.5.2}$$

which is a 4×4 matrix. It only depends on the distance $x^\mu - y^\mu$ and not on the points (events) x^μ and y^μ separately, as we shall show.[4] Expanding in modes and using the anti-commutation rules (2.4.9) to calculate vacuum expectation values, we obtain

$$S_+(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{-ik\cdot(x-y)} \sum_{r=1}^{2} u^{(r)}(\vec{k})\bar{u}^{(r)}(\vec{k}) \tag{2.5.3}$$

confirming translational invariance. Notice also that only positive-energy states contribute to this propagator. The spinorial sum can be written in terms of the projection operator M (eq. (2.3.38)). Using (2.3.41), we have

$$\begin{aligned} S_+(x-y) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, (\gamma\cdot k + m)\, e^{-ik\cdot(x-y)} \\ &= (i\gamma\cdot\partial + m) \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{-ik\cdot(x-y)} \end{aligned} \tag{2.5.4}$$

The last integral is the scalar propagator $D(x-y)$ (eq. (1.4.4)). Therefore,

$$S_+(x-y) = (i\gamma\cdot\partial + m)D(x-y) \tag{2.5.5}$$

Let us now define the propagator (a 4×4 matrix, again)

$$S_-(x-y) = \langle 0|\bar{\psi}(x)\psi(y)|0\rangle \tag{2.5.6}$$

where we merely suppressed indices (no summation over spinor indices is implied). Working as before, we arrive at

$$S_-(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{-ik\cdot(x-y)} \sum_{r=1}^{2} v^{(r)}(\vec{k})\bar{v}^{(r)}(\vec{k}) \tag{2.5.7}$$

showing that only negative-energy states contribute to this propagator. Using the projection matrix (2.3.49), we obtain

$$S_-(x-y) = (i\gamma\cdot\partial - m)D(x-y) \tag{2.5.8}$$

The anti-commutator (2.5.1) is a c-number (not an operator), therefore

$$\begin{aligned} \{\psi(x), \bar{\psi}(y)\} &= \langle 0|\{\psi(x), \bar{\psi}(y)\}|0\rangle \\ &= S_+(x-y) + S_-(y-x) \\ &= (i\gamma\cdot\partial + m)\Big\{D(x-y) - D(y-x)\Big\} \end{aligned} \tag{2.5.9}$$

The validity of (2.5.1) is a direct consequence of the corresponding statement in the scalar case ($D(x-y)-D(y-x) = 0$ for $(x-y)^2 < 0$ (eq. (1.4.8))). Again, causality is due to non-trivial interference between positive-energy modes (particles) propagating

[4]This may also be established by an argument similar to the one we employed in the scalar case, eq. (1.4.5).

in one direction ($x \to y$) and negative-energy modes (anti-particles) propagating in the opposite direction ($y \to x$). Notice that, had we chosen a *commutator* instead, we would have violated causality ($[\psi(x), \bar{\psi}(y)] \neq 0$). The requirement of causality imposes Fermi statistics on spin-1/2 particles.

2.5.2 The Feynman propagator

A propagator which is a physical quantity can be defined similarly to the scalar case in terms of the time-ordered product (*cf.* eq. (1.4.18)),

$$T(\psi(x)\bar{\psi}(y)) = \begin{cases} \psi(x)\bar{\psi}(y) & , \quad x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & , \quad x^0 < y^0 \end{cases} \tag{2.5.10}$$

Notice the (*essential*) minus sign. The Feynman propagator for a Dirac field is (*cf.* eq. (1.4.19))

$$S_F(x-y) = \langle 0|T(\psi(x)\bar{\psi}(y)|0\rangle \tag{2.5.11}$$

Notice that

$$S_F(x-y) = \begin{cases} S_+(x-y) & , \quad x^0 > y^0 \\ -S_-(y-x) & , \quad x^0 < y^0 \end{cases} \tag{2.5.12}$$

Using the expressions (2.5.5) and (2.5.8), as well as the expression for the scalar Feynman propagator (1.4.15), we obtain

$$S_F(x-y) = (i\gamma \cdot \partial + m)D_F(x-y) \tag{2.5.13}$$

This can be written as a fourier transform using (1.4.16),

$$S_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i(k_\mu \gamma^\mu + m)}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \tag{2.5.14}$$

The $i\epsilon$ term ($\epsilon > 0$) places the positive-energy pole ($k_0 = \omega_k$) slightly below the real axis and the negative-energy pole ($k_0 = -\omega_k$) slightly above the real axis. For $x_0 > y_0$ (propagation *forward* in time) we ought to close the contour in the complex k_0-plane in the lower-half plane. Thus only positive-energy states (particles) contribute. On the contrary, for $x_0 < y_0$ (propagation *backwards* in time) the contour closes in the upper-half plane and only negative-energy (anti-particles) contribute.
The Feynman propagator is a Green function for the Dirac equation. Indeed, from eq. (2.5.13) and using the corresponding result in the scalar case (1.4.17), we deduce

$$(i\gamma \cdot \partial - m)S_F(x-y) = i\delta^4(x-y) \tag{2.5.15}$$

2.6 Discrete symmetries

<u>Parity</u>
Under parity, $\vec{x} \to -\vec{x}$, $t \to t$ and $\psi_R \leftrightarrow \psi_L$. It follows from the definition of ψ and γ^0 (2.3.4) that $\psi \to \gamma^0\psi$ and $\bar{\psi} \to \bar{\psi}\gamma^0$. If U_P is the unitary transformation that implements parity ($U_P^2 = I$, $U_P^\dagger = U_P$), then

$$U_P\psi(\vec{x}, t)U_P = \gamma^0\psi(-\vec{x}, t) \tag{2.6.1}$$

From the expansion of ψ (2.4.1) and using (2.4.7), we deduce

$$U_P b^{(r)}(\vec{p}) U_P = b^{(r)}(-\vec{p}) \ , \ U_P c^{(r)}(\vec{p}) U_P = c^{(r)}(-\vec{p}) \tag{2.6.2}$$

i.e., parity reverses the momentum, but does not affect the spin, as expected on physical grounds.
The action of parity on the various biliears and other physical quantities of interest is easily deduced, e.g., for the pseudoscalar S' (eq. (2.3.15)),

$$U_P S' U_P = U_P \bar{\psi} U_P \gamma_5 U_P \psi U_P = \bar{\psi}\gamma^0\gamma_5\gamma^0\psi = -S' \tag{2.6.3}$$

where we used $\{\gamma^0, \gamma_5\} = 0$ (eq. (2.3.16)) and $(\gamma^0)^2 = I$ (eq. (2.3.6)). This confirms that S' is odd under parity.
<u>Time reversal</u>
Under time reversal, $t \to -t$ and $\vec{x} \to \vec{x}$. Consequently, the momentum is flipped ($\vec{p} \to -\vec{p}$) and so is the spin (angular momentum). Recall from our discussion in the scalar case that the operator implementing time reversal, U_T, is *anti*-linear (eq. (1.5.47)) and *anti*-unitary (eq. (1.5.48)).
Consider a left-handed Weyl spinor, ψ_L. If its spin is flipped, then it turns into a right-handed spinor. This is also deduced from the Weyl equation (2.2.19) which can be written as

$$\partial_0 \psi_L = \vec{\sigma} \cdot \vec{\nabla} \psi_L \tag{2.6.4}$$

Indeed, under time reversal, $\partial_0 \to -\partial_0$, so (2.2.19) turns into the Weyl equation for a *right*-handed spinor (2.2.11),

$$\partial_0 \psi_L = -\vec{\sigma} \cdot \vec{\nabla} \psi_L \tag{2.6.5}$$

We need to find a mapping that flips the spin of ψ_L and is anti-linear and anti-unitary. The answer is

$$\psi_L \to -i\sigma^2 \psi_L^* \tag{2.6.6}$$

and similarly for ψ_R. It is easy to check (using the properties of the Pauli matrices (2.1.6)) that $-i\sigma^2\psi_L^*$ satisfies the Weyl equation for a *right*-handed spinor (2.2.11). In particular,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.6.7}$$

Flipping the spin twice doesn't take us back to the original spinor, instead $\psi_L \to -\psi_L$ (since $(-i\sigma^2)^2 = -I$).
Turning to Dirac spinors, we may use the above results if we express the Dirac spinors in terms of Weyl spinors, $u = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$. Then (2.6.6) and its right-handed counterpart imply

$$u \to \begin{pmatrix} -i\sigma^2 & \\ & -i\sigma^2 \end{pmatrix} u^* = -\gamma^1\gamma^3 u^* \tag{2.6.8}$$

Moreover, using (2.3.27) and (2.6.7), we easliy deduce

$$u^{(1)} \to u^{(2)} \ , \ u^{(2)} \to -u^{(1)} \tag{2.6.9}$$

Comparing (2.6.8) and (2.6.9), we obtain

$$u^{(1)*} = \gamma^1\gamma^3 u^{(2)} \ , \ u^{(2)*} = -\gamma^1\gamma^3 u^{(1)} \tag{2.6.10}$$

Notice that the second equation is a consequence of the first, because $\gamma^1\gamma^3$ is a real matrix and we also used the properties of the γ-matrices (2.3.6). The result for negative-energy spinors is identical,

$$v^{(1)*} = \gamma^1\gamma^3 v^{(2)} \ , \ v^{(2)*} = -\gamma^1\gamma^3 v^{(1)} \tag{2.6.11}$$

Let us now apply time reversal on a Dirac spinor. Expanding in modes, we obtain

$$\begin{aligned} U_T\psi(\vec{x},t)U_T = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{r=1}^{2} & \Big\{ U_T b^{(r)}(\vec{k}) U_T u^{(r)*}(\vec{k}) e^{ik\cdot x} \\ & + \ U_T c^{(r)\dagger}(\vec{k}) U_T v^{(r)*}(\vec{k}) e^{-ik\cdot x} \Big\} \end{aligned} \tag{2.6.12}$$

Notice that we had to take the complex conjugates of all c-numbers due to the anti-linearity of U_T. Defining

$$U_T b^{(1)}(\vec{k}) U_T = b^{(2)}(-\vec{p}) \ , \ U_T b^{(2)}(\vec{k}) U_T = -b^{(1)}(-\vec{p}) \tag{2.6.13}$$

and similarly for $c^{(r)}(\vec{p})$, and using (2.6.10) and (2.6.11), after a change of variables $\vec{k} \to -\vec{k}$, we obtain

$$U_T\psi(\vec{x},t)U_T = \gamma^1\gamma^3\psi(\vec{x},-t) \tag{2.6.14}$$

We can easily directly verify that this expression satisfies the time-reversed Dirac equation.

As an example of the action on bilinears, consider the vector current V^μ (eq. (2.3.17)). We have

$$U_T V^\mu U_T = U_T\bar{\psi}U_T(\gamma^\mu)^* U_T\psi U_T \tag{2.6.15}$$

Since all γ matrices are real except for $\gamma^2 = -(\gamma^2)^*$, we deduce

$$U_T V^0 U_T = \bar{\psi}\gamma^3\gamma^1\gamma^0\gamma^1\gamma^3\psi = V^0 \tag{2.6.16}$$

$$U_T V^1 U_T = \bar{\psi}\gamma^3\gamma^1\gamma^1\gamma^1\gamma^3\psi = -V^1 \tag{2.6.17}$$

$$U_T V^2 U_T = -\bar{\psi}\gamma^3\gamma^1\gamma^2\gamma^1\gamma^3\psi = -V^2 \tag{2.6.18}$$

$$U_T V^3 U_T = \bar{\psi}\gamma^3\gamma^1\gamma^3\gamma^1\gamma^3\psi = -V^3 \tag{2.6.19}$$

Thus the charge density V^0 is invariant and the current $\vec{V}$ gets reversed, as expected on physical grounds.

Charge Conjugation

Charge conjugation interchanges particles and antiparticles, so

$$U_C b^{(r)}(\vec{p}) U_C = c^{(r)}(\vec{p}) \ , \ U_C c^{(r)}(\vec{p}) U_C = b^{(r)}(\vec{p}) \tag{2.6.20}$$

It is a unitary operator and $U_C^2 = I$, $U_C^\dagger = U_C$. Its action on the Dirac field ψ is deduced from the expansion (2.4.1),

$$U_C\psi(x)U_C = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{r=1}^{2} \left\{ c^{(r)}(\vec{k})u^{(r)}(\vec{k})e^{-ik\cdot x} + b^{(r)\dagger}(\vec{k})v^{(r)}(\vec{k})e^{ik\cdot x} \right\} \tag{2.6.21}$$

Taking the complex conjugate of (2.4.1), we obtain

$$\psi^*(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{r=1}^{2} \left\{ b^{(r)\dagger}(\vec{k})u^{(r)*}(\vec{k})e^{ik\cdot x} + c^{(r)}(\vec{k})v^{(r)*}(\vec{k})e^{-ik\cdot x} \right\} \tag{2.6.22}$$

To relate these two expressions, we need to relate $u^{(r)*}$ to $v^{(r)}$. This is most easily done in the rest frame using the expressions (2.3.27) and (2.3.44) for the spinors. We find

$$u^{(r)*} = i\gamma^2 v^{(r)} \quad , \quad v^{(r)*} = i\gamma^2 u^{(r)} \quad , \quad r = 1,2 \tag{2.6.23}$$

These two relations imply one another. They are valid in *any* frame, as you can convince yourselves by boosting and rotating the rest frame.
We deduce

$$\psi^* = i\gamma^2 U_C\psi U_C \quad \Rightarrow \quad U_C\psi U_C = -i\gamma^2\psi^* \tag{2.6.24}$$

Do not be fooled by ψ^* into thinking that U_C is anti-linear. $\psi \to \psi^*$ is a *linear* transformation if you think in terms of components (*cf.* the complex scalar case: eq. (1.5.66), which in terms of components (1.5.55) translates into $\phi_1 \to \phi_1$, $\phi_2 \to -\phi_2$). Notice also that we did not take complex conjugates of c-numbers when we acted with U_C in (2.6.21) unlike eq. (2.6.12) where we had acted with the *anti*-linear operator U_T.
Finally the Dirac lagrangian density is invariant under parity (P), time reversal (T) and charge conjugation (C). Be reminded that a general quantum field theory does not possess all three symmetries. However, the product CPT is always a symmetry no matter what the theory is. This can be proved under very broad assumptions (the CPT theorem).

2.7 Low energy fermionic systems

In the low energy limit, anti-fermions can be ignored and only fermions will be created, say by $b^{(r)}(\vec{k})$ ($r = 1, 2$, corresponding to spin up or down). In a finite volume, momenta are quantized and the anti-commutation relations are

$$\{b^{(r)}(\vec{k}) \ , \ b^{(r')\dagger}(\vec{k}')\} = \delta_{\vec{k}\,\vec{k}'}\delta^{rr'} \tag{2.7.1}$$

Defining $\psi^{(r)}(\vec{x})$ and $\psi^{(r)\dagger}(\vec{x})$ as in the bosonic case (with the extra spinorial degree of freedom), we are led to the anti-commutation relations

$$\{\psi^{(r)}(\vec{x}) \ , \ \psi^{(r')\dagger}(\vec{x}')\} = \delta^{rr'}\delta^3(\vec{x} - \vec{x}') \tag{2.7.2}$$

The free Hamiltonian is

$$H = -\int d^3x \psi^\dagger \nabla^2 \psi = \sum_{\vec{k},r} E_k b^{(r)\dagger}(\vec{k})b^{(r)}(\vec{k}) \tag{2.7.3}$$

and interactions can be included as in the bosonic case. However, unlike with bosons, the vacuum $|0\rangle$ (annihilated by all $b^{(r)}(\vec{k})$) is not the ground state of a system of N fermions, because of the Pauli exclusion principle. Ignoring interactions, the minimum energy a system of N fermions can have is when fermions are uniformly distributed in a sphere in momentum space up to energy E_F (Fermi energy). The volume of the Fermi sphere is $\frac{4}{3}\pi k_F^3$, where $k_F = \sqrt{2mE_F}$ is the Fermi momentum. In it, each pair of fermions occupies a cube of side $\frac{2\pi}{L}$. Therefore, the total number of fermions is

$$N = 2\frac{\frac{4}{3}\pi k_F^3}{(\frac{2\pi}{L})^3} \tag{2.7.4}$$

and the Fermi energy is given by

$$E_F = \frac{1}{2m}\left(\frac{3\pi^2 N}{L^3}\right)^{2/3} \tag{2.7.5}$$

Denote the ground state (filled Fermi sea) by $|\Omega\rangle$. Notice that the role of creation and annihilation operators is reversed below the sea level. Indeed, for $k \leq k_F$, $b^{(r)}(\vec{k})$ acts as a creation operator (creating a *hole* in the Fermi sea) whereas $b^{(r)\dagger}(\vec{k})$ annihilates $|\Omega\rangle$.

ONE DIMENSION

Going down to one dimension is interesting because calculations are simplified and explicit results can be often obtained allowing for a better understanding of physical systems in higher dimension where calculations are cumbersome. Moreover, physical systems which behave effectively as one-dimensional exist.

In one dimension the surface of the Fermi sphere consists of two points, $\pm k_F$ (corresponding to left and right moving fermions). Concentrating on dynamics near the Fermi surface (which dominates at low temperatures), the momentum is $k = \pm k_F + q$, where q is small and the energy deviation is $E_k - E_F \approx \pm v_F q$, where $v_F = k_F/m$ is the Fermi velocity.

Ignoring the spin for the moment, the relevant free part of the Hamiltonian (as measured from the Fermi surface) is

$$H_0 = \sum_k (E_k - E_F) b^\dagger(k) b(k) \approx \sum_q v_F q \left[b_+^\dagger(q) b_+(q) - b_-^\dagger(q) b_-(q)\right] \tag{2.7.6}$$

where $b_\pm(q) = b(\pm k_F + q)$.

The interaction part of the Hamiltonian can be written in terms of the Fourier transform of the density operator $\rho = \psi^\dagger\psi$,

$$\widetilde{\rho}(k) = \int dx e^{-ikx}\rho(x) = \sum_{k'} b^\dagger(k+k') b(k') \tag{2.7.7}$$

as

$$V \approx \frac{1}{L}\sum_q \sum_{\epsilon=+,-} [g_4 \widetilde{\rho}_\epsilon(q)\widetilde{\rho}_\epsilon(-q) + g_2 \widetilde{\rho}_\epsilon(q)\widetilde{\rho}_{-\epsilon}(-q)] \tag{2.7.8}$$

where $\widetilde{\rho}_\pm(q) = \widetilde{\rho}(\pm k_F + q)$ and g_2, g_4 are numerical coefficients that can be determined. Also note that

$$\widetilde{\rho}_\pm(-q) = \widetilde{\rho}^\dagger_\pm(q) \tag{2.7.9}$$

The free part of the Hamiltonian can also be written in terms of the operators $\widetilde{\rho}_\pm$ either by direct calculation (hard!) or by the following argument. The commutator of H_0 with $\widetilde{\rho}_\pm$ is

$$[H_0\ ,\ \widetilde{\rho}_\pm(q)] = \pm q v_F \widetilde{\rho}_\pm(q) \tag{2.7.10}$$

Define

$$H_0' = \frac{2\pi v_F}{L}\sum_q [\widetilde{\rho}_+(q)\widetilde{\rho}_+(-q) + \widetilde{\rho}_-(q)\widetilde{\rho}_-(-q)] \tag{2.7.11}$$

One can check that H_0' obeys the same commutation relations (2.7.10) as H_0 near the Fermi surface. To see this, note that

$$\begin{aligned}
[\widetilde{\rho}_+(q)\ ,\ \widetilde{\rho}_+(q')] &\approx \langle\Omega|[\widetilde{\rho}_+(q)\ ,\ \widetilde{\rho}_+(q')]|\Omega\rangle \\
&= \delta_{q,-q'}\sum_k \langle\Omega|\left(b^\dagger_+(k+q)b_+(k+q) - b^\dagger_+(k)b_+(k)\right)|\Omega\rangle
\end{aligned} \tag{2.7.12}$$

Each term in the sum contributes either 1 if $k < 0$ (occupied state) or 0 if $k \geq 0$ (empty state). The two sums seem to cancel each other because one is obtained from the other by a shift $k \to k + q$. However, this naive conclusion is incorrect. Suppose $q > 0$. Then for $k > 0$ both sums give 0s (since $k, k+q > 0$) and for $k < -q$ both sums give 1s (since $k, k+q < 0$). In both cases they cancel each other. For $-q < k < 0$, one sume gives 1s and the other 0s, leading to a mismatch. We deduce

$$[\widetilde{\rho}_+(q)\ ,\ \widetilde{\rho}_+(q')] \approx \delta_{q,-q'}\sum_{-q<k<0}(0-1) = -\frac{qL}{2\pi}\delta_{q,-q'} \tag{2.7.13}$$

where we used the fact that the number of momenta in an interval of length q is $q/(2\pi/L)$.
The above is easily generalized to

$$[\widetilde{\rho}_\epsilon(q)\ ,\ \widetilde{\rho}_{\epsilon'}(q')] \approx -\epsilon\delta_{\epsilon\epsilon'}\delta_{q,-q'}\frac{qL}{2\pi} \tag{2.7.14}$$

Thus, these operators act as *bosonic* creation and annihilation operators offering an alternative (bosonic!) description of our interacting fermionic system (bosonization). Using this result, we easily verify

$$[H_0'\ ,\ \widetilde{\rho}_\pm(q)] = \pm q v_F \widetilde{\rho}_\pm(q) \tag{2.7.15}$$

in agreement with (2.7.10). It follows that H_0' and H_0 differ by a constant. The constant actually vanishes, because

$$\langle\Psi|H_0|\Psi\rangle = \langle\Psi|H_0'|\Psi\rangle\ \ ,\quad |\Psi\rangle = \widetilde{\rho}_+(q)|\Omega\rangle \tag{2.7.16}$$

showing that $H_0' = H_0$.
We have succeeded in writing the entire Hamiltonian in quadratic form in terms of the bosonic operators $\widetilde{\rho}_\pm$. To deduce the spectrum, we perform the Bogoliubov transformation for $q > 0$,

$$\begin{aligned} \widetilde{\rho}_+(q) &= \sqrt{\frac{Lq}{2\pi}}\left[\cosh\theta_q a^\dagger(q) + \sinh\theta_q a(-q)\right] \\ \widetilde{\rho}_-(q) &= \sqrt{\frac{Lq}{2\pi}}\left[\sinh\theta_q a^\dagger(q) + \cosh\theta_q a(-q)\right] \end{aligned} \tag{2.7.17}$$

For $q < 0$, the transformation is deduced from the above and (2.7.9). From (2.7.14) we deduce that the bosonic creation and annihilation operators obey standard commutation relations,

$$[a(q)\ ,\ a^\dagger(q')] = \delta_{qq'} \tag{2.7.18}$$

The Hamiltonian can be written as

$$H = H_0' + V = \frac{2}{L}\sum_{q>0}\sum_{\epsilon=+,-}\left[(2\pi v_F + g_4)\widetilde{\rho}_\epsilon(q)\widetilde{\rho}_\epsilon^\dagger(q) + g_2\widetilde{\rho}_\epsilon(q)\widetilde{\rho}_{-\epsilon}^\dagger(q)\right] \tag{2.7.19}$$

and using (2.7.17), we arrive at

$$H = v\sum_q |q| a^\dagger(q)a(q) \quad , \quad v = \frac{1}{2\pi}\sqrt{(2\pi v_F + g_4)^2 - g_2^2} \tag{2.7.20}$$

under the choice

$$\tanh(2\theta_q) = \frac{g_2}{2\pi v_F + g_4} \tag{2.7.21}$$

showing that the elementary excitations are bosonic quasi-particles (charge density waves) obeying the dispersion relation

$$\omega = v|q| \tag{2.7.22}$$

In the absence of interactions ($g_2 = g_4 = 0$), we have $v = v_F$ and the quasi-particles travel at the speed of free Fermi particles.
Inclusion of spin degrees of freedom does not alter the above conclusion, but in addition to density waves one obtains spin density waves.

2.8 Finite temperature

As in the bosonic case, we define the propagator as the ensemble average

$$D^{rs}(\vec{x},\tau;\vec{x}',\tau') = -\langle T[\psi^{(r)}(\vec{x},\tau)\psi^{(s)\dagger}(\vec{x}',\tau')]\rangle \tag{2.8.1}$$

Similarly, we show that

$$D^{rs}(\vec{x},0;\vec{x}',\tau') = -D^{rs}(\vec{x},\beta;\vec{x}',\tau') \tag{2.8.2}$$

i.e., the propagator is *anti*-periodic in imaginary time with period β. We deduce the Fourier representation

$$D^{rs}(\vec{x},\tau;\vec{x}',\tau') = \frac{1}{\beta}\sum_n e^{-i\omega_n(\tau-\tau')}D^{rs}(\vec{x},\vec{x}';\omega_n) \quad , \quad \omega_n = \frac{(2n+1)\pi}{\beta} \tag{2.8.3}$$

and inversely,

$$D^{rs}(\vec{x},\vec{x}';\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau} D^{rs}(\vec{x},\tau;\vec{x}',0) \tag{2.8.4}$$

in the case where the propagator is only a function of the difference $\tau - \tau'$.
For an ideal gas of fermions, the grand partition function is easily seen to be

$$\mathcal{Z} = \prod_{r=1}^{2}\prod_{\vec{k}}\sum_{N=0}^{1}\left(e^{-\beta(E_k-\mu)}\right)^N = \prod_{\vec{k}}\left(1+e^{-\beta(E_k-\mu)}\right)^2 \tag{2.8.5}$$

leading to the thermodynamic potential

$$\Omega = -\frac{1}{\beta}\ln\mathcal{Z} = -\frac{2}{\beta}\sum_{\vec{k}}\ln\left(1+e^{-\beta(E_k-\mu)}\right) \tag{2.8.6}$$

The mean number of particles is

$$\langle\mathcal{N}\rangle = 2\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} = \sum_{\vec{k}}\frac{1}{1+e^{-\beta(E_k-\mu)}} \tag{2.8.7}$$

This can also be derived by an argument based on the propagator, as in the bosonic case. We also similarly obtain

$$E = \frac{3}{2}pV = 2\frac{V}{4\pi^2}(2m)^{3/2}\int_0^\infty dE_k\frac{E_k^{3/2}}{e^{\beta(E_k-\mu)}+1} \tag{2.8.8}$$

$$N = 2\frac{V}{4\pi^2}(2m)^{3/2}\int_0^\infty dE_k\frac{E_k^{1/2}}{e^{\beta(E_k-\mu)}+1} \tag{2.8.9}$$

At zero temperature ($\beta\to\infty$), the Fermi distribution becomes a step function,

$$\frac{1}{e^{\beta(E_k-\mu)}+1} \to \theta(\mu - E_k) \tag{2.8.10}$$

showing that all energy levels up to the Fermi energy $E_F = \mu$ have been filled.
For the number of particles, we obtain

$$N = 2\frac{V}{4\pi^2}(2m)^{3/2}\int_0^\mu dE_k\, E_k^{1/2} = 2\frac{V}{4\pi^2}(2m)^{3/2}\frac{2}{3}\mu^{3/2} \tag{2.8.11}$$

Solving for μ, we obtain

$$\mu(T=0) = E_F = \frac{1}{2m}\left(\frac{3\pi^2N}{V}\right)^{2/3} \tag{2.8.12}$$

confirming our earlier result (2.7.5).
The energy at $T=0$ is

$$E = 2\frac{V}{4\pi^2}(2m)^{3/2}\int_0^\mu dE_k\, E_k^{3/2} = 2\frac{V}{4\pi^2}(2m)^{3/2}\frac{2}{5}\mu^{5/2} = \frac{3}{5}E_FN \tag{2.8.13}$$

and the pressure is finite ($p = \frac{2}{3}\frac{E}{V}$).
At small but non-zero temperature, the energy can be written as

$$E = 2\frac{V}{4\pi^2}(2m)^{3/2}\left[\int_0^\mu dE_k\, E_k^{3/2} - \int_0^\mu dE_k\,\frac{E_k^{3/2}}{e^{-\beta(E_k-\mu)}+1} + \int_\mu^\infty dE_k\,\frac{E_k^{3/2}}{e^{\beta(E_k-\mu)}+1}\right] \tag{2.8.14}$$

Adding the exponentially small (in the limit $\beta \to \infty$) integral $\int_{-\infty}^0 dE_k\,\frac{E_k^{3/2}}{e^{-\beta(E_k-\mu)}+1}$ to and changing variables $E_k \to 2\mu - E_k$ in the second integral, we have

$$E = 2\frac{V}{4\pi^2}(2m)^{3/2}\left[\frac{2}{5}\mu^{5/2} + \int_\mu^\infty dE_k\,\frac{E_k^{3/2}-(2\mu-E_k)^{3/2}}{e^{\beta(E_k-\mu)}+1} + \dots\right] \tag{2.8.15}$$

The first contribution is similar to the $T = 0$ expression calculated above, except that $\mu \neq E_F$ and needs to be determined. The second contribution contains first-order corrections to the zero-temperature result.
Changing variables to $x = \beta(E_k - \mu)$, we obtain

$$E = 2\frac{V}{4\pi^2}(2m)^{3/2}\left[\frac{2}{5}\mu^{5/2} + \frac{1}{\beta^{5/2}}\int_0^\infty dx\,\frac{(x+\beta\mu)^{3/2}-(\beta\mu-x)^{3/2}}{e^x+1} + \dots\right] \tag{2.8.16}$$

In the limit of large β (and finite $\mu \approx E_F$), we have $(x+\beta\mu)^{3/2} - (\beta\mu - x)^{3/2} \approx 3x(\beta\mu)^{1/2}$, therefore

$$\begin{aligned} E &= 2\frac{V}{4\pi^2}(2m)^{3/2}\left[\frac{2}{5}\mu^{5/2} + \frac{3}{\beta^2}\mu^{1/2}\int_0^\infty dx\,\frac{x}{e^x+1} + \dots\right] \\ &= 2\frac{V}{4\pi^2}(2m)^{3/2}\left[\frac{2}{5}\mu^{5/2} + \mu^{1/2}\frac{\pi^2}{4}(k_BT)^2 + \dots\right] \end{aligned} \tag{2.8.17}$$

The chemical potential is determined from

$$N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} = \frac{2}{3}\left(\frac{\partial E}{\partial\mu}\right) = 2\frac{V}{4\pi^2}(2m)^{3/2}\frac{2}{3}\left[\mu^{3/2} + \frac{\pi^2}{8\mu^{1/2}}(k_BT)^2 + \dots\right] \tag{2.8.18}$$

Solving for μ as a power series in T, we obtain

$$\mu = E_F - \frac{\pi^2}{12}\frac{1}{E_F}(k_BT)^2 + \dots \tag{2.8.19}$$

The entropy is

$$\begin{aligned} S = -\left(\frac{\partial\Omega}{\partial T}\right)_{\mu,V} &= \frac{2}{3}\left(\frac{\partial E}{\partial T}\right)_{\mu,V} = 2\frac{V}{4\pi^2}(2m)^{3/2}\frac{2}{3}\left[\mu^{1/2}\frac{\pi^2}{2}k_B^2T + \dots\right] \\ &= Nk_B\frac{\pi^2}{2}\frac{1}{\beta E_F} + \dots \end{aligned} \tag{2.8.20}$$

and the heat capacity is

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_{V,N} = N k_B \frac{\pi^2}{2} \frac{1}{\beta E_F} + \ldots \tag{2.8.21}$$

vanishing linearly with temperature as $T \to 0$.
At high temperatures ($\beta \to 0$), we obtain the standard result

$$C_V \to \frac{3}{2} k_B N \tag{2.8.22}$$

due to Boltzmann statistics (same as in the bosonic case). There is no discontinuity as we go from low to high temperature (unlike in the bosonic case).

2.9 Superconductors

The microscopic model of Bardeen, Cooper and Schrieffer (BCS) explains the properties of simple superconductors in terms of a few experimental parameters. It is based on the observation that at low temperatures an instability develops and a condensate forms consisting of a pair of electrons of opposite spin forming a system of zero spin (Cooper pairs). These composite particles are bosons and can undergo Bose condensation. Define the gap by

$$\Delta = g \langle \psi^{(2)}(\vec{x}) \psi^{(1)}(\vec{x}) \rangle \tag{2.9.1}$$

Below a certain (critical) temperature, $\Delta \neq 0$. To simplify the discussion, we shall assume a uniform medium so that Δ is constant. $g > 0$ is a coupling constant determined by the effective *attractive* interaction between electrons close to the Fermi surface. We shall approximate this interaction by a contact potential of strength $g > 0$. The grand canonical Hamiltonian is

$$\begin{aligned} H - \mu \mathcal{N} &= -\sum_{r=1}^{2} \int d^3x \psi^{(r)\dagger}(\vec{x}) \left[\frac{1}{2m} \nabla^2 + \mu \right] \psi^{(r)}(\vec{x}) \\ &\quad - \frac{g}{2} \sum_{r,s=1}^{2} \int d^3x \psi^{(r)\dagger}(\vec{x}) \psi^{(s)\dagger}(\vec{x}) \psi^{(s)}(\vec{x}) \psi^{(r)}(\vec{x}) \end{aligned} \tag{2.9.2}$$

Expanding around the condensate, we obtain

$$\begin{aligned} H - \mu \mathcal{N} &= -\sum_{r=1}^{2} \int d^3x \psi^{(r)\dagger}(\vec{x}) \left[\frac{1}{2m} \nabla^2 + \mu \right] \psi^{(r)}(\vec{x}) \\ &\quad - \int d^3x \left[\Delta \psi^{(1)\dagger}(\vec{x}) \psi^{(2)\dagger}(\vec{x}) + \text{h.c.} \right] \\ &\quad - \int d^3x \left[g \psi^{(1)\dagger}(\vec{x}) \psi^{(2)\dagger}(\vec{x}) - \Delta^* \right] \left[g \psi^{(2)}(\vec{x}) \psi^{(1)}(\vec{x}) - \Delta \right] \end{aligned} \tag{2.9.3}$$

where we discarded an irrelevant constant term. The last term is small and can be neglected. We arrive at the BCS effective Hamiltonian

$$\begin{aligned} K_{BCS} &= -\sum_{r=1}^{2}\int d^3x\psi^{(r)\dagger}(\vec{x})\left[\frac{1}{2m}\nabla^2+\mu\right]\psi^{(r)}(\vec{x}) \\ &\quad -\int d^3x\left[\Delta\psi^{(1)\dagger}(\vec{x})\psi^{(2)\dagger}(\vec{x})+\text{h.c.}\right] \end{aligned} \tag{2.9.4}$$

This is in fact a self-consistent formulation *(Hartree-Fock approximation)* because the condensate (2.9.1) is an ensemble average evaluated with the grand canonical Hamiltonian K_{BCS}.
To understand the dynamics, notice that the time-dependent fields obey the field equations

$$\partial_\tau\psi^{(r)} = \left[\frac{1}{2m}\nabla^2+\mu\right]\psi^{(r)}+\Delta\epsilon^{rs}\psi^{(s)\dagger} \tag{2.9.5}$$

where ϵ^{rs} is an anti-symmetric matrix with $\epsilon^{12}=1$.
For the propagator (2.8.1), we deduce

$$\left[-\partial_\tau+\frac{1}{2m}\nabla^2+\mu\right]D^{11}(\vec{x},\tau;\vec{x}',0)+\Delta\hat{D}^{12*}(\vec{x},\tau;\vec{x}',0)=\delta^3(\vec{x}-\vec{x}')\delta(\tau) \tag{2.9.6}$$

where

$$\hat{D}^{rs}(\vec{x},\tau;\vec{x}',\tau') = -\langle T[\psi^{(r)}(\vec{x},\tau)\psi^{(s)}(\vec{x}',\tau')]\rangle \tag{2.9.7}$$

For this new propagator, we find similarly

$$\left[\partial_\tau+\frac{1}{2m}\nabla^2+\mu\right]\hat{D}^{12*}(\vec{x},\tau;\vec{x}',0)-\Delta^*D^{11}(\vec{x},\tau;\vec{x}',0)=0 \tag{2.9.8}$$

We need to solve the system of equations (2.9.6) and (2.9.8) together with the self-consistency condition (gap equation) (2.9.1), which may be written as

$$\mathcal{E}_g = g\hat{D}^{12}(\vec{x},0^+;\vec{x},0) \tag{2.9.9}$$

Introducing the Fourier representation (2.8.3), the three equations to be solved read, respectively,

$$\left[i\omega_n+\frac{1}{2m}\nabla^2+\mu\right]D^{11}(\vec{x},\vec{x}';\omega_n)+\mathcal{E}_g\hat{D}^{12*}(\vec{x},\vec{x}';\omega_n)=\delta^3(\vec{x}-\vec{x}') \tag{2.9.10}$$

$$\left[-i\omega_n+\frac{1}{2m}\nabla^2+\mu\right]\hat{D}^{12*}(\vec{x},\vec{x}';\omega_n)-\mathcal{E}_g^*D^{11}(\vec{x},\vec{x}';\omega_n)=0 \tag{2.9.11}$$

$$\mathcal{E}_g = \frac{g}{\beta}\sum_n\hat{D}^{12}(\vec{x},\vec{x};\omega_n) \tag{2.9.12}$$

The first two equations are easily solved by taking Fourier transforms in $\vec{x}$ turning them into algebraic equations. We deduce

$$D^{11}(\vec{x},\vec{x}';\omega_n) = -\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\frac{i\omega_n+E_k-\mu}{\omega_n^2+(E_k-\mu)^2+|\mathcal{E}_g|^2} \tag{2.9.13}$$

$$\hat{D}^{12}(\vec{x},\vec{x}';\omega_n) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{\mathcal{E}_g}{\omega_n^2 + (E_k-\mu)^2 + |\mathcal{E}_g|^2} \quad (2.9.14)$$

The gap equation reads

$$1 = \frac{g}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_n^2 + (E_k-\mu)^2 + |\mathcal{E}_g|^2} \quad (2.9.15)$$

where we divided both sides by $\mathcal{E}_g$. The series can be evaluated by splitting

$$\frac{1}{\omega_n^2+\delta_k^2} = \frac{1}{2\delta_k}\left[\frac{1}{i\omega_n+\delta_k} - \frac{1}{i\omega_n-\delta_k}\right] \quad , \quad \delta_k = \sqrt{(E_k-\mu)^2 + |\mathcal{E}_g|^2} \quad (2.9.16)$$

and using (1.8.20). We obtain

$$\sum_n \frac{1}{\omega_n^2+\delta_k^2} = \frac{\beta}{2\delta_k}\tanh\frac{\beta\delta_k}{2} \quad (2.9.17)$$

and the gap equation reads

$$1 = g\int \frac{d^3k}{(2\pi)^3 2\delta_k}\tanh\frac{\beta\delta_k}{2} \quad (2.9.18)$$

The range of integration is a narrow shell around the Fermi surface of width $\omega_D = k_B\theta_D \ll E_F$ (Debye energy). Switching variables to $\omega = E_k - \mu$, we obtain

$$1 = g\frac{mk_F}{4\pi^2}\int_{-\omega_D}^{\omega_D} \frac{d\omega}{\sqrt{\omega^2+|\mathcal{E}_g|^2}}\tanh\frac{\sqrt{\omega^2+|\mathcal{E}_g|^2}}{2k_BT} \quad (2.9.19)$$

As $T \to 0$, we obtain

$$|\mathcal{E}_g| \to E_g = 2\omega_D e^{-\frac{2\pi^2}{mk_Fg}} \quad (2.9.20)$$

The gap vanishes at the critical temperature T_c, so

$$1 = g\frac{mk_F}{2\pi^2}\int_0^{\omega_D}\frac{d\omega}{\omega}\tanh\frac{\omega}{2k_BT_c} = g\frac{mk_F}{2\pi^2}\int_0^{\omega_D/(2k_BT_c)}\frac{dx}{x}\tanh x \quad (2.9.21)$$

Integrating by parts, we obtain

$$\frac{2\pi^2}{mk_Fg} = [\ln x\tanh x]_0^{\omega_D/(2k_BT_c)} - \int_0^{\omega_D/(2k_BT_c)} dx\frac{\ln x}{\cosh^2 x} \quad (2.9.22)$$

In the last integral we can replace $\omega_D/(2k_BT_c)$ with ∞, because it's large. We finally arrive at

$$k_BT_c = \frac{2e^\gamma}{\pi}\omega_D e^{-\frac{2\pi^2}{mk_Fg}} \approx 1.13\omega_D e^{-\frac{2\pi^2}{mk_Fg}} \quad (2.9.23)$$

Notice that the ratio

$$\frac{E_g}{k_BT_c} = \pi e^{-\gamma} \approx 1.76 \quad (2.9.24)$$

is a *universal constant* independent of the material.

UNIT 3

Photons (spin 1)

3.1 Massive fields

3.1.1 The classical theory

We have already seen how to build a theory of relativistic scalar (spin-0) and spinor (spin-1/2) fields. Including relativity was simple for the scalar - the Klein-Gordon equation (1.2.33) provided a relativistic description. The spinor was a little trickier. A non-relativistic spinor has two degrees of freedom (e.g., $\uparrow$ (up) or $\downarrow$ (down)), but the relativistic spinor had to have *four* components (massive Dirac field). We saw that the extra two degrees of freedom had a physical meaning: they represented the *anti*-particle. Thus, including relativity led to the necessary existence of anti-particles for spinors.[1]

Now we wish to include photons into the picture, which are *vectors* (spin-1 particles). Naïvely, they have *three* degrees of freedom, however they only come with transverse polarization, so they really only have two degrees of freedom. This is possible only because they are massless, so they always travel at the speed of light. There is no non-relativistic description of photons!

To draw on familiar concepts, imagine that the photon has a small mass. Then, being a vector, it will have to have three degrees of freedom ("transversality" is a meaningless concept, because the photon can now be at rest). If we include relativity, the smallest object that constains a vector (and can therefore accommodate the degrees of freedom of the photon) is a four-vector A^μ. This has *four* components: $\vec{A}$ is the actual vector and A^0 is a scalar (from a three-dimensional (non-relativistic) point of view). Therefore, it appears that a relativistic vector particle must have four degrees of freedom. What is the physical meaning of the fourth degree of freedom?

To answer this question, let us construct a Lagrangian density. For the "kinetic-energy" term, we have two possibilities (*cf.* the scalar case (1.2.32)),

$$\partial_\mu A^\nu \partial^\mu A_\nu \ , \ \ \partial_\mu A^\nu \partial_\nu A^\mu \tag{3.1.1}$$

[1]The *real* scalar field is its own anti-particle, which is why nothing exciting occurs when one includes relativity.

All others may be obtained from the above by adding total derivatives. For the mass term, we only have one possibility,

$$m^2 A_\mu A^\mu \tag{3.1.2}$$

Thus, the most general Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A^\nu \partial^\mu A_\nu + a\partial_\mu A^\nu \partial_\nu A^\mu - m^2 A_\mu A^\mu) \tag{3.1.3}$$

We shall assume that $m \neq 0$. The massless case ($m = 0$) will be discussed later. The field equation is

$$\partial_\mu \partial^\mu A_\nu + a\partial_\mu \partial_\nu A^\mu + m^2 A_\nu = 0 \tag{3.1.4}$$

It admits plane-wave solutions,

$$A_\mu = e_\mu e^{-ip\cdot x} \tag{3.1.5}$$

where e_μ is the polarization vector. Plugging into the field equation, we obtain

$$-p^2 e_\nu - ap_\nu e\cdot p + m^2 e_\nu = 0 \tag{3.1.6}$$

This is easily solved in the rest frame in which $p^\mu = (M, \vec{0})$, where M does not necessarily coincide with m. We obtain

$$(m^2 - M^2)e_\nu = ap_\nu e\cdot p \tag{3.1.7}$$

Four linearly independent polarizations are

$$e_\mu^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \;,\; e_\mu^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \;,\; e_\mu^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \;,\; e_\mu^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.1.8}$$

$e_\mu^{(0)}$ represents *longitudinal* polarization ($e_\mu^{(0)} \propto p_\mu$), the rest are *transverse* ($e^{(i)} \cdot p = 0$). For the longitudinal polarization, we deduce from (3.1.7), setting $\nu = 0$, that $p_\mu = Me_\mu^{(0)}$ (valid in arbitrary frame) and

$$M^2 = \frac{m^2}{1+a} \tag{3.1.9}$$

Thus we get a scalar particle of mass M (you can tell it is a Lorentz scalar because it has no direction other than p_μ). The other three transverse polarizations all have (since the right-hand side of (3.1.7) vanishes)

$$M^2 = m^2 \tag{3.1.10}$$

They form a vector particle of mass m.

If we want to build a theory of vector particles only, we'd better get rid of the scalar. That's easy: let

$$M \to \infty \tag{3.1.11}$$

then the scalar becomes infinitely heavy and decouples. Thus we ought to choose

$$a = -1 \tag{3.1.12}$$

In this case, the longitudinal polarization $e_\mu^{(0)}$ is no longer a possibility: eq. (3.1.7) leads to $m^2 - M^2 = -M^2$ and so $m = 0$, which is a case we are not currently considering. Therefore, only transverse polarizations may exist, i.e., $p \cdot e = 0$. In terms of A_μ (Fourier transform), we have

$$\partial_\mu A^\mu = 0 \tag{3.1.13}$$

This is a constraint subtracting one degree of freedom from the original four. We end up with three degrees of freedom, as desired. How did it come about?
The field equation with $a = -1$ reads

$$\partial^\mu F_{\mu\nu} = -m^2 A_\nu \ , \ \ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.1.14}$$

If $m = 0$, these are the *Maxwell* equations (without currents) in electrodynamics. Taking the divergence of both sides, we obtain

$$\partial^\nu \partial^\mu F_{\mu\nu} = -m^2 \partial^\nu A_\nu \tag{3.1.15}$$

Since $F_{\mu\nu}$ is anti-symmetric, the left-hand side vanishes, showing that the constraint (3.1.13) is necessary for the consistency of the field equations if $a = -1$. The latter may be written as (eq. (3.1.14) with (3.1.13))

$$\partial_\mu \partial^\mu A_\nu + m^2 A_\nu = 0 \tag{3.1.16}$$

showing that each component of A_μ satisfies the Klein-Gordon equation (1.2.33).

3.1.2 Canonical quantization

The Lagrangian density (3.1.3) with $a = -1$ may be written as

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m^2 A_\mu A^\mu \tag{3.1.17}$$

The conjugate momenta are

$$E_0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0 \ , \ \ E_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = F_{0i} \tag{3.1.18}$$

Thus, there are only three momenta, as expected, since there are only three degrees of freedom. I called them E_i instead of the customary Π_i, because they coincide with the "electric" field F_{0i}. Since $E_0 = 0$, it follows that A_0 is not a dynamical field. It can be expressed through the other fields by using the field equation (3.1.14) for $\nu = 0$,

$$A_0 = -\frac{1}{m^2} \vec{\nabla} \cdot \vec{E} \tag{3.1.19}$$

The Hamiltonian density is

$$\mathcal{H} = \vec{E} \cdot \partial_0 \vec{A} - \mathcal{L} \tag{3.1.20}$$

The Lagrangian density may be written as

$$\mathcal{L} = \tfrac{1}{2}\vec{E}^2 - \tfrac{1}{2}\vec{B}^2 - \tfrac{1}{2}m^2\vec{A}^2 + \tfrac{1}{2}m^2(A^0)^2 \tag{3.1.21}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$ is the "magnetic" field. Then the Hamiltonian density is given by

$$\mathcal{H} = \tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}\vec{B}^2 + \tfrac{1}{2}m^2\vec{A}^2 - \tfrac{1}{2}m^2(A_0)^2 - \vec{\nabla}\cdot\vec{E}A_0 \tag{3.1.22}$$

where I omitted a divergence, $\vec{\nabla}\cdot(A_0\vec{E})$. Using (3.1.19) to eliminate the non-dynamical field A_0 , we arrive at

$$\mathcal{H} = \tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}\vec{B}^2 + \tfrac{1}{2}m^2\vec{A}^2 + \frac{1}{2m^2}(\vec{\nabla}\cdot\vec{E})^2 \tag{3.1.23}$$

a manifestly positive quantity written explicitly in terms of the three fields $\vec{A}$ and their conjugate momenta $\vec{E}$. It is a quadratic expression, so quantization proceeds along the same lines as in the Klein-Gordon case.

3.1.3 The massless limit

If $m = 0$, we should recover the photon which has two degrees of freedom. However, the massive vector has three degrees of freedom no matter how small m is. Is there a discontinuity? If so, we would easily be able to tell if the photon had a mass (even tiny) by looking at the specific heat of a black body (e.g., the Sun). Since only two degrees of freedom contribute to the specific heat of the Sun (three degrees of freedom would produce a 50% higher measurement), are we to conclude that the photon is exactly massless? Or, does the third degree of freedom (the longitudinal one, i.e., $\vec{e} \propto \vec{p}$, not to be confused with the "longitudinal" polarization discarded earlier, $e_\mu \propto p_\mu$) somehow decouple as $m \to 0$? It turns out that the latter is the case, so no measurement can ever tell us for sure that the photon is exactly massless. Let's see how this decoupling occurs.

Couple A_μ to matter represented by a current J^μ. The Lagrangian density (3.1.17) gets an additional term,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2A_\mu A^\mu - A_\mu J^\mu \tag{3.1.24}$$

The field equation (3.1.14) is modified to

$$\partial^\mu F_{\mu\nu} + m^2 A_\nu = J_\nu \tag{3.1.25}$$

Taking its divergence and using the constraint (3.1.13), we conclude

$$\partial_\nu J^\nu = 0 \tag{3.1.26}$$

showing that the current J^μ must be conserved. The current will emit "photons" due to its coupling. The amplitude for the emission of a "photon" with momentum p^μ and polarization e^μ is

$$\mathcal{A} \sim e\cdot\tilde{J}(p) \tag{3.1.27}$$

where $\tilde{J}^\mu$ is the Fourier transform of J^μ. Choose axes so the $\vec{p}$ is along the z-axis. Then

$$p_\mu = (E\ \ 0\ \ 0\ \ p)\ ,\ \ E = \sqrt{m^2+p^2} \tag{3.1.28}$$

Since $p \cdot e = 0$ (from the constraint (3.1.13)), there are three possible polarizations (linearly independent). Choose

$$e^{(1)}_\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad e^{(2)}_\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad e^{(3)}_\mu = \frac{1}{m} \begin{pmatrix} p \\ 0 \\ 0 \\ E \end{pmatrix} \tag{3.1.29}$$

normalized by $e^2 = -1$. $e^{(3)}_\mu$ represents *longitudinal* polarization ($\vec{e}^{(3)} \propto \vec{p}$), the other two are *transverse* ($\vec{e}^{(i)} \perp \vec{p}$, $i = 1, 2$).
From current conservation (3.1.26) we have $p \cdot \tilde{J} = 0$ and so

$$E\tilde{J}^0 - p\tilde{J}^3 = 0 \tag{3.1.30}$$

The amplitude for the emission of a transverse mode is

$$\mathcal{A}^{(i)} \sim e^{(i)} \cdot \tilde{J} = \tilde{J}^i \quad (i = 1, 2) \tag{3.1.31}$$

whereas for the longitudinal mode we obtain

$$\mathcal{A}^{(3)} \sim e^{(3)} \cdot \tilde{J} = \frac{1}{m}(p\tilde{J}^0 - E\tilde{J}^3) = -\frac{m}{p}\tilde{J}^0 \tag{3.1.32}$$

where we used (3.1.30) to eliminate $\tilde{J}^3$. Thus, as $m \to 0$, $\mathcal{A}^{(3)} \to 0$, i.e., the emission of a longitudinal mode is suppressed. In a black body, these modes cannot be in thermal equilibrium with the other two (transverse) states, unless we wait for a very long time ($o(1/m)$).

3.2 Electrodynamics

3.2.1 Gauge invariance

Set $m = 0$ in (3.1.24). We obtain

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \tag{3.2.1}$$

leading to the field equations

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{3.2.2}$$

which are the Maxwell equations of electrodynamics. Taking divergences of both sides, we deduce

$$\partial_\nu J^\nu = 0 \tag{3.2.3}$$

i.e., the current must be conserved. This is necessary for the consistency of the field equations and not an input to the theory. We know from our discussion above that one of the degrees of freedom decouples in this massless limit leaving only two (transverse) degrees of freedom. We would like to understand this in the context of the Lagrangian (3.2.1) without reference to limits of other (physically irrelevant) theories.

To this end, observe that the theory defined by (3.2.1) has a *local* symmetry (gauge symmetry). Under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \omega \tag{3.2.4}$$

where $\omega(x^\mu)$ is an arbitrary function of spacetime, $F_{\mu\nu}$ is invariant and the Lagrangian density changes by

$$\delta\mathcal{L} = \partial_\mu \mathcal{F}^\mu \ , \ \mathcal{F}^\mu = -\omega J^\mu \tag{3.2.5}$$

where I used (3.2.3). The Noether current is found from (1.5.5),

$$K^\mu = F^{\mu\nu}\partial_\nu \omega + \omega J^\mu \tag{3.2.6}$$

The conserved charge is

$$Q = \int d^3 K^0 = \int d^3 x (\vec{\nabla} \cdot \vec{E} - \rho)\omega \tag{3.2.7}$$

where $\vec{E}$ is the electric field, $\rho = J^0$ is the electric charge density and I discarded a total divergence. This charge generates gauge transformations and is given by Gauss's Law. This also implies that the charge vanishes identically!
To see what this means, compare with a *global* symmetry, e.g., translational invariance, which implies conservation of momentum, P^μ, which is the Noether charge that generates the transformations (translations). To be concrete, consider our solar system and let $\vec{x}_\odot$ be the position of the Sun. If the Sun had made a primordial decision to locate at $\vec{x}_\odot + \vec{a}$, it would be OK, as long as all the planets moved by $\vec{a}$ as well. This would have resulted in an *equivalent* system, albeit a different one. There is nothing special about $\vec{x}_\odot$, it just happened to be our Sun's choice.
As another example, consider the elliptical orbit of the Earth around the Sun. The axis of the ellipse is arbitrary. The Earth would be just as happy with any other choice of axis. All axes are related to each other by rotations which are generated by angular momentum (the Noether charge). They correspond to distinct but *equivalent* systems.
In the case of gauge transformations, since $Q = 0$, when applied to an observable, it does not change it,

$$\delta\mathcal{O} \sim [Q, \mathcal{O}] = 0 \tag{3.2.8}$$

Thus, two vector potentials related to each other by a gauge transformation (3.2.4) (gauge-equivalent) represent the same physical system, not merely two equivalent systems. We may select one A_μ out of all possible vector potentials in an equivalence class and the choice is arbitrary. This can be done by imposing a constraint, which is often referred to as *choosing a gauge.* This is what reduces the number of degrees of freedom from three to the observed two for the photon.
EXAMPLE: Choose the axial gauge,

$$A_3 = 0 \tag{3.2.9}$$

Given A_μ, we can find the potential $\bar{A}_\mu$ which is gauge-equivalent to A_μ and obeys (3.2.9), by choosing

$$\omega = -\int_{-\infty}^{z} A_3 \tag{3.2.10}$$

Using (3.2.4), we find $\bar{A}_3 = 0$, as desired. The choice of $\bar{A}_\mu$ is unique if we assume ω vanishes at infinity. Indeed, if there were two different gauge-equivalent potentials satisfying (3.2.9), then $\partial_3\omega = 0$ for some $\omega \neq 0$, so ω would be independent of z and equal to its value as $z \to \infty$ which, by assumption is zero; a contradiction.
In this case the number of degrees of freedom are explicitly two: A_1 and A_2 (A_0 is *not* dynamical).

3.2.2 Canonical quantization

Working in the axial gauge (3.2.9), there are only two dynamical fields, A_1 and A_2. Switching to the Hamiltonian formalism, we have two conjugate momenta,

$$E_i = F_{0i} \ , \ \ i = 1, 2 \tag{3.2.11}$$

These are two of the components of the electric field. The z-component is given in terms of the time component of the vector potential,

$$E_3 = -\partial_3 A_0 \tag{3.2.12}$$

which in turn is expressed in terms of the dynamical fields via Gauss's Law,

$$\partial_3^2 A_0 = J_0 + \partial_1 E_1 + \partial_2 E_2 \tag{3.2.13}$$

The Hamiltonian density is

$$\mathcal{H} = \vec{E} \cdot \partial_0 \vec{A} - \mathcal{L} \tag{3.2.14}$$

(recall that there is no third component in $\vec{A}$). The Lagrangian density may be written as

$$\mathcal{L} = \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) - A_\mu J^\mu \tag{3.2.15}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$ is the magnetic field. We obtain

$$\mathcal{H} = \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2) + \vec{A} \cdot \vec{J} \tag{3.2.16}$$

which is the standard form of the electromagnetic energy. It is written explicitly in terms of the dynamical fields A_1 and A_2 and their conjugate momenta E_1 and E_2. Being quadratic in the fields, its quantization proceeds as in the scalar case. Unitarity is manifest, but Lorentz invariance is not (even rotational invariance is elusive). Fortunately, due to the gauge symmetry, to establish Lorentz invariance, we may work in a different gauge. All gauge choices lead to the same physical results.

3.2.3 The Lorentz gauge

Here is a gauge choice that will lead to manifestly Lorentz-invariant results (the Lorentz gauge)

$$\partial_\mu A^\mu = 0 \tag{3.2.17}$$

The Lagrangian density (3.2.1) may be modified to

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\lambda}(\partial_\mu A^\mu)^2 - A_\mu J^\mu \tag{3.2.18}$$

where λ is an arbitrary parameter. The additional term vanishes on account of the gauge-fixing condition (3.2.17) and should not affect any physical quantities (they should all be independent of λ). The field (Maxwell) equations now read

$$\partial_\mu F^{\mu\nu} - \frac{1}{\lambda}\partial^\nu \partial_\mu A^\mu = J^\nu \tag{3.2.19}$$

They can be solved in terms of the Green function *(Feynman propagator)* $D_F^{\mu\nu}$ satisfying

$$\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{\lambda}\partial_\mu\partial_\nu\right) D_F^{\nu\rho}(x) = i\delta_\mu^\rho \delta^4(x) \tag{3.2.20}$$

We obtain

$$A^\mu(x) = i\int d^4y D_F^{\mu\nu}(x-y)J_\nu(y) \tag{3.2.21}$$

The propagator can be written as (*cf.* eq. (1.4.19))

$$D_F^{\mu\nu}(x-y) = \langle 0|T(A^\mu(x)A^\nu(y))|0\rangle \tag{3.2.22}$$

To calculate it, take Fourier transforms,

$$D_F^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \tilde{D}_F^{\mu\nu}(k) \tag{3.2.23}$$

We deduce

$$\left(\eta_{\mu\nu}k^2 - k_\mu k_\nu + \frac{1}{\lambda}k_\mu k_\nu\right)\tilde{D}_F^{\nu\rho} = -i\delta_\mu^\rho \tag{3.2.24}$$

Notice that in the absence of λ ($\lambda \to \infty$), eq. (3.2.24) cannot be solved because the matrix multiplying the propagator is not invertible (dot both sides of (3.2.24) with k^μ to obtain a contradiction). The general form of $\tilde{D}_F^{\mu\nu}$ dictated by its tensor structure is

$$\tilde{D}_F^{\mu\nu} = \frac{-i}{k^2+i\epsilon}\left(A\eta^{\mu\nu} + B\frac{k^\mu k^\nu}{k^2}\right) \tag{3.2.25}$$

where A and B are dimensionless coefficients and $i\epsilon$ was added as in the scalar case (eq. (1.4.16) with $m=0$). It determines which poles contribute and defines the boundary conditions for the Green function.
Plugging (3.2.25) into (3.2.24), we obtain the unique solution $A = 1$, $B = \lambda - 1$. Therefore,

$$\tilde{D}_F^{\mu\nu} = \frac{-i}{k^2+i\epsilon}\left(\eta^{\mu\nu} + (\lambda-1)\frac{k^\mu k^\nu}{k^2}\right) \tag{3.2.26}$$

Standard choices:

- *Feynman gauge:* $\lambda = 1$,

$$\tilde{D}_F^{\mu\nu} = \frac{-i}{k^2+i\epsilon}\,\eta^{\mu\nu} \tag{3.2.27}$$

  It gives the simplest expression for the propagator.

- *Landau gauge:* $\lambda = 0$,

$$\tilde{D}_F^{\mu\nu} = \frac{-i}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \tag{3.2.28}$$

  It shows transversality explicitly: $k_\mu \tilde{D}_F^{\mu\nu} = 0$, or $\partial_\mu D_F^{\mu\nu} = 0$.

The propagator depends on the arbitrary parameter λ. This is not a problem, because unlike in the scalar and spinor cases, the propagator (3.2.22) is *not* a physical quantity (since A^μ isn't). Here are some physical quantities:

- The coupling of two currents:

$$\mathcal{A} = \tilde{J}_\mu^{(1)} \tilde{D}_F^{\mu\nu} \tilde{J}_\nu^{(2)} \tag{3.2.29}$$

  For a conserved current, $k_\mu \tilde{J}^\mu(k) = 0$, so

$$\mathcal{A} = \tilde{J}_\mu^{(1)} \frac{-i}{k^2 + i\epsilon} \tilde{J}^{(2)\mu} \tag{3.2.30}$$

  In the static case, this gives the *Coulomb potential*.

- The two-point function

$$\mathcal{D}_{\mu\nu;\rho\sigma}(x,y) = \langle 0|T(F_{\mu\nu}(x)F_{\rho\sigma}(y))|0\rangle \tag{3.2.31}$$

  Taking Fourier transforms, we obtain

$$\tilde{\mathcal{D}}_{\mu\nu;\rho\sigma} = \frac{-i}{k^2 + i\epsilon}\left(\eta_{\nu\sigma}k_\mu k_\rho - \eta_{\mu\sigma}k_\nu k_\rho - \eta_{\nu\rho}k_\mu k_\sigma + \eta_{\mu\rho}k_\nu k_\sigma\right) \tag{3.2.32}$$

Both physical quantities are independent of λ.
While Lorentz invariance has been manifest throughout our discussion in the Lorentz gauge, unitarity is far from obvious. Indeed, if you compare the propagator in the Feynman gauge (eq. (3.2.27) to the massless scalar propagator (1.4.16), you realize that the temporal component of the former (D_F^{00}) has the wrong sign! To see the significance of this observation, let us have a closer look at the scalar propagator (1.4.19). To simplify the notation, arrange the ordering so that $x^0 > y^0$ and insert a complete set of states,

$$\mathbb{I} = \int d^3\Sigma_k \frac{|\vec{k}\rangle\langle\vec{k}|}{\langle\vec{k}|\vec{k}\rangle} + \sum_n \frac{|n\rangle\langle n|}{\langle n|n\rangle} \tag{3.2.33}$$

where I separated the one-particle states (eq. (1.3.22)) from the rest (with two or more particles). Moreover, instead of assuming that the states are normalized, I explicitly divided by their respective norms. Eq. (1.4.19) may be written as

$$D_F(x-y) = \int d^3\Sigma_k \frac{\langle 0|\phi(x)|\vec{k}\rangle\langle\vec{k}|\phi(y)|0\rangle}{\langle\vec{k}|\vec{k}\rangle} + \sum_n \frac{\langle 0|\phi(x)|n\rangle\langle n|\phi(y)|0\rangle}{\langle n|n\rangle} \tag{3.2.34}$$

A state $|n\rangle$ contains two or more creation operators, therefore it is orthogonal to $\phi|0\rangle$ which only contains one creation operator. It follows that only one-particle states contribute to the propagator. Moreover, by translation invariance of the vacuum,

$$\langle 0|\phi(x)|\vec{k}\rangle = e^{-ik\cdot x}\langle 0|\phi(0)|\vec{k}\rangle \tag{3.2.35}$$

where I used (1.3.35) and (1.3.33). Therefore,

$$D_F(x-y) = \int d^3\Sigma_k e^{-ik\cdot(x-y)} \frac{|\langle 0|\phi(x)|\vec{k}\rangle|^2}{\langle\vec{k}|\vec{k}\rangle} \tag{3.2.36}$$

This agrees with the expression we derived before (eq. (1.4.4) and use (1.3.19)) if the inner products are equal to 1 (as was the case for a scalar). Now this expression is to be compared with the propagator (3.2.23) in the Feynman gauge (expression (3.2.27)),

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i\eta^{\mu\nu}}{k^2+i\epsilon} \tag{3.2.37}$$

Since $x^0 > y^0$, to integrate over k^0, we ought to close the contour in the upper-half plane. We obtain

$$D_F^{\mu\nu}(x-y) = -\eta^{\mu\nu}\int d^3\Sigma_k\, e^{-ik\cdot(x-y)} \tag{3.2.38}$$

where we expressed the measure as in (1.3.19). The only way the temporal component D_F^{00} can agree with (3.2.36) is if the states contributing have negative norm ($\langle\vec{k}|\vec{k}\rangle < 0$)! These states are called ghosts and lead to negative probabilities and a potential breakdown of unitarity. To show that no calamities occur one needs to show that ghosts do not contribute to physical quantities. This is a tedious exercise, but fortunately, it is unnecessary. Since we have already established the unitarity of the theory working in a different gauge (tha axial gauge), there is no need to repeat this exercise here (in the Lorentz gauge). On the other hand, Lorentz invariance is manifest making it unnecessary to prove it in the axial gauge, where it was far from obvious. Gauge invariance guarantees that no matter which gauge one adopts, the theory will be unitary and Lorentz invariant.

3.2.4 Adding scalars

Gauge invariance is such a powerful symmetry that it will guide us in building interactive theories. We start with a *complex* scalar for which the Lagrangian density is given by (1.5.62) and repeated for convenience here,

$$\mathcal{L} = \partial_\mu\psi\partial^\mu\psi^* - m^2\psi^*\psi \tag{3.2.39}$$

It possesses an internal $U(1)$ symmetry

$$\psi(x) \to e^{i\theta}\psi(x) \tag{3.2.40}$$

leading to the (conserved) Noether current

$$j_\mu = i(\psi\partial_\mu\psi^* - c.c.) \tag{3.2.41}$$

and charge $Q = \int d^3xj^0$. We would like to couple this scalar to the vector potential so that j^μ can be considered an electric current. Notice that we could not have chosen a *real* field because it has no similar current (its excitations are *neutral* particles which are their own anti-particles).
To make use of gauge symmetry, we shall *gauge* the global U(1) symmetry (3.2.40) allowing θ to depend on spacetime,

$$\psi(x) \to e^{-ie\omega(x)}\psi(x) \tag{3.2.42}$$

where e is an arbitrary constant and ω is the same function that appeared in the gauge transformation of the vector potential (3.2.4), $A_\mu \to A_\mu + \partial_\mu\omega$. The Lagrangian is not invariant under the *local* transformation (3.2.42). This may easily be remedied by introducing the gauge derivative

$$D_\mu = \partial_\mu + ieA_\mu \tag{3.2.43}$$

Under a gauge transformation, it transforms as

$$D_\mu \to e^{-ie\omega}D_\mu e^{ie\omega} \tag{3.2.44}$$

Therefore, $D_\mu\psi$ transforms like ψ. If we replace $\partial_\mu \to D_\mu$ in the Lagrangian (3.2.39), the modified Lagrangian density

$$\mathcal{L} = |D_\mu\psi|^2 - m^2|\psi|^2 \tag{3.2.45}$$

is obviously gauge-invariant. The electric current is defined by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial A_\mu} = ie(\psi(D_\mu\psi)^* - c.c.) \tag{3.2.46}$$

It may also be obtained from j_μ (eq. (3.2.41)) by the replacement $\partial_\mu \to D_\mu$ together with the additional factor e. It is a gauge-invariant quantity. It is also conserved, as a short calculation shows,

$$\partial_\mu J^\mu = ie\{|D_\mu\psi|^2 - m^2|\psi|^2 - c.c.\} = 0 \tag{3.2.47}$$

The stress-energy tensor may also be deduced from the Noether current under translations ($x^\mu \to x^\mu + a^\mu$). In the absence of a vector potential, this is a symmetry of the theory and we obtain (*cf.* eq. (1.5.16))

$$\begin{aligned} T_\mu^\nu &= \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi)}\partial_\mu\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi^*)}\partial_\mu\psi^* - \delta_\mu^\nu\mathcal{L} \\ &= \partial_\mu\psi\partial^\nu\psi^* + \partial_\mu\psi^*\partial^\nu\psi - \delta_\mu^\nu(|\partial_\alpha\psi|^2 - m^2|\psi|^2) \end{aligned} \tag{3.2.48}$$

In the presence of A_μ, there is no symmetry under translations, in general, so the stress-energy tensor cannot be obtained as a Noether current. It is easy to turn the above expression into a gauge-invariant quantity by replacing $\partial_\mu \to D_\mu$. We obtain

$$T_\mu^\nu = D_\mu\psi(D^\nu\psi)^* + (D_\mu\psi)^*D^\nu\psi - \delta_\mu^\nu(|D_\alpha\psi|^2 - m^2|\psi|^2) \tag{3.2.49}$$

This is no longer expected to be conserved. A short calculation yields

$$\partial_\nu T^\nu_\mu = F_{\mu\nu} J^\nu \tag{3.2.50}$$

The right-hand side is the *Lorentz four-force density*. To bring it to a more familiar form, integrate over space. The left-hand side gives the time-derivative of the total four-momentum,

$$\int d^3x \partial_\nu T^\nu_\mu = \frac{dP_\mu}{dt} \ , \ P_\mu = \int d^3x T^0_\mu \tag{3.2.51}$$

where we discarded a divergence. For the spatial components, we obtain the total force,

$$\vec{F} = \frac{d\vec{P}}{dt} = \int d^3x(\rho\vec{E} + \vec{J} \times \vec{B}) \tag{3.2.52}$$

which is the familiar form of the Lorentz force.

3.2.5 The Higgs mechanism

As we already know, the complex scalar field can couple to the electromagnetic field. The presence of the Mexican-hat potential does not alter this. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu\phi|^2 - \lambda \left(|\phi|^2 - \frac{v^2}{2} \right)^2 \tag{3.2.53}$$

where D_μ is the gauge derivative (3.2.43). It has a $U(1)$ *gauge* (local) symmetry. In terms of the field σ, the gauge transformation is

$$\sigma \to \sigma - e\omega \tag{3.2.54}$$

therefore different values of σ correspond to the *same* unique ground state. All previously distinct ground states are now in the same gauge equivalency class. It follows that σ is not a dynamical field; in other words the Goldstone boson has disappeared! This is the Higgs mechanism: once the symmetry is gauged, there is no spontaneously broken symmetry and no massless boson.

In terms of ρ and σ, the Lagrangian density reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \rho^2 (\partial_\mu \sigma + eA_\mu)^2 - \frac{\lambda}{4} (\rho^2 - v^2)^2 \tag{3.2.55}$$

and is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \omega \ , \quad \sigma \to \sigma - e\omega \ , \quad \rho \to \rho \tag{3.2.56}$$

To fix the gauge, we may simply choose

$$\sigma = 0 \tag{3.2.57}$$

showing explicitly that σ is not a dynamical field, since it is not even present.

In terms of ρ',

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 v^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \rho' \partial^\mu \rho' - \lambda v^2 \rho'^2 + \text{interactions} \tag{3.2.58}$$

so the system consists of a *massive* vector field of mass

$$m_V = ev \tag{3.2.59}$$

and a scalar of mass

$$m_S = \sqrt{2\lambda v^2} \tag{3.2.60}$$

What happens in the limit $e \to 0$? The vector field becomes massless reducing its degrees of freedom from 3 to 2. We have already seen what happens to the third degree of freedom: it decouples from the other two turning into a scalar. In the present case, it turns into the Goldstone boson! Thus, in the limit $e \to 0$, the massive vector spits off (or "throws up") a Goldstone boson and turns into a transverse photon.
Following this in reverse, as we switch on e ($e \neq 0$), the photon "eats" the Goldstone boson and becomes massive increasing its degrees of freedom by one.

3.2.6 Adding spinors

A massive spinor field $\psi(x)$ is described by the Dirac Lagrangian density (2.3.5) which possesses a global symmetry (eq. (2.3.19)). As with the scalar field, we need to gauge this symmetry to add electromagnetic interactions. The modification is simple; replace $\partial_\mu \to D_\mu$. The resulting Lagrangian density is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - A_\mu J^\mu \quad , \quad J^\mu = eV^\mu \tag{3.2.61}$$

where V^μ is the Noether current (2.3.17) under the symmetry (2.3.19). The electromagnetic current is obtained by multiplying the Noether current by the unit charge e.

3.2.7 Superconductors

As an example, we shall use ψ to describe Cooper pairs in a superconductor (BCS theory). These are pairs of electrons which (unlike spin-1/2 electrons) obey Bose-Einstein statistics, since they have integer spin. This way, they manage to avoid the Pauli exclusion principle and be in the same state at low enough temperatures. Suppose ψ is given by a plane wave, in the rest frame of which we simply have

$$\psi = \psi_0 e^{-imt} \tag{3.2.62}$$

since $p_\mu = (m, \vec{0})$. Immerse this in a static magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ with $\frac{d\vec{A}}{dt} = 0$ and assume vanishing electric field, $\vec{E} = 0$, so $A_0 = 0$. The charge density is given by the time component of (3.2.47),

$$\rho = J^0 = 2em|\psi_0|^2 \tag{3.2.63}$$

The current density is similarly found to be

$$\vec{J} = -e^2|\psi_0|^2\vec{A} = -\frac{\rho e}{2m}\vec{A} \tag{3.2.64}$$

which is the London equation. It is curious that a physical quantity ($\vec{J}$) is proportional to an unphysical quantity ($\vec{A}$). This is not a cause for alarm, because with my assumptions I have actually fixed the gauge (Lorentz gauge, $\partial_\mu A^\mu = 0$, same as the *Coulomb*

gauge $\vec{\nabla} \cdot \vec{A} = 0$, since I set $A_0 = 0$). It shows that $\vec{A}$ has physical meaning by itself and is not merely a useful mathematical devise to define electric and magnetic fields.
Ampère's Law (three of the Maxwell equations (3.2.2)) becomes

$$\vec{\nabla} \times \vec{B} = -\frac{\rho e}{2m}\vec{A} \tag{3.2.65}$$

Taking curls and using $\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}$ and $\vec{\nabla} \cdot \vec{B} = 0$, we obtain

$$\nabla^2 \vec{B} = \frac{1}{\lambda^2}\vec{B} \ , \quad \lambda = \sqrt{\frac{\rho e}{2m}} \tag{3.2.66}$$

where λ is a length scale. If ψ occupies the region $z > 0$ and a constant magnetic field $\vec{B} = B_0 \hat{x}$ is applied in empty space $z < 0$, then the solution for $z > 0$ is

$$B = B_0 e^{-z/\lambda} \hat{x} \tag{3.2.67}$$

(the other solution is discarded because it blows up as $z \to +\infty$). This shows that the magnetic field cannot penetrate the superconductor (Meissner effect).
You may be wondering why this description is not valid for an ordinary conductor. After all, they are the same system - only the temperature changes. The answer is that an ordinary conductor is not in a *coherent state* and cannot be described by a quantum field. This is why it develops *resistance* when a current runs through it. Superconductors are examples of rare *macroscopic quantum* systems.

3.2.8 The Aharonov-Bohm effect

As we showed in eq. (3.2.42), by gauge invariance $\psi(x)$ and $e^{-ie\omega(x)}\psi(x)$ represent the *same* physical system. Therefore, the *phase* of a field has no physical meaning. On the other hand, from interference experiments we know that the relative phase does have physical meaning. We need a gauge-invariant way to compare phases of particles, *e.g.,* electrons. [2]
Suppose we move the particle from x^μ to $x^\mu + \epsilon^\mu$ (infinitesimally). Then the wavefunction changes from $\psi(x)$ to $\psi(x+\epsilon) \approx \psi(x) + \epsilon^\mu \partial_\mu \psi$. The phase will change, but

$$\delta\psi = \epsilon^\mu \partial_\mu \psi \tag{3.2.68}$$

cannot tell us how, because it is not a gauge-invariant quantity.
This is similar to parallel transport along a curved surface. You do your best to keep a vector parallel to itself, but that is impossible due to curvature. Instead of the ordinary derivative, the *covariant derivative* vanishes, which defines parallel transport.
Similarly, in the presence of a vector potential A_μ, the phase of the particle changes so that the gauge derivative (3.2.43) vanishes,

$$\epsilon^\mu D_\mu \psi = 0 \tag{3.2.69}$$

It follows that

$$\psi(x+\epsilon) = \psi(x) + \epsilon^\mu \partial_\mu \psi = \psi(x) - ie\epsilon_\mu A^\mu \psi(x) \approx e^{-ie\epsilon \cdot A}\psi(x) \tag{3.2.70}$$

[2] Electrons are actually described by spinors, but their properties under Lorentz transformations are not important for the purposes of this discussion.

This can be easily exponentiated: for transport from x to y, we have

$$\psi(y) = e^{-ie\int_x^y dx\cdot A}\psi(x) \tag{3.2.71}$$

The phase clearly depends on the path. For a closed loop, we obtain a phase (Wilson loop)

$$e^{-ie\oint A\cdot dx} \tag{3.2.72}$$

which is a gauge-invariant quantity.
This can be verified experimentally: send electrons from point $\vec{x}$ to $\vec{y}$ in the presence of a vector potential created by a solenoid of flux

$$\Phi = \int d\vec{S}\cdot\vec{B} = \oint d\vec{x}\cdot\vec{A} \tag{3.2.73}$$

where we used Stokes' theorem.
Electrons going through different paths, $\mathcal{C}_1$ and $\mathcal{C}_2$ on either side of the solenoid will interfere. The relative phase is

$$e^{-ie\int_{\mathcal{C}_1}\vec{A}\cdot d\vec{x}}e^{+ie\int_{\mathcal{C}_2}\vec{A}\cdot d\vec{x}} = e^{-ie\Phi} \tag{3.2.74}$$

It depends on the flux. The electrons *never* see an electric or magnetic field (which vanish outside the solenoid), yet they feel the vector potential $\vec{A}$. This shows that A_μ plays a physical role and it is not merely a convenient mathematical device for the fields $\vec{E}$ and $\vec{B}$. It is not gauge-invariant, but one may build gauge-invariant quantities from it affecting charged particles (*e.g.,* electrons).

3.2.9 Phonons

Here is an example of a system which can be described in terms of *quasi-particles* that are not particles in the common usage of the word, but excitations of collective motion. Consider a solid consisting of a lattice of atoms each of mass m. Let a be the spacing between neighboring atoms. First, suppose that the solid is one-dimensional and model the forces between atoms by springs, each of spring constant K, connecting atoms, thus forming a one-dimensional chain. The chain is so long (with $\sim 10^{23}$ atoms), that its ends are immaterial. Let A_i be the (longitudinal) displacement of the ith atom from its equilibrium position. Then the Lagrangian of the system is

$$L = \frac{1}{2}\sum_i\left[m\dot{A}_i^2 - K(A_{i+1}-A_i)^2\right] \tag{3.2.75}$$

If we are only interested in macroscopic properties of the solid (i.e., at distances $d \gg a$), then we can approximate

$$ia \approx x \ , \quad A_i \approx A(x) \ , \quad \sum_i \approx \frac{1}{a}\int dx \ , \quad A_{i+1}-A_i \approx a\partial_x A \tag{3.2.76}$$

The Lagrangian can be written in terms of a Lagrangian density as

$$L = \int dx\mathcal{L} \ , \quad \mathcal{L} = \frac{m}{2a}(\partial_t A)^2 - \frac{Ka}{2}(\partial_x A)^2 \tag{3.2.77}$$

We deduce the equation of motion (wave equation)

$$\frac{1}{c^2}\partial_t^2 A = \partial_x^2 A \ , \quad c = \sqrt{\frac{Ka^2}{m}} \tag{3.2.78}$$

The solutions are *phonons* obeying the dispersion relation

$$\omega = \omega_k \equiv ck \ . \tag{3.2.79}$$

The Hamiltonian density is

$$\mathcal{H} = \frac{a}{2m}\pi^2 + \frac{Ka}{2}(\partial_x A)^2 \ , \quad \pi = \frac{\partial \mathcal{L}}{\partial(\partial_t A)} = \frac{m}{a}\partial_t A \tag{3.2.80}$$

The quantization of this system proceeds as in the Klein-Gordon case. We expand

$$A(x,t) = \sqrt{\frac{a}{m}} \int \frac{dk}{2\pi\sqrt{2\omega_k}} \left[a(k) e^{-i(\omega_k t - kx)} + a^\dagger(k) e^{i(\omega_k t - kx)} \right] \tag{3.2.81}$$

In terms of creation and annihilation operators, the Hamiltonian reads

$$H = \int \frac{dk}{2\pi} \omega_k a^\dagger(k) a(k) \tag{3.2.82}$$

where we subtracted the zero-point (ground state) energy.
The above simple-minded model can be generalized to three dimensions rather straightforwardly. The displacement A turns into a vector $\vec{A}$, which in the absence of shear strength and vorticity has vanishing circulation,

$$\oint d\vec{l} \cdot \vec{A} = 0 \tag{3.2.83}$$

and therefore, by Stokes' theorem,

$$\vec{\nabla} \times \vec{A} = \vec{0} \tag{3.2.84}$$

It obeys the same wave equation,

$$\frac{1}{c^2}\partial_t^2 \vec{A} = \nabla^2 \vec{A} \ , \quad c = \sqrt{\frac{Ka^2}{m}} = \sqrt{\frac{B}{\rho}} \tag{3.2.85}$$

where $B = Ka$ is the bulk modulus and $\rho = m/a$ is the density of the solid, and c is the speed of sound. The dispersion relation is

$$\omega = \omega_k \equiv c|\vec{k}| \tag{3.2.86}$$

The Hamiltonian reads

$$H = \int \frac{d^3k}{(2\pi)} \omega_k a^\dagger(\vec{k}) a(\vec{k}) \tag{3.2.87}$$

Since there is no restriction on the number of phonons, the chemical potential vanishes,

$$\mu = 0 \tag{3.2.88}$$

and we have a *canonical* ensemble. The partition function is

$$Z = \text{Tr} e^{-\beta H} = \prod_{\vec{k}} \frac{1}{1 - e^{-\beta\omega_k}} \tag{3.2.89}$$

The free energy is

$$F = -\frac{1}{\beta} \ln Z = \frac{V}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln\left[1 - e^{-\beta\omega_k}\right] \tag{3.2.90}$$

where $V = L^3$ is the volume.
The integral has a cutoff, because wavelengths must be longer than the interatomic spacing a for the wave to propagate (and the collective motion - quasiparticle - to exist), therefore $k \lesssim 1/a$ and $\omega_k \lesssim c/a$. Switching variables to ω_k, we obtain

$$F = \frac{V}{2\pi^2 \beta c^3} \int_0^{\omega_D} d\omega_k \omega_k^2 \ln\left[1 - e^{-\beta\omega_k}\right] \tag{3.2.91}$$

with the cutoff (Debye frequency) $\omega_D \sim c/a$.
The energy is

$$E = \frac{\partial(\beta F)}{\partial \beta} = \frac{V}{2\pi^2 c^3} \int_0^{\omega_D} d\omega_k \frac{\omega_k^3}{e^{\beta\omega_k} - 1} \tag{3.2.92}$$

and the heat capacity is

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = k_B \frac{V}{2\pi^2 c^3} (k_B T)^3 \int_0^{\theta_D/T} dx \frac{x^4 e^x}{(e^x - 1)^2} \tag{3.2.93}$$

where we introduced the dimensionless variable $x = \beta\omega_k$ and the Debye temperature

$$\theta_D = \frac{\omega_D}{k_B} \tag{3.2.94}$$

The Debye temperature can be given a precise definition by going to the high temperature limit in which purely classical behavior is expected.
As $T \to \infty$, we have

$$C_V = k_B \frac{V}{2\pi^2 c^3} (k_B T)^3 \int_0^{\theta_D/T} dx \left[x^2 + \ldots\right] = k_B \frac{V}{6\pi^2 c^3} (k_B \theta_D)^3 + \ldots \tag{3.2.95}$$

so C_V approaches a constant. If the crystal has N ions, it has $3N$ degrees of freedom, therefore, we expect $C_V = 3N k_B$ as $T \to \infty$. It follows that

$$k_B \theta_D = (18\pi^2 N)^{1/3} \frac{\hbar c}{L} \tag{3.2.96}$$

where we restored $\hbar$ to show explicitly that both sides have dimensions of energy.
As $T \to 0$, we obtain

$$C_V \approx k_B \frac{V}{2\pi^2 c^3}(k_B T)^3 \int_0^\infty dx \frac{x^4 e^x}{(e^x - 1)^2} = \frac{2\pi^2}{15} k_B \left(\frac{k_B T L}{\hbar c}\right)^3 = \frac{12\pi^4}{5} N k_B \left(\frac{T}{\theta_D}\right)^3 \tag{3.2.97}$$

i.e., $C_V \propto T^3$ at low temperatures (Debye Law), in excellent agreement with experimental results.

3.2.10 The running coupling constant

The constant e represents the charge of a single fermion and can be measured by measuring the electric force between two fermions which is proportional to e^2. The quantity

$$\alpha = \frac{e^2}{4\pi} \tag{3.2.98}$$

is the fine-structure constant and is a dimensionless number (if you restore $\hbar$ and c). Experimentally,

$$\alpha \approx \frac{1}{137} \tag{3.2.99}$$

a famous constant that belongs to Nature! For a long time people thought there was something magic about it, otherwise why would Nature choose it? We now know that it is not a constant at all. This is because of vacuum polarization: the vacuum surrounding a charge is full of pairs of particles and anti-particles which are created and annihilated evading detection, as allowed by the Heisenberg uncertainty principle (virtual particles). A *(real)* charged particle attracts virtual anti-particles and repels virtual particles thus polarizing the surrounding space, just like a dielectric. The bare charge e_0 of the real particle is screened and the measured charge e is

$$e^2 = \frac{e_0^2}{\varepsilon} \quad , \quad \varepsilon > 1 \tag{3.2.100}$$

where ε is the dielectric constant of the vacuum.
The closer we get to the particle, the less e_0 is screened. To probe shorter distances, we need to increase the energy E (at a fundamental level, $E \sim \frac{1}{\text{distance}}$), therefore the measured charge e is energy-dependent. It *increases* at high energies. Experimentally, we have seen $\alpha \approx \frac{1}{128}$ near the mass of the weakly interacting Z particle (about 100 GeV) - a very slow change.
To understand how e changes, we need to study interactions. Until we do that, we shall be content with a heuristic argument that leads to a quantitative understanding of $e = e(E)$.
It is simpler to understand the magnetic properties of the vacuum and then deduce the dielectric constant from

$$\mu\varepsilon = 1 \tag{3.2.101}$$

(where μ is the magnetic permeability) which ought to be true in the vacuum.
If we switch on a uniform magnetic field $\vec{B}$, the magnetic moments of the virtual pairs tend to align along $\vec{B}$ producing a magnetization

$$\vec{M} = \chi \vec{H} \quad , \quad \vec{B} = \vec{H} + \vec{M} \quad , \quad \chi = \mu - 1 \tag{3.2.102}$$

where χ is the magnetic susceptibility.[3] Under a change in the magnetic field, the energy density changes by

$$d\rho = -MdH = -\chi H dH \tag{3.2.103}$$

therefore, under the magnetic field B, the vacuum stores energy of density

$$\Delta\rho = -\frac{1}{2}\chi H^2 \tag{3.2.104}$$

We wish to understand this energy the same way we understood the vacuum energy of a scalar (1.3.7) and of a fermion (2.4.15). Let us start with scalars.

Scalars

We will set the mass $m = 0$ to simplify the calculation. This does not limit the generality of the result, because the mass does not contribute, as can be checked. For a free massless scalar, the energy is

$$\omega_k = |\vec{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} \tag{3.2.105}$$

Switching on a uniform magnetic field in the z-direction,

$$\vec{B} = B\hat{z} = \vec{\nabla} \times \vec{A} \ , \quad \vec{A} = Bx\hat{y} \tag{3.2.106}$$

changes the energy to

$$\omega_{k,B} = |\vec{k} - e_0\vec{A}| = \sqrt{k_x^2 + (k_y - e_0 Bx)^2 + k_z^2} \tag{3.2.107}$$

where e_0 is the bare charge of the scalar. By writing

$$\omega_{k,B}^2 = k_x^2 + e_0^2 B^2 \left(x - \frac{k_y}{e_0 B} \right)^2 + k_z^2 \tag{3.2.108}$$

we see that k_y and k_z commute with $\omega_{k,B}^2$ and their spectrum has not changed. The system in the x-direction has turned into a harmonic oscillator of frequency $e_0 B$. We deduce the energy levels (Landau levels)

$$\omega_{k,B} = \sqrt{(2n+1)e_0 B + k_z^2} \tag{3.2.109}$$

We obtain the energy density by summing over all states,

$$\rho = \sum_n \int \frac{dk_z}{2\pi} \, g_n \, \omega_{k,B} \tag{3.2.110}$$

where g_n is the degeneracy of the nth level. It can be determined by taking the limit $B \to 0$. In this case, we should recover the free expression

$$\rho_0 = \int \frac{d^3k}{(2\pi)^3} \, \omega_k \tag{3.2.111}$$

[3] In standard textbooks, such as Jackson's *Classical Electrodynamics*, $\vec{B} = \vec{H} + 4\pi\vec{M}$ and so $\chi = \frac{\mu - 1}{4\pi}$. Our choice of units is somewhat unusual, *e.g.,* the Maxwell equations are given by (3.2.2) instead of the more standard form $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$.

To see that we do, let us write

$$\rho_0 = \int \frac{dk_z dk_\perp\, k_\perp}{(2\pi)^2} \sqrt{k_z^2 + k_\perp^2} \tag{3.2.112}$$

where $k_\perp = \sqrt{k_x^2 + k_y^2}$. In the limit $B \to 0$, the sum in (3.2.110) turns into an integral,

$$\rho \to \frac{1}{2e_0 B} \int \frac{dk_z}{2\pi}\, dk_\perp^2 g(k_\perp) \sqrt{k_\perp^2 + k_z^2} \tag{3.2.113}$$

where $k_\perp^2 = (2n+1)e_0 B$. By comparing (3.2.112) and (3.2.113), we deduce

$$g_n = \frac{1}{2\pi}\, e_0 B \tag{3.2.114}$$

To find the magnetic susceptibility, we need to calculate

$$\Delta\rho = \rho - \rho_0 \tag{3.2.115}$$

We have

$$\Delta\rho = \frac{1}{2} \int \frac{dk_z}{(2\pi)^2} \left[\sum_{n=0}^{\infty} 2eB \sqrt{k_z^2 + (2n+1)e_0 B} - \int dk_\perp^2 \sqrt{k_z^2 + k_\perp^2} \right] \tag{3.2.116}$$

Using

$$\begin{aligned}
\int_0^\infty dx f(x) &= \sum_{n=0}^{\infty} \int_{n\epsilon}^{(n+1)\epsilon} dx f(x) \\
&= \sum_{n=0}^{\infty} \int_{n\epsilon}^{(n+1)\epsilon} dx \left[f(x_n) + (x - x_n) f'(x_n) + \ldots \right]_{x_n = (n+1/2)\epsilon} \\
&= \sum_{n=0}^{\infty} \epsilon f((n+1/2)\epsilon) + \frac{\epsilon^3}{24} f''((n+1/2)\epsilon) + \ldots \\
&= \sum_{n=0}^{\infty} \epsilon f((n+1/2)\epsilon) + \frac{\epsilon^2}{24} \int_0^\infty dx f''(x) + O(\epsilon^4)
\end{aligned} \tag{3.2.117}$$

with $\epsilon = 2e_0 B$, $f(x) = \sqrt{k_z^2 + x}$, $x = k_\perp^2$, we may expand $\Delta\rho$ in powers of $e_0 B$ as

$$\Delta\rho = \frac{(e_0 B)^2}{48} \int \frac{dk_z dk_\perp^2}{(2\pi)^2} \frac{1}{(k_z^2 + k_\perp^2)^{3/2}} + \ldots = \frac{(e_0 B)^2}{24} \int \frac{d^3k}{(2\pi)^3 \omega_k^3} + \ldots \tag{3.2.118}$$

For weak coupling, $H \approx B$ (or $\mu \approx 1$), so the magnetic susceptibility is

$$\chi = -\frac{e_0^2}{12} \int \frac{d^3k}{(2\pi)^3 \omega_k^3} \tag{3.2.119}$$

The energy ω_k should not exceed a cutoff limit Λ beyond which QED is not expected to be valid. Also, if we are performing the experiment at energy E, then $\omega_k \gtrsim E$, since energies below E are not accessible by our apparatus. Therefore,

$$\chi = -\frac{e_0^2}{12}\int_E^\Lambda \frac{4\pi\, dk\, k^2}{(2\pi)^3 k^3} = -\frac{e_0^2}{24\pi^2}\ln\frac{\Lambda}{E} \tag{3.2.120}$$

Fermions

The above discussion may be extended to fermions. The same Landau levels are obtained but with *opposite sign* for the same reason as in the free case (*cf.* with the bosonic (1.3.7) and fermionic (2.4.15) contributions to the vacuum energy). Moreover, the spin also couples to the magnetic field. We obtain

$$\omega_{k,B,S}^2 = \omega_{k,B}^2 - \vec{m}\cdot\vec{B} \ , \quad \vec{m} = ge_0\vec{S} \tag{3.2.121}$$

where $g = 2$ (Thomas precession). [4] It follows that

$$\omega_{k,B,S} = \omega_{k,B} - \frac{ge_0BS_z}{2\omega_{k,B}} - \frac{(ge_0BS_z)^2}{8\omega_{k,B}^3} + \dots \tag{3.2.122}$$

where $S_z = \pm\frac{1}{2}$. The first term is familiar from the scalar case. The second term does not contribute to the vacuum energy, because $\langle S_z\rangle = 0$. In the third term we can replace $\omega_{k,B}$ by ω_k, because the difference is of higher order in $(e_0B)^2$. Thus, it becomes a function of the momentum. We deduce the energy density for a fermion

$$\rho_f = -2\sum_n\int\frac{dk_z}{2\pi}g_n\omega_{k,B} + 2\int\frac{d^3k}{(2\pi)^3}\frac{(e_0B)^2}{8\omega_k^3} + \dots \tag{3.2.123}$$

and using (3.2.115) and (3.2.118),

$$\Delta\rho_f = -2\Delta\rho + \frac{(e_0B)^2}{4}\int\frac{d^3k}{(2\pi)^3\omega_k^3} + \dots = \frac{(e_0B)^2}{6}\int\frac{d^3k}{(2\pi)^3\omega_k^3} + \dots \tag{3.2.124}$$

leading to the magnetic susceptibility of fermions

$$\chi_f = 4\chi = -\frac{e_0^2}{6\pi^2}\ln\frac{\Lambda}{E} \tag{3.2.125}$$

The corresponding dielectric constant is

$$\varepsilon = \frac{1}{\mu} \approx 1 - \chi \approx 1 + \frac{e_0^2}{6\pi^2}\ln\frac{\Lambda}{E} \tag{3.2.126}$$

[4] After restoring the mass, it is easy to check that this reproduces the correct non-relativistic limit

$$E = m_e + \frac{(\vec{p} - e_0\vec{A})^2}{2m_e} - \vec{\mu}\cdot\vec{B}$$

where $\vec{\mu} = g\frac{e_0}{2m_e}\vec{S}$ is the magnetic moment and m_e is the mass of the electron.

where we ignored terms of higher order in e_0^2, showing that the charge increases slowly as we increase the energy,

$$e^2 \approx \frac{e_0^2}{1 + \frac{e_0^2}{6\pi^2} \ln \frac{\Lambda}{E}} \tag{3.2.127}$$

as expected.
Can our final expression for the running coupling constant be tested? e_0 is a charge which can be measured only if we go infinitely close to the point charge. This is equivalent to performing the experiment at infinitely high energy. On the other hand, we cannot really go above the cutoff Λ, so we should associate "infinity" with the cutoff and then e_0 is the charge at that energy. This will make our result (3.2.127) consistent, for $e = e_0$ at $E = \Lambda$.
But even Λ is probably too high to be attained any time soon (or ever). So it looks like our conclusion might not be testable. This is in fact not the case. Let us take E as low as it can get. As far as we can tell, the lightest charged particle is the electron, so $E \gtrsim m_e$. Going to low energies is equivalent to looking at the particle from far away (a distance $\sim 1/m_e$) in which case we know the value of e^2 - it is given by (3.2.98) and (3.2.99). Going further away will not change this value appreciably. Call this minimum value $\tilde{e}$ ($\tilde{e}^2 = \frac{4\pi}{137}$). We have

$$\tilde{e}^2 = \frac{e_0^2}{1 + \frac{e_0^2}{6\pi^2} \ln \frac{\Lambda}{m_e}} \tag{3.2.128}$$

We may now eliminate *both* e_0 and Λ from (3.2.127) and (3.2.128) and obtain

$$e^2 = \frac{\tilde{e}^2}{1 + \frac{\tilde{e}^2}{6\pi^2} \ln \frac{m_e}{E}} \tag{3.2.129}$$

expressed entirely in terms of measurable quantities. This expression is valid as long as the denominator is close to 1 which is the case if

$$\frac{\tilde{e}^2}{6\pi^2} \ln \frac{E}{m_e} \lesssim 1 \tag{3.2.130}$$

We deduce an upper limit

$$E \lesssim m_e e^{6\pi^2/\tilde{e}^2} \approx 10^{281} m_e \tag{3.2.131}$$

which for all practical purposes is infinite!

UNIT 4

The Standard Model

4.1 Non-abelian gauge theories

4.1.1 From $U(1)$ (QED) to $SU(2)$ (Yang-Mills)

Gauge invariance played a central role in our development of QED. It turns out that it plays an equally important role for all other forces of Nature (weak, strong, as well as gravitational, although gauge invariance is not sufficient for the quantization of gravity).

Let us recall what we did to build QED. Firstly, we realized that the Lagrangian density for a photon (3.2.1) had a local symmetry (under the gauge transformation (3.2.4)). Unlike a global symmetry, this implied that the associated charge vanished identically and so two systems related by a gauge transformation had to be identical and not just equivalent. Thus, to describe dynamics, we had to choose a single representative from each equivalence class. This was extremely important in the proof of unitarity of the theory. When we added other fields interacting with photons (representing charged particles), the guiding principle was gauge invariance. The global transformations of complex scalar (3.2.40) or Dirac (2.3.19) fields were promoted to local ones (3.2.42) and we built Lagrangian densities with local $U(1)$ symmetry (since $e^{-ie\omega} \in U(1)$). In general, we replaced derivatives with gauge derivatives (3.2.43) which had nice transformation properties,

$$D_\mu \to e^{-ie\omega} D_\mu e^{ie\omega} \tag{4.1.1}$$

Thus, QED is a $U(1)$ gauge theory. We may construct more complicated theories if we replace $U(1)$ by another group. The simplest case is the group of rotations in three dimensions, $SO(3)$, or its covering $SU(2)$. This idea was originally proposed by Yang and Mills. The simplest non-trivial object that can "rotate" is a spinor (doublet)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{4.1.2}$$

which can be a boson or a fermion. Its "spin" is in an abstract three-dimensional space (isospin). Yang and Mills originally proposed that a neutron and a proton be viewed

as an isospin doublet $\binom{n}{p}$, which allowed them to view β-decay as an isospin rotation $(n \to p)$. Let us not be specific and simply study the consequences without worrying about Nature for the moment.
A rotation matrix for the doublet ψ is

$$\Omega = e^{ig\vec{\omega}\cdot\vec{S}} \quad , \quad \vec{S} = \frac{1}{2}\,\vec{\sigma} \tag{4.1.3}$$

where $\vec{S}$ is the spin (generator of rotations in the spinor representation) and σ^i $(i = 1, 2, 3)$ are the Pauli matrices. In infinitesimal form,

$$\Omega = 1 + ig\vec{\omega}\cdot\vec{S} \tag{4.1.4}$$

Under a gauge transformation,

$$\psi \to \Omega^{-1}\psi \tag{4.1.5}$$

which generalizes (3.2.42) in QED. The arbitrary constant g will play a role similar to the role e plays in QED.
The generalization of the transformation law of the vector potential (3.2.4) can be inferred from the transformation of a gauge derivative (4.1.1) which may be written as

$$D_\mu \to \Omega^{-1}D_\mu\Omega \tag{4.1.6}$$

We deduce

$$A_\mu \to \Omega^{-1}A_\mu\Omega - \frac{i}{g}\Omega^{-1}\partial_\mu\Omega \tag{4.1.7}$$

Evidently, A_μ is a 2×2 matrix, since $\Omega \in SU(2)$. Therefore, it has four components. They may be conveniently chosen as

$$A^0_\mu = \mathrm{tr}A_\mu \quad , \quad A^i_\mu = \mathrm{tr}(A_\mu\sigma^i) \quad i = 1, 2, 3 \tag{4.1.8}$$

In infinitesimal form, (4.1.7) reads

$$A_\mu \to A_\mu + \partial_\mu\omega - ig[\omega\ ,\ A_\mu] \quad , \quad \omega = \vec{\omega}\cdot\vec{S} \tag{4.1.9}$$

In terms of components,

$$A^0_\mu \to A^0_\mu \quad , \quad A^i_\mu \to A^i_\mu + \partial_\mu\omega^i + g\epsilon^{ijk}\omega^j A^k_\mu \tag{4.1.10}$$

where we used the commutation relations of the Pauli matrices (2.1.7) and the fact that they are traceless. Thus, A^0_μ is a scalar in isospin space and can be considered separately. We shall concentrate on the other three components of A_μ (effectively setting $A^0_\mu = 0$) which together form a *vector* in isospin space (*spin-1* or *adjoint representation*), $\vec{A}_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$. The gauge transformation is then the standard rotation in three-dimensional space,

$$\vec{A}_\mu \to \vec{A}_\mu + \partial_\mu\vec{\omega} + g\vec{\omega}\times\vec{A}_\mu \tag{4.1.11}$$

The first two terms on the right-hand side are the same as in QED, except that now we have three different types of photons. If the group were $U(1)\times U(1)\times U(1)$, i.e., three copies of QED, then we would not have the last term on the right-hand side. The latter is the novel feature of our model and implies that the three varieties of photons interact with each other. They themselves have *charge*!

To see how they interact, we need to construct electric and magnetic fields, i.e., generalize $F_{\mu\nu}$ (3.1.14). We cannot use the expression $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for each type of photon, because it does not transform nicely under gauge transformations. Instead, let us re-write it as

$$F_{\mu\nu} = -\frac{i}{g}[D_\mu \ , \ D_\nu] \tag{4.1.12}$$

which is true in QED (with $g = e$) and transforms nicely. Using (4.1.6), we readily obtain

$$F_{\mu\nu} \to \Omega^{-1} F_{\mu\nu} \Omega \tag{4.1.13}$$

This definition allows us to interpret $F_{\mu\nu}$ as some kind of *"curvature"* and A_μ as a *"connection."* On the other hand, notice that electric and magnetic fields are no longer physical quantities since they are not gauge-invariant! One has to work harder to define physical quantities in this system. Here are some examples:

(a) Traces of a string of Fs,

$$\mathrm{tr} F_{\mu_1\nu_1} \cdots F_{\mu_n\nu_n} \tag{4.1.14}$$

Since $\mathrm{tr}F_{\mu\nu} = 0$, the first non-trivial gauge-invariant quantity is $\mathrm{tr}F_{\mu\nu}F_{\rho\sigma}$.

(b) The Wilson loop

$$W(\mathcal{C}) = \mathrm{tr}\mathcal{P}e^{ig\oint_{\mathcal{C}} dx^\mu A_\mu} \tag{4.1.15}$$

generalizing (3.2.72). $\mathcal{P}$ denotes path-ordering: the curve $\mathcal{C}$ ought to be cut up into infinitesimal segments. For each segment we calculate the matrix $e^{igdx^\mu A_\mu}$ and then multiply the matrices following the ordering of the segments in the path.[1]

For an infinitesimal loop,

$$W(\mathcal{C}) = 2 - \frac{g^2\mathcal{A}}{2}\,\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \dots \tag{4.1.16}$$

where $\mathcal{A}$ is the area of the surface enclosed by $\mathcal{C}$. The first non-trivial contribution is proportional to the Lagrangian density. This is why the Wilson loop forms the basis of a *lattice definition* of a gauge theory and can be handled by a computer.

Explicitly, we obtain from (4.1.12)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu \ , \ A_\nu] \tag{4.1.17}$$

where the last contribution is non-linear and is not present in QED. This may also be written in terms of components ($F_{\mu\nu} = \vec{F}_{\mu\nu} \cdot \vec{S}$),

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu \tag{4.1.18}$$

[1] This careful ordering is not necessary in QED, because each piece yields a number (phase) and they all commute with each other, so we can use $e^A e^B = e^{A+B}$ (not true if A and B are matrices).

or equivalently

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^j_\mu A^k_\nu \tag{4.1.19}$$

We may now define the Lagrangian density (Yang-Mills)

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr} F_{\mu\nu} F^{\mu\nu} - 2\mathrm{tr} J^\mu A_\mu = -\frac{1}{4}\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} - \vec{J}^\mu \cdot \vec{A}^\mu \tag{4.1.20}$$

generalizing the QED Lagrangian density (3.2.1) in an obvious way. It is manifestly gauge-invariant and includes interactions between photons (through its cubic and quartic terms), as expected. We also added an interaction term with matter. Of course, we needed three currents since we have three different photons and therefore three different types of charges. This Lagrangian is gauge-invariant provided

$$\partial_\mu \vec{J}^\mu + g\vec{J}^\mu \times \vec{A}_\mu = \vec{0} \tag{4.1.21}$$

This may also be written in matrix form as

$$[D_\mu \ , \ J^\mu] = 0 \tag{4.1.22}$$

So the current is *not* conserved. This is because charge may flow from matter (J^μ) to the photons (A_μ) and vice versa, which is impossible in electromagnetism because the photons have no charge.
Finally, the Maxwell equations are non-linear,

$$\partial_\mu \vec{F}^{\mu\nu} + g\vec{F}^{\mu\nu} \times \vec{A}_\mu = \vec{J}^\nu \tag{4.1.23}$$

or in matrix form,

$$[D_\mu \ , \ F^{\mu\nu}] = J^\nu \tag{4.1.24}$$

Fermions
Matter fields comprising the current J^μ may be added in the same way as with electromagnetism. We shall concentrate on the physically relevant case of spinors. The Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{4.1.25}$$

is gauge invariant (using (4.1.5) and (4.1.6)). Writing it in the form (3.2.61), we deduce the current

$$\vec{J}^\mu = g\bar{\psi}\gamma^\mu \vec{S}\psi \tag{4.1.26}$$

which is not conserved (show!). On the other hand, it is not gauge invariant either so it is not observable. If the charge density $J^0 = g\bar{\psi}\gamma^0\vec{S}\psi = g\psi^\dagger\vec{S}\psi$ is not gauge invariant, can we find a gauge invariant quantity that will measure matter density? The answer is: remove $\vec{S}$ which is obstructing gauge invariance. We obtain the gauge invariant quantity

$$\psi^\dagger\psi \tag{4.1.27}$$

which can be thought of as a particle-antiparticle pair. It does not couple to the vector potential because it has zero charge.

Two other such gauge invariant quantities are (recall ψ^a ($a = 1, 2$) are anti-commuting fields)

$$\epsilon^{ab}\psi^a_\alpha\psi^b_\beta = \begin{vmatrix} \psi^1_\alpha & \psi^1_\beta \\ \psi^2_\alpha & \psi^2_\beta \end{vmatrix} \quad , \quad \epsilon^{ab}\psi^{a\dagger}_\alpha\psi^{b\dagger}_\beta = \begin{vmatrix} \psi^1_\alpha & \psi^1_\beta \\ \psi^2_\alpha & \psi^2_\beta \end{vmatrix} \tag{4.1.28}$$

a two-particle and two-antiparticle state, respectively. We have also shown the Dirac indices α, β for clarity.
Under the gauge transformation (4.1.5) for a general matrix Ω,

$$\epsilon^{ab}\psi^a_\alpha\psi^b_\beta \to \det\Omega^{-1}\epsilon^{ab}\psi^a_\alpha\psi^b_\beta \tag{4.1.29}$$

Invariance follows from $\det\Omega = 1$ for $\Omega \in SU(2)$.
In all these cases the total charge (isospin) vanishes. Unlike in electromagnetism, charged observable objects do not exist.

4.1.2 From $SU(2)$ to $SU(N)$

The above construction may be generalized to an arbitrary *Lie group.* We just need to replace the generators $S^i = \frac{\sigma^i}{2}$ of the $SU(2)$ algebra by the appropriate generators in the Lie group of choice. Let T^a be the generators of a Lie group satisfying the *Lie algebra*

$$[T^a \, , \, T^b] = if^{abc}T^c \tag{4.1.30}$$

where f^{abc} are the *structure constants* generalizing ϵ^{ijk} in $SU(2)$ (eq. (2.1.7)). An element of the group close to the identity may be expanded as

$$\Omega = \mathbb{I} + ig\omega^a T^a + \dots \tag{4.1.31}$$

generalizing (4.1.4).
Let us concentrate on $SU(N)$ for definiteness. It is the most physically relevant group (at least for $N = 2, 3$). The generators T^a are $N \times N$ *hermitian* matrices (since Ω is unitary) satisfying the orthogonality relation

$$\mathrm{tr} T^a T^b = \frac{1}{2}\delta^{ab} \tag{4.1.32}$$

They form the fundamental representation of the algebra. There are other representations, just as with $SU(2)$ where higher representations correspond to larger spin. Of particular interest is the adjoint representation corresponding to the spin-1 (vector) representation of $SU(2)$. The latter has three dimensions ($SU(2)$ is the covering of the group of three-dimensional rotations $SO(3)$) because there are three Pauli matrices which count the number of degrees of freedom of $SU(2)$ matrices.
How many generators T^a are there? An $N \times N$ unitary matrix has N^2 components so it needs $2N^2$ *real* parameters. Unitarity imposes N^2 constraints

$$U^\dagger U = \mathbb{I}$$

and being special (the S in $SU(N)$) is one more constraint

$$\det U = +1$$

Therefore the number of independent real parameters is

$$2N^2 - (N^2+1) = N^2 - 1$$

This is the number of generators,

$$T^a \;\;,\quad a = 1, \dots, N^2-1$$

For $N = 2$, $a = 1, 2, 3$ (three-dimensional vector; three 2×2 Pauli matrices).
For $N = 3$, $N^2 - 1 = 8$; we obtain eight 3×3 Gell-Mann matrices.

Adjoint representation

Let $\hat{T}^a$ be the generators in the adjoint representation ($(N^2-1) \times (N^2-1)$ matrices) generalizing the generators of rotations in three dimensions (2.1.9). We have

$$(\hat{T}^a)^{bc} = -if^{abc} \tag{4.1.33}$$

Let us check that they satisfy the Lie algebra (4.1.30). We need to show

$$(\hat{T}^a\hat{T}^b - \hat{T}^b\hat{T}^a)^{de} = if^{abc}(\hat{T}^c)^{de}$$

Using the definition (4.1.33), this can be written as

$$f^{adc}f^{bce} + f^{bdc}f^{cae} + f^{abc}f^{cde} = 0 \tag{4.1.34}$$

The validity of this equation is a direct consequence of the Jacobi identity

$$[T^d\,,\,[T^a\,,\,T^b]] + [T^a\,,\,[T^b\,,\,T^d]] + [T^b\,,\,[T^d\,,\,T^a]] = 0 \tag{4.1.35}$$

Another important property of the structure constants is that they are totally *antisymmetric*.

PROOF: Using (4.1.32) and (4.1.30), we obtain

$$f^{abc} = -2i\mathrm{tr}[T^a\,,\,T^b]T^c$$

Antisymmetry is now obvious.

The Casimir

The operator

$$\mathbf{C} = T^aT^a \tag{4.1.36}$$

commutes with all generators.

PROOF: We have

$$[\mathbf{C}\,,\,T^b] = \{[T^a\,,\,T^b]\,,\,T^a\} = if^{abc}\{T^a\,,\,T^c\}$$

The last expression is a product of a factor antisymmetric in ac (f^{abc}) and a factor symmetric in ac ($\{T^a\,,\,T^c\}$). Therefore, it vanishes.
It follows that

$$\mathbf{C} \propto \mathbb{I}$$

In the fundamental representation, let

$$\mathbf{C} = C_F \mathbb{I}_{N\times N}$$

Taking traces of both sides and using (4.1.32), we deduce

$$C_F = \frac{\frac{1}{2}\,\delta^{aa}}{N} = \frac{N^2-1}{2N} \tag{4.1.37}$$

In the adjoint representation, we have

$$\hat{T}^a\hat{T}^a = C_A \mathbb{I}_{(N^2-1)\times(N^2-1)} \tag{4.1.38}$$

Calculating C_A is harder. We need to build the adjoint representation from the fundamental. So consider a spinor ψ in the fundamental (N) representation. Its conjugate, $\psi^\dagger$ belongs to the $\bar{N}$ representation. The tensor product $N\otimes\bar{N}$ is spanned by $\psi^A\psi^{\dagger B}$ ($A, B = 1,\dots,N$), a total of N^2 objects. They can be grouped into the adjoint (N^2-1 objects) and the trivial (singlet) representation as follows. Define

$$V^a = \psi^\dagger T^a\psi = \mathrm{tr}T^a\psi\psi^\dagger \quad , \quad S = \psi^\dagger\psi = \mathrm{tr}\psi\psi^\dagger \tag{4.1.39}$$

From (4.1.5) and (4.1.31), under an infinitesimal transformation, ψ transforms as

$$\psi \to \psi - ig\omega^a T^a\psi \quad , \quad \psi^\dagger \to \psi^\dagger + ig\psi^\dagger T^a\omega^a \tag{4.1.40}$$

It follows that $S \to S$ (invariant) and

$$\delta V^a = -ig\psi^\dagger[T^a\ ,\ T^b]\psi = -ig\omega^b(\hat{T}^b)^{ac}V^c$$

or in matrix notation,

$$\delta V = -ig\omega^b\hat{T}^b V \tag{4.1.41}$$

which shows that V^a transforms as a "vector" (adjoint representation; *cf.* with the rotation (2.1.13)).
The action of a generator on the $N\otimes\bar{N}$ representation may be written as

$$T^a_{N\otimes\bar{N}} = T^a_N\otimes\mathbb{I}_{\bar{N}} + \mathbb{I}_N\otimes T^a_{\bar{N}} \tag{4.1.42}$$

where the first factor on each term acts on ψ whereas the second factor acts on $\psi^\dagger$. Evidently (from (4.1.40)),

$$T^a_N = T^a \quad , \quad T^a_{\bar{N}} = -(T^a)^T \tag{4.1.43}$$

$T^a_{N\otimes\bar{N}}$ may be written as a matrix in the basis $\{V^a, S\}$,

$$T^a_{N\otimes\bar{N}} = \begin{pmatrix} \hat{T}^a & \\ & 0 \end{pmatrix} \tag{4.1.44}$$

where the 0 in the diagonal is due to the trivial action of $T^a_{N\otimes\bar{N}}$ on the singlet S. Squaring and taking the trace, we obtain

$$\mathrm{tr}T^a_{N\otimes\bar{N}}T^a_{N\otimes\bar{N}} = \mathrm{tr}\hat{T}^a\hat{T}^a = (N^2-1)C_A \tag{4.1.45}$$

where in the last step we used (4.1.38).
Alternatively, from (4.1.42) we obtain

$$T^a_{N\otimes\bar{N}}T^a_{N\otimes\bar{N}} = T^a_N T^a_N \otimes \mathbb{I}_{\bar{N}} + \mathbb{I}_N \otimes T^a_{\bar{N}}T^a_{\bar{N}} + 2T^a_N \otimes T^a_{\bar{N}}$$

and taking traces,

$$\mathrm{tr}T^a_{N\otimes\bar{N}}T^a_{N\otimes\bar{N}} = 2N\mathrm{tr}T^aT^a = N(N^2-1) \quad (4.1.46)$$

where we used $\mathrm{tr}T^a = 0$ and in the last step (4.1.32).
Comparing the two expressions for the trace (4.1.45) and (4.1.46),we deduce

$$C_A = N \quad (4.1.47)$$

The quadratic Casimirs in the fundamental (4.1.37) and adjoint (4.1.47) representations will play important physical roles.
Physics

Having understood various technical details about the group $SU(N)$, let us return to our main endeavour: physics. To generalize the Yang-Mills theory ($SU(2)$ gauge theory) we need do *nothing*. The vector potential, field $F_{\mu\nu}$, Lagrangian density, etc, are as in $SU(2)$ as long as we remember to replace $S^i \to T^a$.

4.1.3 The BRST transformation

To quantize the $SU(N)$ gauge theory, we first need to fix the gauge, as in QED. Which gauge you choose should not affect physical results. In QED, showing unitarity was easy in one gauge (axial gauge (3.2.9)) whereas Lorentz invariance was manifest in a different gauge (Lorentz gauge (3.2.17)). Applying the discussion in QED to a non-abelian gauge theory is not straightforward, because of the non-linearity of the theory. Quantization is greatly facilitated by a trick discovered by Becchi, Rouet and Stora, and independently by Tyutin (working in apparent isolation in the Soviet Union - a country some of you may have never heard of): the BRST transformation. This trick also provides a neat definition of the Hilbert space and a proof of unitarity of the theory within the Lorentz gauge. I shall describe it in some detail for all the above reasons and also because it plays a central role in the development of *string theory.*
For definiteness, let us adopt the Lorentz gauge

$$\partial^\mu A^a_\mu = 0 \quad (4.1.48)$$

although any other gauge will do.
We may write the Lagrangian density (4.1.20) as we did in electrodynamics (eq. (3.2.18)),

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}' \quad (4.1.49)$$

where

$$\mathcal{L}_0 = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \quad , \quad \mathcal{L}' = -\frac{1}{2\lambda}(\partial^\mu A^a_\mu)^2 \quad (4.1.50)$$

and I omitted the matter part; it is not essential for the introduction of the BRST transformation and may always be added later. The parameter λ is arbitrary and no physical quantities should depend on it. Since we fixed the gauge, this Lagrangian is not invariant under the gauge transformation (4.1.9). The suggestion of BRST amounted to promoting the gauge parameter ω to a field $\omega = \epsilon \mathbf{c}$ (where ϵ keeps track of the smallness of the transformation and $\mathbf{c}$ is the new field) and write

$$A_\mu^a \to A_\mu^a + \epsilon(\partial_\mu \mathbf{c}^a + igf^{abc}\mathbf{c}^b A_\mu^c) \tag{4.1.51}$$

and instead of thinking of the gauge-fixing condition in terms of a reduced number of fields, we should *enlarge* the field content of our theory. This sounds like a step in the wrong direction, but it is a brilliant suggestion, as we shall see.
We want the final construct to possess a local symmetry. It turns out that in addition to $\mathbf{c}$, we need one more field, $\mathbf{b}$ and they should both be *anti*-commuting fields. The new fields are the ghosts. They are Lorentz scalars, yet they obey fermi statistics. This will turn out not to be a problem, because they correspond to no physical particle.
The original Lagrangian $\mathcal{L}_0$ is gauge-invariant, hence invariant under the transformation (4.1.51). The additional piece $\mathcal{L}'$ is not invariant. To make it invariant, we shall add terms involving the new fields $\mathbf{b}$ and $\mathbf{c}$. The augmented Lagrangian density

$$\mathcal{L}' = -\frac{1}{2\lambda}(\partial^\mu A_\mu^a)^2 + \partial^\mu \mathbf{b}^a(\partial_\mu \mathbf{c}^a + igf^{abc}\mathbf{c}^b A_\mu^c) \tag{4.1.52}$$

is invariant under the BRST transformation

$$\begin{aligned}
\delta A_\mu^a &= \epsilon(\partial_\mu \mathbf{c}^a + igf^{abc}\mathbf{c}^b A_\mu^c) \\
\delta \mathbf{c}^a &= -\frac{ig}{2}\,\epsilon f^{abc}\mathbf{c}^b\mathbf{c}^c \\
\delta \mathbf{b}^a &= \frac{1}{\lambda}\,\epsilon \partial^\mu A_\mu^a
\end{aligned} \tag{4.1.53}$$

The first line is (4.1.51) and of course $\mathcal{L}_0$ is also BRST-invariant. Notice also that since $\mathbf{b}$ and $\mathbf{c}$ are anti-commuting fields, ϵ must be an anti-commuting parameter. Thus the BRST symmetry is a kind of supersymmetry mixing bosonic and fermionic fields.
PROOF:

$$\begin{aligned}
\delta\mathcal{L}' &= \partial^\mu \mathbf{b}^a \delta(\partial_\mu \mathbf{c}^a + igf^{abc}\mathbf{c}^b A_\mu^c) \\
&= \frac{ig^2}{2}\epsilon\partial^\mu \mathbf{b}^a \left\{f^{abc}f^{cde} + 2f^{acd}f^{cbe}\right\} A_\mu^b \mathbf{c}^d \mathbf{c}^e \\
&= 0
\end{aligned}$$

where we used the fact that f^{abc} is anti-symmetric and the last step was a consequence of the Jacobi identity (4.1.34). We also discarded a total divergence. We actually have

$$\delta\mathcal{L}' = \epsilon\partial^\mu \mathbf{F}_\mu \ , \quad \mathbf{F}_\mu = -\frac{1}{\lambda}\partial^\nu A_\nu^a(\partial_\mu \mathbf{c}^a + igf^{abc}\mathbf{c}^b A_\mu^c) \tag{4.1.54}$$

The beauty of the BRST transformation can be seen through two important observations:
OBSERVATION A: *Its square vanishes.*

PROOF: Omitting the parameter ϵ for simplicity (note that the two δ have different ϵ parameters),

$$\delta^2 A^a_\mu = -\frac{g^2}{2}\left\{f^{abc}f^{cde} + 2f^{acd}f^{cbe}\right\} A^b_\mu \mathbf{c}^d\mathbf{c}^e = 0$$

where once again we used the Jacobi identity (4.1.34). This identity is also responsible for the vanishing of $\delta^2\mathbf{c}$,

$$\delta^2\mathbf{c}^a = \frac{g^2}{2} f^{abc}f^{cde}\mathbf{c}^b\mathbf{c}^d\mathbf{c}^e = 0$$

Finally,

$$\delta^2\mathbf{b}^a = \frac{1}{\lambda}\partial^\mu(\partial_\mu\mathbf{c}^a + igf^{abc}\mathbf{c}^b A^c_\mu)$$

This vanishes due to the field equation of $\mathbf{c}$.
In the quantum theory, we would like to have $\delta^2 = 0$ without invoking the (classical) field equations. This can be achieved by introducing an auxiliary field $\mathcal{F}^a$ (a common trick in supersymmetric theories), replacing the Lagrangian density (4.1.52) by

$$\mathcal{L}' = \frac{\lambda}{2}\,\mathcal{F}^a\mathcal{F}^a - \mathcal{F}^a\partial^\mu A^a_\mu + \partial^\mu\mathbf{b}^a(\partial_\mu\mathbf{c}^a + igf^{abc}\mathbf{c}^b A^c_\mu) \tag{4.1.55}$$

and augmenting the BRST transformation of $\mathbf{b}$ (4.1.53) as

$$\delta\mathbf{b}^a = \epsilon\mathcal{F}^a \quad , \quad \delta\mathcal{F}^a = 0 \tag{4.1.56}$$

It follows immediately that

$$\delta^2\mathbf{b}^a = 0 \ , \quad \delta^2\mathcal{F}^a = 0$$

whereas the rest of our conclusions above remain the same.
OBSERVATION B: *The Lagrangian density (4.1.55) may be written as*

$$\epsilon\mathcal{L}' = \delta\mathbf{L} \ , \quad \mathbf{L} = \mathbf{b}^a\left\{-\partial^\mu A^a_\mu + \frac{\lambda}{2}\,\mathcal{F}^a\right\} \tag{4.1.57}$$

up to a total divengence.
PROOF: It is easy to see that

$$\delta\mathbf{L} = \epsilon\mathcal{L}' - \partial^\mu(\mathbf{b}^a\delta A^a_\mu)$$

The above two observations lead to an interesting way of determining whether $\mathcal{L}_0$ and $\mathcal{L}_0 + \mathcal{L}'$ are the same physical quantity. Firstly, $\delta\mathcal{L}_0 = \delta(\mathcal{L}_0 + \mathcal{L}') = 0$, so $\delta = 0$ may be adopted as the criterion of being a physical quantity. Secondly, the difference between the two Lagrangians can be written as δ(something) which may serve as the definition of being the same physical quantity. This points to a *cohomology* and a neat definition of the Hilbert space.

The BRST charge

Since the BRST transformation is a symmetry of the theory, there is an associated conserved Noether current,

$$\mathbf{J}_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^a_\nu)}\,\delta A^a_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu \mathbf{b}^a)}\,\delta\mathbf{b}^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu \mathbf{c}^a)}\,\delta\mathbf{c}^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu \mathcal{F}^a)}\,\delta\mathcal{F}^a + \mathbf{F}_\mu \quad (4.1.58)$$

where $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$, and a conserved Noether charge (BRST charge)

$$\mathbf{Q} = \int d^3x \mathbf{J}_0 \quad (4.1.59)$$

where

$$\mathbf{J}_0 = -F^{0ia}\,(\partial_i\mathbf{c}^a + igf^{abc}\mathbf{c}^b A^c_i) - \frac{ig}{2}f^{abc}\partial_0\mathbf{b}^a\,\mathbf{c}^b\mathbf{c}^c - \frac{1}{\lambda}\partial^\nu A^a_\nu(\partial_0\mathbf{c}^a + igf^{abc}\mathbf{c}^b A^c_0) \quad (4.1.60)$$

After discarding a total divergence, we deduce

$$\mathbf{Q} = \int d^3x\left\{(D_iF^{0i})^a\mathbf{c}^a - \frac{1}{\lambda}\partial^\nu A^a_\nu(D_0\mathbf{c})^a - \frac{ig}{2}f^{abc}\partial_0\mathbf{b}^a\,\mathbf{c}^b\mathbf{c}^c\right\} \quad (4.1.61)$$

$\mathbf{Q}$ is the generator of BRST transformations in the sense

$$\delta\Phi = i[\epsilon\mathbf{Q}\ ,\ \Phi] \quad (4.1.62)$$

for any field Φ.
Hilbert space
$\mathbf{Q}$ is a *nilpotent* operator,

$$\mathbf{Q}^2 = 0 \quad (4.1.63)$$

This follows from $\delta^2 = 0$ but may also be shown directly from (4.1.61). We may now define physical states as a *cohomology.*
The Hilbert space of all physical states is defined as the set of states annihilated by $\mathbf{Q}$,

$$\mathbf{Q}|\Psi\rangle = 0 \quad (4.1.64)$$

i.e., they are in the *kernel* $\mathcal{K}$ of $\mathbf{Q}$.
Two states whose difference is in the *range* $\mathcal{R}$ of $\mathbf{Q}$ (BRST exact) represent the same physical system,

$$|\Psi_1\rangle \equiv |\Psi_2\rangle = |\Psi_1\rangle + \mathbf{Q}|\mathbf{X}\rangle \quad (4.1.65)$$

Thus, the Hilbert space is defined as the *quotient* $\mathcal{K}/\mathcal{R}$ (notice that $\mathcal{R}\subseteq\mathcal{K}$ because $\mathbf{Q}$ is nilpotent).
Notice that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ have the same inner product with any physical state $|\Psi\rangle$ (easy to see if you use $\mathbf{Q}^\dagger = \mathbf{Q}$). It may also be shown that

$$\langle\Psi|\Psi\rangle > 0 \quad (4.1.66)$$

for all physical states which are not equivalent to the trivial state (i.e., not in $\mathcal{R}$) and that, despite appearances, the Hilbert space contains the right degrees of freedom (two polarizations). We shall not show these important facts here. Instead, we shall limit ourselves to the simplest example: QED.

EXAMPLE (QED)

Ignoring the unnecessary for our discussion complications due to the auxiliary field $\mathcal{F}$, we may write $\mathcal{L}'$ using (4.1.52) as

$$\mathcal{L}' = -\frac{1}{2\lambda}(\partial_\mu A^\mu)^2 + \partial_\mu \mathbf{b} \partial^\mu \mathbf{c} \tag{4.1.67}$$

The BRST transformation (4.1.53) reads

$$\delta A_\mu = \epsilon \partial_\mu \mathbf{c} \ , \quad \delta \mathbf{c} = 0 \ , \quad \delta \mathbf{b} = \frac{1}{\lambda} \epsilon \partial_\mu A^\mu \tag{4.1.68}$$

The BRST charge (4.1.61) is

$$\mathbf{Q} = \int d^3x \, \left\{ \mathbf{c} \vec{\nabla} \cdot \vec{E} - \frac{1}{\lambda} \partial_0 \mathbf{c} \partial_\mu A^\mu \right\} \tag{4.1.69}$$

where $E^i = F^{0i}$ is the electric field.

Quantization proceeds most straightforwardly if we choose $\lambda = 1$ (Feynman gauge). Of course, any choice of λ will lead to the same physical results. In the Feynman gauge, the Maxwell equations ((3.2.19) with no current) reduce to four independent massless Klein-Gordon equations

$$\partial^2 A_\mu = 0 \tag{4.1.70}$$

Any other choice of λ will lead to coupled equations which we would then have to solve following a procedure along the lines of the quantization of the Dirac equation. This is best left as an exercise for the brave reader. Notice that the ghosts also obey the massless Klein-Gordon equation. We may therefore expand in modes,

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(e^{ik\cdot x} a_\mu^\dagger(\vec{k}) + e^{-ik\cdot x} a_\mu(\vec{k}) \right)$$

where $k_0 = \omega_k = |\vec{k}|$,

$$\mathbf{b}(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(e^{ik\cdot x} \mathbf{b}^\dagger(\vec{k}) + e^{-ik\cdot x} \mathbf{b}(\vec{k}) \right)$$

and similarly for $\mathbf{c}$.

We impose the commutation relations

$$[a_\mu(\vec{k}) \ , \ a_\nu^\dagger(\vec{k}')] = -\eta_{\mu\nu} \ (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \tag{4.1.71}$$

which are standard except for $\mu = \nu = 0$ for which the sign is wrong, leading to negative norm states. This is an inevitable conclusion because the choice of $\eta_{\mu\nu}$ is dictated by Lorentz invariance. Our formalism is supposed to fix this.

For the ghosts, notice that the momentum conjugate to $\mathbf{b}$ is $\partial_0\mathbf{c}$ and *not* $\partial_0\mathbf{b}$. Similarly for the other ghost field. Thus, we need to impose the anti-commutation relations

$$\{\mathbf{b}(\vec{k})\ ,\ \mathbf{c}^\dagger(\vec{k}')\} = (2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

The BRST charge (4.1.69) may also be expanded in modes. To this end, let us first use the field equations (4.1.70) to bring it to the form

$$\mathbf{Q} = \int d^3x\ \{\mathbf{c}\partial_0\partial_\mu A^\mu - \partial_0\mathbf{c}\partial_\mu A^\mu\}$$

which makes it clearer that time-dependent terms do not contribute. We obtain

$$\mathbf{Q} = \int \frac{d^3k}{(2\pi)^3}\ \left\{k^\mu a_\mu(\vec{k})\mathbf{c}^\dagger(\vec{k}) + k^\mu a_\mu^\dagger(\vec{k})\mathbf{c}(\vec{k})\right\} \tag{4.1.72}$$

Therefore,

$$[\mathbf{Q}\ ,\ a_\mu(\vec{p})] = -p_\mu\mathbf{c}(\vec{p})\ ,\quad [\mathbf{Q}\ ,\ a_\mu^\dagger(\vec{p})] = p_\mu\mathbf{c}^\dagger(\vec{p})$$
$$\{\mathbf{Q}\ ,\ \mathbf{b}(\vec{p})\} = p^\mu a_\mu(\vec{p})\ ,\quad \{\mathbf{Q}\ ,\ \mathbf{b}^\dagger(\vec{p})\} = p^\mu a_\mu^\dagger(\vec{p})$$
$$\{\mathbf{Q}\ ,\ \mathbf{c}(\vec{p})\} = \{\mathbf{Q}\ ,\ \mathbf{c}^\dagger(\vec{p})\} = 0 \tag{4.1.73}$$

They lead to the expected transformation properties (4.1.68) of the corresponding fields. By applying $\mathbf{Q}$ again in (4.1.73), it is obvious that it is a nilpotent operator ($\mathbf{Q}^2 = 0$). This also follows directly from the mode expansion (4.1.72).
Turning to the Hilbert space, consider a single photon state of momentum $\vec{p}$ and polarization e^μ,

$$|\vec{p}, e\rangle = e^\mu a_\mu^\dagger(\vec{p})|0\rangle$$

For it to be a physical state, it should be annihilated by $\mathbf{Q}$. Since

$$\mathbf{Q}|\vec{p}, e\rangle = p_\mu e^\mu \mathbf{c}^\dagger(\vec{p})|0\rangle \tag{4.1.74}$$

this requirement yields

$$p_\mu e^\mu = 0 \tag{4.1.75}$$

which is the Lorentz gauge (Fourier transform of (3.2.17)).
If we were to perform a gauge transformation $A_\mu \to A_\mu + \partial_\mu\omega$, we would shift $e_\mu \to e_\mu + \hat{\omega}p_\mu$, where $\hat{\omega}$ is a number (since $A_\mu = e_\mu e^{ip\cdot x}$). That would change the state to

$$|\vec{p}, e\rangle \to |\vec{p}, e\rangle + \hat{\omega}p^\mu a_\mu^\dagger(\vec{p})|0\rangle \tag{4.1.76}$$

The additional piece may be written as

$$p^\mu a_\mu^\dagger(\vec{p})|0\rangle = \mathbf{Q}\mathbf{b}^\dagger(\vec{p})|0\rangle \tag{4.1.77}$$

Therefore the new state (4.1.76) describes the same photon. A gauge transformation does not alter the physics!
What about “ghost particles”? We could define states

$$|\vec{p}, \mathbf{b}\rangle = \mathbf{b}^\dagger(\vec{p})|0\rangle\ \ ,\ \ |\vec{p}, \mathbf{c}\rangle = \mathbf{c}^\dagger(\vec{p})|0\rangle$$

representing particles of type $\mathbf{b}$ or $\mathbf{c}$. However, acting with the BRST charge,

$$\mathbf{Q}|\vec{p},\mathbf{b}\rangle = p^\mu a_\mu^\dagger(\vec{p})|0\rangle \neq 0 \ , \quad \mathbf{Q}|\vec{p},\mathbf{c}\rangle = 0$$

Thus $|\vec{p},\mathbf{b}\rangle$ is not a physical state. On the other hand, from (4.1.74), for any polarization that does not obey (4.1.75) we have

$$|\vec{p},\mathbf{c}\rangle = \frac{1}{p\cdot e}\mathbf{Q}|\vec{p},e\rangle$$

i.e., this state is in the range of $\mathbf{Q}$, therefore it describes a physically empty system. As promised, there are no ghost particles!
Finally, let us check unitarity. From the commutation relations (4.1.71), we obtain negative inner products,

$$\langle 0|a_0(\vec{p})a_0^\dagger(\vec{p}^{\,\prime})|0\rangle = -(2\pi)^3\delta^3(\vec{p}-\vec{p}^{\,\prime})$$

However these states possess polarization vectors which do not obey (4.1.75) and are therefore not physical. For two physical one-photon states, we obtain

$$\langle\vec{p},e|\vec{p}^{\,\prime},e'\rangle = -e\cdot e'\ (2\pi)^3\delta^3(\vec{p}-\vec{p}^{\,\prime}) \tag{4.1.78}$$

Since e_μ is orthogonal to the null vector p^μ, it is spacelike ($e^2 < 0$ and we usually set $e^2 = -1$).[2] Similarly for e'_μ. The inner product of two spacelike vectors in negative, so

$$e\cdot e' < 0$$

and the inner product (4.1.78) of the two states is *positive*, as required by unitarity.

4.1.4 Chromodynamics (strong interactions)

The electromagnetic force (holding atoms together) acts on particles which have electric charge and is described by a gauge theory based on the group $U(1)$. The strong nuclear force (holding nuclei together) is described by a gauge theory based on the group $SU(3)$. It acts on particles with *color* charge called *quarks*; hence the name *chromodynamics* (from the Greek word *chroma* for color) for the theory of the strong force. There are three types of color charges (often chosen as red, blue, green) and $3^2-1=8$ types of color photons called *gluons* (from the English word *glue* for glue). Quarks come in 6 varieties (*flavors*), u (up), d (down), s (strange), c (charm), b (bottom) and t (top), so in addition to $N=3$ (as in $SU(3)$), we have

$$N_f = 6\ . \tag{4.1.79}$$

More flavors may exist, but only six have been observed. The world around us is made mostly of the two lightest quarks, u and d (with a little bit of s). The rest of the quarks are heavy and easily decay.

[2]To see this, use $0 = p\cdot e = p_0e_0 - \vec{p}\cdot\vec{e} \geq |\vec{p}|(e_0 - |\vec{e}|)$. It follows that $e_0 - |\vec{e}| \leq 0$ and so $e^2 \leq 0$. The equality holds only if $e \propto p$ which would make the state BRST-exact (eq. (4.1.77)) describing an empty system. Therefore, $e^2 < 0$.

The Lagrangian density describing each of these quark flavors is given by (4.1.25) with a corresponding color current

$$J^a_\mu = g\bar{\psi}\gamma_\mu T^a \psi \tag{4.1.80}$$

(*cf.* with eq. (4.1.26)) where $\psi = u, d, s, c, b, t$. As with $SU(2)$, these currents are not gauge invariant quantities and cannot be observed. Only *colorless (white)* objects are observable called *hadrons*. They include

$$\psi^\dagger \psi \tag{4.1.81}$$

(*cf.* with eq. (4.1.27)), a quark-antiquark state *(meson)* and

$$\epsilon^{abc}\psi^{(1)a}\psi^{(2)b}\psi^{(3)c} = \begin{vmatrix} \psi^{(1)1} & \psi^{(2)1} & \psi^{(3)1} \\ \psi^{(1)2} & \psi^{(2)2} & \psi^{(3)2} \\ \psi^{(1)3} & \psi^{(2)3} & \psi^{(3)3} \end{vmatrix} \quad , \quad \epsilon^{abc}\psi^{(1)a\dagger}\psi^{(2)b\dagger}\psi^{(3)c\dagger} \tag{4.1.82}$$

(*cf.* with eq. (4.1.28)), a 3-quark state *(baryon)* and its antiparticle *(anti-baryon)*. For $\psi^{(1)} = \psi^{(2)} = u$, $\psi^{(3)} = d$, we have a *proton* whereas if we change $\psi^{(2)} = d$, we obtain a *neutron*. Other combinations lead to more exotic baryons. That these objects are "white" follows from extending (4.1.29) to $SU(3)$.

We may also construct multi-quark states consisting of quarks which are not at the same point. E.g., a state representing a meson consisting of a quark at x and an anti-quark at y is

$$\psi^\dagger(y)\mathcal{P}e^{ig\int_x^y dx^\mu A_\mu}\psi(x) \tag{4.1.83}$$

The insertion of the Wilson line (*cf.* eq. (4.1.15)) ensures gauge invariance. It can be thought of as a string of glue holding the quark-antiquark pair together. As they try to separate, the string develops tension making separation hard. If enough energy is supplied, the string breaks and a quark-antiquark pair forms at the breaking point, resulting in the formation of two mesons. Thus the meson decays into two other mesons, which are also "white" objects; never to quarks (colored objects).

Similarly for the baryons, if we separate the three quarks, we need to introduce three strings attached to them and joining at an intermediate point. As the quarks fly apart, all three strings develop tension and may break, so a baryon (just like a meson) will decay into hadrons (observed at accelerators as *jets*).

It is instructive to compare the above picture to electrodynamics, e.g., the Hydrogen atom consisting of an electron described by $\psi_e(x)$ and a proton described by $\psi_p(y)$ (ignoring strong force effects). Under a gauge transformation,

$$\psi_e(x) \to (1 - ie\omega(x))\psi_e(x) \quad , \quad \psi_p(y) \to (1 + ie\omega(x))\psi_p(x) \tag{4.1.84}$$

Notice the difference in signs due to opposite electric charges. A gauge-invariant quantity for the system (Hydrogen atom) is

$$\psi_p^T(y)e^{ie\int_x^y dx^\mu A_\mu}\psi_e(x) \tag{4.1.85}$$

where we inserted a photon string connecting the two charges. We know that we only need 13.6 eV to separate them (ionization energy), so as they try to fly apart the string does not develop a tension. On the contrary, as we know, the Coulomb force weakens. Surely, the string can break and a particle-antiparticle can form, but the probability for that to happen is very low. Once the electron and the proton are far apart, they are both happy, because the corresponding electric currents ($\bar{\psi}_e \gamma^\mu \psi_e$ for the electron and $\bar{\psi}_p \gamma^\mu \psi_p$ for the proton) are gauge invariant.

The running coupling constant

To understand the (drastic) difference in behavior between the strong and electromagnetic forces, we need to understand the energy dependence of the strong fine stucture constant

$$\alpha_s = \frac{g^2}{4\pi} \tag{4.1.86}$$

We shall do that by repeating the steps leading to eq. (3.2.129) exhibiting the energy dependence of the electric fine structure constant.

Turning on a uniform magnetic field (3.2.106), the energy levels of a scalar field change to the Landau levels (3.2.109) resulting in an overall change in energy density (3.2.118)

$$\Delta\rho = \frac{(QB)^2}{24} \int \frac{d^3k}{(2\pi)^3 \omega_k^3} + \dots \tag{4.1.87}$$

and corresponding magnetic susceptibility (eq. (3.2.120))

$$\chi = -\frac{Q^2}{24\pi^2} \ln \frac{\Lambda}{E} \tag{4.1.88}$$

where Q is the (color) charge of the scalar field, Λ is a cutoff hiding our ignorance of (very) high energy effects and E is the energy at which we are performing measurements.

For a particle with spin $\vec{S}$, the energy levels receive additional contributions from the coupling of the spin with the magnetic field (3.2.122),

$$\omega_{k,B,S} = \omega_{k,B} - \frac{QBS_z}{2\omega_{k,B}} - \frac{(QBS_z)^2}{2\omega_k^3} + \dots \tag{4.1.89}$$

For a fermion, this leads to a change in energy density (eq. (3.2.124))

$$\Delta\rho_S^{(fermion)} = -2\Delta\rho + (QBS_z)^2 \int \frac{d^3k}{(2\pi)^3 \omega_k^3} + \dots = -2\Delta\rho(1 - 12S_z^2) + \dots \tag{4.1.90}$$

whereas for a boson, we obtain (notice the opposite sign)

$$\Delta\rho_S^{(boson)} == 2\Delta\rho(1 - 12S_z^2) + \dots \tag{4.1.91}$$

The corresponding contributions to the magnetic susceptibility are

$$\chi_S = \pm 2(1 - 12S_z^2)\chi \tag{4.1.92}$$

showing that fermions (bosons) have a diamagnetic (paramagnetic) effect.

In the case of electromagnetism, only fermions contributed, but in our case both fermions (quarks of spin 1/2) and bosons (gluons of spin 1) contribute. We have N_f quarks, therefore a total magnetic susceptibility

$$\chi_{total} = -\frac{1}{24\pi^2}(2N_f Q^2_{quark} - 11Q^2_{gluon}) \ln \frac{\Lambda}{E} \qquad (4.1.93)$$

The color charge of a quark may be found by considering the creation and annihilation of a quark-antiquark virtual pair contributing to the vacuum polarization. It is described by a current (4.1.80). Let us fix the group index a to concentrate on a particular gluon pointing in the direction a in color space. The amplitude of this process is $\sim J^a J^a$ (ignoring spacetime indices) with no summation on a. After summing over all possible colors of the virtual quark and antiquark, the amplitude becomes $\sim g^2 \mathrm{tr} T^a T^a = \frac{1}{2} g^2$, where we used (4.1.32). This is to be compared with the QED result $\sim e^2$. Therefore,

$$Q^2_{quark} = \frac{1}{2} g_0^2 \qquad (4.1.94)$$

where g_0 is the bare coupling constant.
A similar argument for the gluons yields [3]

$$Q^2_{gluon} = \frac{C_A}{2} g_0^2 \qquad (4.1.95)$$

where C_A is the Casimir in the adjoint representation (4.1.47).
The running strong coupling constant is

$$g^2 \approx \frac{g_0^2}{1 - \chi_{total}} \approx \frac{g_0^2}{1 + (N_f - \frac{11}{2} C_A) \frac{g_0^2}{12\pi^2} \ln \frac{\Lambda}{E}} \qquad (4.1.96)$$

in quantum chromodynamics (QCD).
This appears to be a dull result, but be careful with signs: $C_A = N = 3$ and $N_f = 6$, so the coefficient of the logarithm is negative ($N_f - \frac{11}{2} C_A < 0$)! This is opposite to QED (eq. (3.2.127)) and has far reaching consequences. It is perhaps the last significant theoretical result in high energy physics that has received expresimental verification. It is due to 't Hooft, Politzer, Gross and Wilczek (3 independent groups). It implies the strong interactions get weaker at high energies, unlike electromagnetic interactions which get stronger. Thus, quarks become free at high energies or short distances, e.g., inside a hardon (asymptotic freedom). At high energies, it is easy to do calculations doing perturbation theory as in QED because the coupling is weak. This is the regime in which QCD has received experimental verification. At low energies the coupling is strong; QCD is very complicated and feebly understood.
Recall that in QED we went to low energies where we knew $\frac{e^2}{4\pi} = \frac{1}{137}$ (small). This allowed us to express e^2 at any scale E in terms of measurable quantities (eq. (3.2.128)). The mass of the electron m_e provided a natural scale to rely upon. This is not possible in QCD. As we go to low energies the coupling constant increases and perturbation theory (hence also our result (4.1.96)) becomes unreliable. There is no natural scale

[3] The argument is awkwardly lengthly (similar to the derivation of C_A (eq. (4.1.47)) and will be omitted.

similar to m_e in QED. To define the quantum theory, it is necessary to introduce a scale. We shall call it Λ_{QCD}. You may argue that we could use the quark masses instead, but they are not observable particles, like the electron is in QED. Alternatively, we could use the mass of a "white" observable state, such as the proton or the neutron, but it is hard (so far impossible) to relate their masses to the strong coupling constant. Of course, after introducing Λ_{QCD} (a *physical* scale - not a cutoff), we expect all other scales (hardonic masses, etc) to be determined in terms of it.
It should also be pointed out that Λ_{QCD} is necessary even in the absence of fermions because gluons interact with each other. Thus bound states may exist *(glueballs)* and the quantum theory possesses a scale even though the classical theory does not.
We shall define Λ as the scale at which the coupling constant diverges. According to our approximate expression (4.1.96) this occurs when

$$1 + \left(N_f - \frac{11}{2} C_A \right) \frac{g_0^2}{12\pi^2} \ln \frac{\Lambda}{\Lambda_{QCD}} = 0 \tag{4.1.97}$$

Eliminating Λ, we obtain the strong fine structure constant

$$\alpha_s = \frac{g^2}{4\pi} = \frac{3\pi}{(\frac{11}{2} C_A - N_f) \ln \frac{E}{\Lambda_{QCD}}} \tag{4.1.98}$$

in terms of physical quantities. By fitting this function to experimental data, we obtain $\Lambda_{QCD} \approx 200$ MeV. For perturbation theory to be valid, we need to stay well above Λ_{QCD}, say $E \gtrsim 1$ GeV (the mass of the proton), so $\alpha_s \lesssim 0.4$ (since $C_A = 3$, $N_f = 6$). At distances $\gtrsim 1/\Lambda_{QCD}$ (size of light hadrons), forces become strong.

4.2 Current algebra and pions

The pion (π) is much lighter than all other hadrons ($m_\pi \approx 140$ MeV to be compared with $m_p \approx 1,000$ MeV). Nambu (Nobel prize 2008) speculated that strong interactions have an approximate symmetry which is spontaneously broken and the pion is the Goldstone boson. This led to the development of current algebra and remarkable predictions based on simple assumptions. Of course, the pion is a bound state of quark-antiquark pairs; that's why it is too complicated to describe in terms of QCD. In practice, calculations from first principles are possible using perturbation theory and this requires that we stay at energies $E \gg \Lambda_{QCD}$, where $\Lambda_{QCD} \approx 200$ MeV $\approx m_\pi$! If one goes to low energies ($E \ll \Lambda_{QCD}$), then the pion looks like a point and its complicated structure matters little. It turns out that its properties can be understood by an *effective low-energy* Lagrangian.
Let us restrict attention to the two lightest quarks, u (up) and d (down) which is what most of *our* world is made of. Ignoring interactions, the Lagrangian density is

$$\mathcal{L} = \bar{u}(i\gamma^\mu \partial_\mu - m_u)u + \bar{d}(i\gamma^\mu \partial_\mu - m_d)d \tag{4.2.1}$$

In the limit $m_u, m_d \to 0$, this Lagrangian has lots of symmetries. These are the symmetries we are after. They are only approximate, because $m_u, m_d \neq 0$, but the masses are small, which justifies the approximation. Define the isospin doublet

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{4.2.2}$$

Then we may write

$$\mathcal{L} \approx \bar{\psi} i\gamma^\mu \partial_\mu \psi \tag{4.2.3}$$

Splitting ψ into Weyl spinors,

$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi \ , \quad \psi_R = \frac{1}{2}(1+\gamma_5)\psi \tag{4.2.4}$$

we may write

$$\mathcal{L} \approx \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R \tag{4.2.5}$$

Evidently, it possesses the symmetry

$$\psi_L \to U_L \psi_L \ , \quad \psi_R \to U_R \psi_R \tag{4.2.6}$$

where $U_{L,R}$ are *independent* 2×2 unitary matrices ($U_{L,R} \in U(2)$).
We have $U(2) \simeq SU(2) \times U(1)$, where $U(1)$ is the transformation

$$\psi \to e^{i\theta}\psi \tag{4.2.7}$$

whose Noether current is the fermion number. The two $U(1)$ symmetries can be combined into the vector $U(1)$,

$$U(1)_V \ : \ \psi_L \to e^{i\theta}\psi_L \ , \quad \psi_R \to e^{i\theta}\psi_R \tag{4.2.8}$$

and the axial $U(1)$,

$$U(1)_A \ : \ \psi_L \to e^{i\theta}\psi_L \ , \quad \psi_R \to e^{-i\theta}\psi_R \tag{4.2.9}$$

The former survives the inclusion of masses whereas the latter doesn't.
We shall concentrate on $SU(2)_L \times SU(2)_R$ which does not survive the inclusion of masses. It is called chiral symmetry (for the Greek word *chir* for hand). To break the symmetry spontaneously, suppose

$$\langle 0|\bar{\psi}_L^a \psi_R^b|0\rangle = -v^3\delta^{ab} \tag{4.2.10}$$

where $a, b = 1, 2$ are isospin indices. To see what part of the symmetry is left unbroken, observe that under a chiral transformation,

$$\bar{\psi}_L^a \psi_R^b \to \bar{\psi}_L^c U_L^{*ac} U_R^{bd} \psi_R^d \tag{4.2.11}$$

It follows that

$$-v^3\delta^{ab} \to -v^3 U_L^{*ac} U_R^{bd}\delta^{cd} = -v^3(U_R U_L^\dagger)^{ba} \tag{4.2.12}$$

Thus the vacuum expectation value (4.2.10) is invariant under a chiral transformation with $U_R U_L^\dagger = \mathbb{I}$, i.e., $U_R = U_L$. This is the vector $SU(2)$,

$$SU(2)_V \ : \ \psi_L \to U\psi_L \ , \quad \psi_R \to U\psi_R \ ; \quad \psi \to U\psi \tag{4.2.13}$$

The broken symmetry is the axial $SU(2)$. It has three generators and therefore three Goldstone bosons - the three pions $\pi^\pm, \pi^0$.

The corresponding Noether currents of the various symmetries are

$$\begin{aligned} SU(2)_L &: \quad \vec{J}_L^\mu = \bar{\psi}_L \gamma^\mu \vec{S} \psi_L \\ SU(2)_R &: \quad \vec{J}_R^\mu = \bar{\psi}_R \gamma^\mu \vec{S} \psi_R \\ SU(2)_V &: \quad \vec{J}_V^\mu = \vec{J}_R^\mu + \vec{J}_L^\mu = \bar{\psi} \gamma^\mu \vec{S} \psi \\ SU(2)_A &: \quad \vec{J}_A^\mu = \vec{J}_R^\mu - \vec{J}_L^\mu = \bar{\psi} \gamma^\mu \gamma_5 \vec{S} \psi \end{aligned} \tag{4.2.14}$$

where $\vec{S} = \frac{1}{2}\vec{\sigma}$.
By Goldstone's theorem, the pions couple to the currents of the broken symmetry $\vec{J}_A^\mu$. As we showed earlier, the fourier transform of the Green function (1.6.34) had a pole at $p^2 = 0$ (1.6.41). The residue of this pole is proportional to the coupling of the current to the Goldstone boson. [4] In our case, this implies that the fourier transform

$$\tilde{G}^{\mu ij}(p) = -if_\pi p^\mu \delta^{ij} \tag{4.2.15}$$

where

$$G^{\mu ij}(x) = \langle 0 | J_A^{\mu i}(x) | \pi^j \rangle \tag{4.2.16}$$

and we used $G^{\mu ij}(x) \propto \delta^{ij}$ which follows from $SU(2)_V$ invariance.
The proportionality constant f_π can be determined from experiment. It is found

$$f_\pi = 93 \text{ MeV} \tag{4.2.17}$$

We may also confirm that the pion is indeed massless. From conservation of the axial current, $\partial_\mu \vec{J}_A^\mu = 0$, we deduce

$$0 = p_\mu \tilde{G}^{\mu ij}(p) = -if_\pi p^2 \delta^{ij} \tag{4.2.18}$$

therefore $p^2 = 0$.

Chiral Lagrangian

To obtain an effective Lagrangian for *soft* (low energy) pions, let us extend the assumption (4.2.10) to

$$\langle \bar{\psi}_L^a \psi_R^b \rangle = -v^3 \Sigma^{ab} \quad , \quad \Sigma \in SU(2) \tag{4.2.19}$$

where the expectation value is now computed in the presence of a soft pion. The origin of (4.2.19) is hard (perhaps impossible) to understand from QCD, but if we stay at energies $E \ll \Lambda_{QCD}$, the details are not important; only the symmetry matters.
Σ describes the orientation of the vacuum *(order parameter)* and corresponds to (isospin) rotations. The pions are associated with infinitesimal chiral rotations, so

$$\Sigma = e^{i\vec{\pi}\cdot\vec{\sigma}/F} \tag{4.2.20}$$

where $\vec{\pi}$ consists of the three pion fields and F is a constant to be determined. We have

$$\Sigma^\dagger \Sigma = \mathbb{I} \quad , \quad \det \Sigma = 1 \tag{4.2.21}$$

[4] Of course, it is also proportional to the coupling of the field ϕ to the Goldstone boson.

and under an $SU(2)_L \times SU(2)_R$ transformation (using (4.2.6) and (4.2.19)),

$$\Sigma \to U_L^\dagger \Sigma U_R \tag{4.2.22}$$

We wish to build a Lagrangian for Σ. It should contain a quadratic term $\frac{1}{2}\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}$, since the pions are massless scalars. We shall choose

$$\mathcal{L} = \frac{1}{4} F^2 \mathrm{tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + \ldots \tag{4.2.23}$$

where the dots represent higher-order derivative terms. This is the simplest choice. Other possibilities one might try are:

- $(\mathrm{tr}\Sigma^\dagger \partial_\mu \Sigma)^2$

  However, this vanishes. To see this, define

  $$U(\epsilon) = \Sigma^\dagger(x)\Sigma(x+\epsilon) \tag{4.2.24}$$

  Expanding in ϵ,

  $$\epsilon^\mu \Sigma^\dagger \partial_\mu \Sigma = \mathbb{I} - U(\epsilon) + \mathcal{O}(\epsilon^2) \tag{4.2.25}$$

  On the other hand, $U(\epsilon) \in SU(2)$, therefore

  $$U(\epsilon) = e^{i\vec{u}\cdot\vec{\sigma}} = \mathbb{I} + i\vec{u}\cdot\vec{\sigma} + \mathcal{O}(\epsilon^2) \tag{4.2.26}$$

  since $\vec{u} \sim \mathcal{O}(\epsilon)$. It follows that

  $$\epsilon^\mu \Sigma^\dagger \partial_\mu \Sigma = -i\vec{u}\cdot\vec{\sigma} \tag{4.2.27}$$

  Taking traces and using $\mathrm{tr}\vec{\sigma} = 0$, we deduce

  $$\mathrm{tr}\Sigma^\dagger \partial_\mu \Sigma = 0 \tag{4.2.28}$$

- $\mathrm{tr}(\Sigma^\dagger \partial_\mu \Sigma)^2$

  We have

  $$\mathrm{tr}(\Sigma^\dagger \partial_\mu \Sigma)^2 = -\mathrm{tr}\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \tag{4.2.29}$$

  where we used $\partial_\mu \Sigma^\dagger = \partial_\mu \Sigma^{-1} = -\Sigma^{-1}\partial_\mu \Sigma \Sigma^{-1} = -\Sigma^\dagger \partial_\mu \Sigma \Sigma^\dagger$.

  Therefore, this is not a new term.

Clearly the Lagrangian density (4.2.23) is invariant under the transformation (4.2.22). The corresponding Noether currents are:
For $\Sigma \to U_L^\dagger \Sigma$, we obtain

$$\vec{J}_L^\mu = \frac{i}{4} F^2 \mathrm{tr}\left[\Sigma^\dagger \vec{S} \partial^\mu \Sigma - \mathrm{h.c.}\right] \tag{4.2.30}$$

For $\Sigma \to \Sigma U_R$, we obtain

$$\vec{J}_R^\mu = \frac{i}{4} F^2 \mathrm{tr}\left[\Sigma \vec{S} \partial^\mu \Sigma^\dagger - \mathrm{h.c.}\right] \tag{4.2.31}$$

where $\vec{S} = \frac{1}{2}\vec{\sigma}$.
Notice that the left current is obtained from the right current by $\Sigma \to \Sigma^\dagger$, or $\vec{\pi} \to -\vec{\pi}$, showing that the pion is odd under parity (pseudoscalar).
Expanding Σ, we obtain

$$\vec{J}^\mu_L = -\frac{1}{2}F\partial^\mu\vec{\pi} + \dots \quad , \quad \vec{J}^\mu_L = \frac{1}{2}F\partial^\mu\vec{\pi} + \dots \tag{4.2.32}$$

The axial current is

$$\vec{J}^\mu_A = \vec{J}^\mu_R - \vec{J}^\mu_L = F\partial^\mu\vec{\pi} + \dots \tag{4.2.33}$$

For the one-particle state $|\pi^i\rangle$, we deduce

$$\langle 0|\tilde{J}^{\mu j}_A(p)|\pi^i\rangle = -iFp^\mu\delta ij \tag{4.2.34}$$

Comparing with our earlier result (4.2.15), we obtain

$$F = f_\pi = 93 \text{ MeV} \tag{4.2.35}$$

Thus the single free parameter in our theory is fixed by experiment. Expanding the Lagrangian density (4.2.23) in the pion field, we have

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \dots \tag{4.2.36}$$

where $\mathcal{L}_i$ $(i = 2, 3, 4, \dots)$ describes the (effective) interaction of i pions. We obtain

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} \tag{4.2.37}$$

as desired,

$$\mathcal{L}_3 = 0 \tag{4.2.38}$$

$$\mathcal{L}_4 = -\frac{1}{6F^2}\left(\vec{\pi}\times\partial_\mu\vec{\pi}\right)^2 \tag{4.2.39}$$

etc. Our theory has a lot of predictive power, because all interactions involving pions are determined (and can be compared with experiment).

4.3 Electroweak interactions

4.3.1 Gauge bosons and the Higgs

We wish to build a theory that unifies weak and electromagnetic interactions. Weak interactions were first discovered in β-decay,

$$n \to pe\bar{\nu}_e \tag{4.3.1}$$

This looked like a point interaction coupling a hadronic current (consisting of quarks, as we know now) to a leptonic current (consisting of an electron and its neutrino, both of which do not interact strongly). The strength is (F for Fermi)

$$G_F \sim \frac{10^{-5}}{m_p^2} \sim \frac{1}{(300\text{ GeV})^2} \tag{4.3.2}$$

This is similar to the gravitational interaction, where the coupling constant is Newton's constant

$$G_N \sim \frac{1}{m^2_{Planck}} \sim \frac{1}{(10^{16}\ \mathrm{GeV})^2} \tag{4.3.3}$$

Thus weak interactions appear to be associated with a mass scale $O(100\ \mathrm{GeV})$ (the rest of G_F being a dimensionless coupling constant). This can be the mass of an intermediate vector particle mediating the interaction. The scattering amplitude includes a factor corresponding to the propagator whose Fourier transform behaves like

$$\sim \frac{i}{k^2 - m_V^2} \tag{4.3.4}$$

In the low energy limit $k^2 \to 0$, yielding a factor $\sim 1/m_V^2$, which explains why β-decay looks like a point interaction of strength

$$G_F \sim \frac{g^2}{m_V^2} \ , \quad g \sim O(1) \ , \quad m_V \sim O(100\ \mathrm{GeV}) \tag{4.3.5}$$

After Fourier transforming (4.3.4), we obtain the screened static potential (Yukawa potential)

$$V_{\mathrm{weak}} \sim g^2 \frac{e^{-m_V|\vec{x}|}}{|\vec{x}|} \tag{4.3.6}$$

An application of this idea to gravity fails, even though the strength is similar in dimensions to weak interactions, because gravity is a *long-range* force ($V_{\mathrm{gravity}} \sim 1/|\vec{x}|$).
Another peculiar property of weak interactions is that they do not conserve parity: the electrons and neutrinos involved in β-decay are *left-handed*.
Since weak interactions transform electrons to neutrinos (or create electrons and antineutrinos), we should put them in a doublet,

$$\psi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \tag{4.3.7}$$

where L indicates that the spinors are left-handed (recall that if e is a Dirac field, then $e_L = \frac{1}{2}(1-\gamma_5)e$ is a left-handed Weyl field). ψ_L is a spinor living in an abstract (*weak isospin*) three-dimensional space. Rotations in this space form a $SU(2)$ group and the gauge theory corresponding to the weak charge should be a non-abelian $SU(2)$ gauge theory. To accommodate electromagnetic forces, we need to enlarge the gauge group. Indeed, the electric charge Q cannot be a generator of $SU(2)$, because then it would be traceless,

$$\mathrm{tr}Q = \mathrm{tr}Q^a \frac{\sigma^a}{2} = 0 \tag{4.3.8}$$

whereas

$$\mathrm{tr}Q = Q_\nu + Q_e = 0 + (-1) \neq 0 \tag{4.3.9}$$

We need to have $SU(2) \times U(1)$ at least. It turns out that the minimal choice suffices.
What is the physical meaning of this $U(1)$? Let Y be the corresponding charge (called "weak hYpercharge"). The electric charge Q will be a linear combination of Y and

an $SU(2)$ generator. The latter may be chosen to be in the 3-direction without loss of generality. Thus

$$Q = a\frac{\sigma^3}{2} + Y = \begin{pmatrix} \frac{a}{2} + Y & 0 \\ 0 & -\frac{a}{2} + Y \end{pmatrix} \tag{4.3.10}$$

Acting on the doublet ψ_L, we deduce

$$Q_\nu = \frac{a}{2} + Y \ , \quad Q_e = -\frac{a}{2} + Y \tag{4.3.11}$$

Demanding $Q_\nu = 0$, $Q_e = -1$, we obtain

$$a = 1 \ , \quad Y = -\frac{1}{2} \tag{4.3.12}$$

Let B_μ be the vector potential for the hypercharge. We have three more gauge bosons,

$$W_\mu = W_\mu^a \frac{\sigma^a}{2} \tag{4.3.13}$$

The photon field A_μ is a linear combination of B_μ and W_μ^3. The other three gauge fields will have to be given masses via the Higgs mechanism in order to describe weak interactions.
To build interactions, we need to introduce the gauge derivative,

$$D_\mu = \partial_\mu + i\mathbb{A}_\mu \tag{4.3.14}$$

where I absorbed the coupling constant into the definition of the $SU(2) \times U(1)$ vector potential,

$$\mathbb{A}_\mu = gW_\mu^a \frac{\sigma^a}{2} + g' B_\mu Y \tag{4.3.15}$$

This is because the $U(1)$ coupling constant can be different from the $SU(2)$ coupling constant (however all generators of $SU(2)$ must have the same coupling constant).
To reveal the photon, rotate in the "space" of W_μ^3 and B_μ by an angle θ_W (W for weak or Weinberg),

$$W_\mu^3 = \cos\theta_W Z_\mu + \sin\theta_W A_\mu \ , \quad B_\mu = -\sin\theta_W Z_\mu + \cos\theta_W A_\mu \tag{4.3.16}$$

where A_μ is the photon and Z_μ is a weak boson. The vector potential reads

$$\mathbb{A}_\mu = \left(g\sin\theta_W \frac{\sigma^3}{2} + g'\cos\theta_W Y \right) A_\mu + \dots \tag{4.3.17}$$

where I only included the terms involving the photon. The coefficient of A_μ should be eQ, where e is the electromagnetic coupling (not to be confused with the electron field e!). Therefore,

$$e\left(\frac{\sigma^3}{2} + Y\right) = g\sin\theta_W \frac{\sigma^3}{2} + g'\cos\theta_W Y \tag{4.3.18}$$

imposing two constraints on the three parameters g, g', θ_W,

$$\tan\theta_W = \frac{g'}{g} \ , \quad e = \frac{gg'}{\sqrt{g^2 + g'^2}} \tag{4.3.19}$$

For the Higgs mechanism, we need to add a scalar field ϕ of Lagrangian density

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - \frac{\lambda}{2}\left(\phi^\dagger \phi - \frac{v^2}{2}\right)^2 \tag{4.3.20}$$

Since the gauge derivative is a 2×2 matrix, ϕ must be a doublet (complex scalar) field. The classical ground state is at

$$\phi^\dagger \phi = \frac{v^2}{2} \tag{4.3.21}$$

All choices are gauge-equivalent. We want to make a choice that will leave the photon massless. This can be achieved if our choice obeys

$$Q\phi = 0 \tag{4.3.22}$$

so that the $U(1)$ symmetry generated by Q is *unbroken*. If we are dumb enough to make a different choice (for which $Q\phi \neq 0$), then we will have to redefine the electric charge and the photon. Eq. (4.3.22) is obeyed by the ground state

$$\phi = \frac{v}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.3.23}$$

(same as a doublet representing a (chargeless) neutrino). Let us expand around the ground state,

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} v + \phi' \\ 0 \end{pmatrix} \tag{4.3.24}$$

where ϕ' is a *real* scalar field. We made a gauge choice (unitary gauge, equivalent to setting $\sigma = 0$ in the abelian case). Originally, ϕ had 4 degrees of freedom (complex doublet). After gauge-fixing, only 1 degree of freedom remains, ϕ', the Higgs particle. The other 3 degrees of freedom are expected to be eaten up by 3 of the four gauge bosons, hopefully by $W_\mu^{1,2}$ and Z_μ (*not* A_μ which should remain massless).
Let's watch. The Lagrangian density reads

$$\mathcal{L} = \frac{1}{2}(v\ \ 0)\mathbb{A}^\mu \mathbb{A}_\mu \begin{pmatrix} v \\ 0 \end{pmatrix} + \dots \tag{4.3.25}$$

where I only included terms which were quadratic in the gauge fields. Explicitly,

$$\mathcal{L} = \frac{e^2 v^2}{8\sin^2\theta_W \cos^2\theta_W} Z_\mu Z^\mu + \frac{e^2 v^2}{4\sin^2\theta_W} W_\mu^+ W^{\mu -} + \dots \tag{4.3.26}$$

where

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm i W_\mu^2) \tag{4.3.27}$$

so that W_μ^- is the complex conjugate of W_μ^+ (particle and anti-particle). There is no term involving the photon A_μ showing that the photon is massless. The mass of Z_μ is

$$m_Z = \frac{ev}{2\sin\theta_W \cos\theta_W} \tag{4.3.28}$$

and the mass of $W^\pm_\mu$ is

$$m_W = \frac{ev}{2\sin\theta_W} \tag{4.3.29}$$

Numerically, using $\frac{e^2}{4\pi} = \frac{1}{129}$ (*not* $\frac{1}{137}$ - see eq. (3.2.129)), eq. (4.3.46) and (4.3.50), we obtain

$$m_W = 79.5\,GeV \quad , \quad m_Z = 90\,GeV \tag{4.3.30}$$

These values receive *loop* corrections (to be discussed later) $\sim 3\%$. They agree amazingly well with experiment (hence the Nobel prize to the experimentalists at CERN who first observed $W^\pm$ and Z^0).

4.3.2 Leptons

Leptons are expected to be described by a Lagrangian density

$$\mathcal{L}_L = \bar{\psi}_L i\gamma^\mu D_\mu \psi_L \tag{4.3.31}$$

Since ψ_L is a Weyl spinor, it must be massless. To give a mass to the electron, we need to have a term involving e_R. The latter does not couple to W^a_μ; it is an $SU(2)$ *singlet.* [5] The Lagrangian density for e_R is

$$\mathcal{L}_R = \bar{e}_R i\gamma^\mu(\partial_\mu + ig' B_\mu Y)e_R \tag{4.3.32}$$

i.e., it only includes the part of $\mathbb{A}_\mu$ involving the $U(1)$ field B_μ. The hypercharge differs from the hypercharge of the left-handed doublet. Since e_R is an $SU(2)$ singlet, $Y = Q$, therefore

$$Y = -1 \tag{4.3.33}$$

For a mass term, we need to include $\bar{e}_L e_R$ in the Lagrangian density. Since e_L and e_R transform differently under $SU(2)$, such a term is forbidden by gauge invariance - weak interactions violate parity in an essential way! So who gives mass to the electron? The answer is the same as with gauge bosons: the Higgs field. This is because gauge invariance allows triple Higgs-electron-electron couplings.
To see this, first note the transformation properties of ϕ under $SU(2)$ and $U(1)$, respectively,

$$\delta\phi = i\omega^a \frac{\sigma^a}{2}\phi \quad , \quad \delta\phi = i\omega Y\phi \quad , \quad Y = -\frac{1}{2} \tag{4.3.34}$$

ϕ has the same hypercharge as the neutrino because it was deliberately made to look like it.
Next, we introduce its charge conjugate

$$\phi^C = -i\sigma_2\phi^* \tag{4.3.35}$$

Standard algebra yields

$$\delta\phi^C = i\omega^a \frac{\sigma^a}{2}\phi^C \quad , \quad \delta\phi^C = i\omega Y\phi^C \quad , \quad Y = +\frac{1}{2} \tag{4.3.36}$$

[5] We shall ignore the (much smaller) mass of the neutrino - it can be added in the same way.

It follows that $\bar{\psi}_L \phi^C e_R$ is an $SU(2)$ singlet and has $Y = \frac{1}{2} + \frac{1}{2} - 1 = 0$. Therefore, it is gauge-invariant and may be part of the Lagrangian density (*Yukawa coupling*),

$$\mathcal{L}_{\text{Yukawa}} = -f_e \bar{\psi}_L \phi^C e_R + \text{h.c.} \quad (4.3.37)$$

where f_e is the electron Yukawa coupling constant. Since

$$\phi^C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \phi' \end{pmatrix} \quad (4.3.38)$$

we obtain

$$\mathcal{L}_{\text{Yukawa}} = -\frac{f_e v}{\sqrt{2}} (\bar{e}_L e_R + \bar{e}_R e_L) + \text{interactions} \quad (4.3.39)$$

showing that the Higgs field has given a mass to the electron,

$$m_e = \frac{f_e v}{\sqrt{2}} \quad (4.3.40)$$

Knowing the mass of the electron ($m_e = 0.511$ MeV) and v (eq. (4.3.46)), we deduce

$$f_e = 2.9 \times 10^{-6} \quad (4.3.41)$$

This is a *dimensionless* number, as fundamental as the fine structure constant $\alpha = \frac{1}{137}$. Why did Nature choose such a small number? I don't know.

Moreover, Nature seems to have copied the leptons (at least) twice for a total of three *generations* (*families*)

$$\{e, \nu_e\} \ , \quad \{\mu, \nu_\mu\} \ , \quad \{\tau, \nu_\tau\} \quad (4.3.42)$$

They are identical except for their masses. Since the masses are different, so are the corresponding Yukawa coupling constants f_e, f_μ and f_τ. Why are they different? Again, I don't know.

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From the left and right pieces of the Lagrangian density (eqs. (4.3.31) and (4.3.32), respectively), we can read off the currents coupled to the various gauge bosons.

- charged current coupled to W_μ^+,

$$J_W^\mu = \frac{e}{\sqrt{2}\sin\theta_W} \bar{\nu}_e \gamma^\mu e_L \quad (4.3.43)$$

  and similarly for W_μ^- (obtained by charge conjugation).

  The interaction of two currents at low energies (as in β-decay) has strength

$$\left(\frac{e}{\sqrt{2}\sin\theta_W} \right)^2 \frac{1}{m_W^2} = \frac{2}{v^2} \quad (4.3.44)$$

  where the second factor comes from the W-boson propagator, as we discussed above and we used eq. (4.3.29). This must equal the measured strength

$$2\sqrt{2} G_F = 3.28 \times 10^{-5} \ (\text{GeV})^{-2} \quad (4.3.45)$$

  We deduce

$$v = \frac{1}{\sqrt{\sqrt{2} G_F}} = 247 \text{ GeV} \quad (4.3.46)$$

- electromagnetic current coupled to A_μ,

$$J^\mu_A = e(\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R) \tag{4.3.47}$$

No surprises here and of course no current for the neutrino.

- neutral current coupled to Z_μ,

$$J^\mu_Z = \frac{e}{\cos\theta_W \sin\theta_W}\left(\frac{1}{2}\bar{\nu}_{eL}\gamma^\mu\nu_{eL} + \left(\sin^2\theta_W - \frac{1}{2}\right)\bar{e}_L\gamma^\mu e_L + \sin^2\theta_W\bar{e}_R\gamma^\mu e_R\right) \tag{4.3.48}$$

The strength of the effective low-energy interaction is

$$\sim \left(\frac{e}{\cos\theta_W \sin\theta_W}\right)^2 \frac{1}{m_Z^2} = \frac{2}{v^2} \tag{4.3.49}$$

where we used (4.3.28) - same as $W^\pm_\mu$.

By performing $e\nu_e$ scattering experiments, we can determine θ_W, since this is the only free (unknown) parameter. Moreover, the model has a lot of predictive power: *all* cross-sections measured experimentally have to fit the theoretical predictions based on a single parameter. The fit is embarassingly good. We obtain

$$\sin^2\theta_W = 0.23 \;, \therefore \quad \theta_W = 29^o \tag{4.3.50}$$

4.3.3 Quarks

From β-decay, we know that we need to put the u and d quarks in a doublet

$$\psi_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \tag{4.3.51}$$

just as we did with e and ν_e ($p \to n$ entails $u \to d$).
The charges are $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$ so the charge matrix is

$$Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \tag{4.3.52}$$

Since $Q = \frac{\sigma^3}{2} + Y$ (the coefficient of $\frac{\sigma^3}{2}$ must match the leptonic case (4.3.11) for gauge invariance), we deduce

$$Y = \frac{1}{6} \tag{4.3.53}$$

Notice that we actually have two constraints for Y and they are both satisfied. A coincidence? No, as we will see later when we discuss *chiral anomalies*: quarks and leptons need each other for a deep reason (gauge invariance).

To give masses to the quarks, first we need to introduce the right-handed partners u_R and d_R which are $SU(2)$ singlets, therefore $Y = Q$,

$$Y(u_R) = \frac{2}{3} \ , \quad Y(d_R) = -\frac{1}{3} \tag{4.3.54}$$

The Yukawa interaction for the u quark is given by the Lagrangian density

$$\mathcal{L}_u = -f_u \bar{\psi}_L \phi u_R + \text{h.c.} \tag{4.3.55}$$

It is obviously an $SU(2)$ singlet and has $Y = -\frac{1}{6} - \frac{1}{2} + \frac{2}{3} = 0$, hence it is gauge-invariant.
For the d quark, we need to involve ϕ^C, instead,

$$\mathcal{L}_d = -f_d \bar{\psi}_L \phi^C d_R + \text{h.c.} \tag{4.3.56}$$

It is obviously an $SU(2)$ singlet and has $Y = -\frac{1}{6} + \frac{1}{2} - \frac{1}{3} = 0$, hence it is gauge-invariant.
Explicitly,

$$\mathcal{L}_u + \mathcal{L}_d = -\frac{f_u v}{\sqrt{2}}(\bar{u}_L u_R + \bar{u}_R u_L) - \frac{f_d v}{\sqrt{2}}(\bar{d}_L d_R + \bar{d}_R d_L) + \text{interactions} \tag{4.3.57}$$

showing that the masses of the two quarks are, respectively,

$$m_u = \frac{f_u v}{\sqrt{2}} \ , \quad m_d = \frac{f_d v}{\sqrt{2}} \tag{4.3.58}$$

Similarly to leptons, Nature has made copies of these quarks (three *families*),

$$\{u, d\} \ , \quad \{c, s\} \ , \quad \{t, b\} \tag{4.3.59}$$

which only differ in their masses.
Why did Nature make the same number of copies of quarks and leptons? She had a deep reason, to be revealed when we discuss *chiral anomalies.*
CURRENTS
The currents involving the quarks may be found from the Lagrangian density of left-handed quarks

$$\mathcal{L}_L = (\bar{u}\ \bar{d})_L i\gamma^\mu D_\mu \begin{pmatrix} u \\ d \end{pmatrix}_L + (\bar{c}\ \bar{s})_L i\gamma^\mu D_\mu \begin{pmatrix} c \\ s \end{pmatrix}_L + \dots \tag{4.3.60}$$

together with its right-handed counterpart

$$\mathcal{L}_R = \sum_{q=u,d,c,s,\dots} \bar{q}_R i\gamma^\mu(\partial_\mu + ig' B_\mu Y(q_R)) q_R \tag{4.3.61}$$

where I omitted the third generation in order to focus on the first two.
Unfortunately, Nature decided to play a game here which complicated things. Why? Because she could.

The d_L and s_L fields you see above are not the same as those in the Yukawa couplings. This is allowed by gauge invariance. Indeed, we may rotate them in a two-dimensional (*flavor*) space,

$$\begin{pmatrix} d' \\ s' \end{pmatrix}_L = \mathcal{O} \begin{pmatrix} d \\ s \end{pmatrix}_L \quad , \quad \mathcal{O} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \tag{4.3.62}$$

$\mathcal{O}$ is an orthogonal matrix and θ_C is the Cabibbo angle. Nature decided to adopt the Lagrangian density

$$\mathcal{L}'_L = (\bar{u}\,\bar{d}')_L i\gamma^\mu D_\mu \begin{pmatrix} u \\ d' \end{pmatrix}_L + (\bar{c}\,\bar{s}')_L i\gamma^\mu D_\mu \begin{pmatrix} c \\ s' \end{pmatrix}_L + \dots \tag{4.3.63}$$

instead, which is more complicated, but still gauge-invariant. [6] With this choice, generations get mixed and mass eigenstates are not the same as weak eigenstates. This has observable consequences. [7]

- Charged current coupled to W^+_μ,

$$J^\mu_W = \frac{e}{\sqrt{2}\sin\theta_W}(\bar{u}_L\gamma^\mu d'_L + \bar{c}_L\gamma^\mu s'_L) \tag{4.3.64}$$

  It includes a strangeness-changing component proportional to $\sin\theta_C$ (absent when $\theta_C = 0$). This allows the annihilation of a u quark and an s anti-quark (into a W-boson), so e.g., a K^+-meson ($u\bar{s}$ bound state) can decay into leptons. The suppression factor for such interactions is $\sin^2\theta_C$ and is found experimentally to be (Cabibbo suppression)

$$\sin^2\theta_C = 0.05 \ ,\therefore \quad \theta_C = 13^o \tag{4.3.65}$$

  β-decay is also suppressed because the hadronic current involved ($\bar{u}_L\gamma^\mu d_L$, similar to its leptonic counterpart) has a factor $\cos\theta_C$, so due to mixing, we get a suppression of the decay rate by a factor

$$\cos^2\theta_C = 0.95 \tag{4.3.66}$$

- neutral current coupled to Z_μ,

$$J^\mu_Z = \frac{e}{\cos\theta_W\sin\theta_W}(\ \bar{u}\ \bar{d}'\ )_L\left(\frac{\sigma^3}{2} - \sin^2\theta_W Q\right)\gamma^\mu \begin{pmatrix} u \\ d' \end{pmatrix}_L + \dots \tag{4.3.67}$$

  is independent of θ_C because the matrix $\frac{\sigma^3}{2} - \sin^2\theta_W Q$ is diagonal, so we could simply replace $d' \to d$, $s' \to s$. So interactions involving Z-boson exchange do not mix generations (*preserve flavor*).

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[6] We could also rotate $\begin{pmatrix} u \\ c \end{pmatrix}_L$, but there is no point. It's the *relative* rotation of $\begin{pmatrix} u \\ c \end{pmatrix}_L$ *vs* $\begin{pmatrix} d \\ s \end{pmatrix}_L$ that makes a difference and this is what $\mathcal{O}$ represents.

[7] For a long time, it was thought that no Cabibbo mixing occurred with leptons. Recent experimental observations (*neutrino oscillations*) showed that leptons also mix. We shall not discuss this mixing here. It can be introduced (together with neutrino masses) in much the same way as with quarks.

EXAMPLE: $K^0 \rightarrow \mu^+\mu^-$ (K^0 is a $d\bar{s}$ bound state) is not allowed. In fact $K_L^0 \rightarrow \mu^+\mu^-$ has a branching ratio of $\sim 10^{-8}$ due to quantum (*loop*) effects (to be discussed later).

HISTORICAL REMARK: The fourth quark (c) was postulated theoretically by Glashow-Iliopoulos-Maiani in 1970 (GIM mechanism). It was much heavier than the first three quarks (u, d, s) to have been observed experimentally, but was needed to explain the above experimental results on currents.

KOBAYASHI-MASKAWA MATRIX

If Nature can mix two generations, why not mix all three of them? Indeed, she did. The Lagrangian density is given in terms of

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = U \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{4.3.68}$$

where U is a unitary matrix (Kobayashi-Maskawa matrix - Nobel prize 2008). It contains additional (to θ_C) angles and more interestingly, it includes an unremovable *phase* which implies *CP violation*. I won't discuss this any further.

4.3.4 The Higgs

The Higgs field plays a fundamental role being solely responsible for the masses of all particles. Yet it hasn't been observed. Even though all parameters have been figured out, there is one parameter about which we haven't a clue: the mass of the Higgs particle. Why is this so? The Higgs potential may be written as

$$V = \frac{\lambda}{2}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 = \frac{1}{2}\lambda v^2 \phi'^2 + \text{interactions} \tag{4.3.69}$$

showing that the mass of the Higgs particle is

$$m_H = \sqrt{\lambda v^2} \tag{4.3.70}$$

We know v, but λ does not enter any of the observed parameters nor does it contribute to cross sections (except through small quantum corrections). Worse yet, quantum effects make infinite contributions to the mass of the Higgs particle (which may explain why no fundamental spin-0 particles have been observed in Nature). Various solutions have been proposed but Nature has yet to reveal her choice. The issue should be settled at the Large Hadron Collider (LHC). We all look forward to the date when it will start producing data.

4.4 The Standard Model

If we combine electroweak interactions based on $SU(2) \times U(1)_Y$ with strong interactions (QCD) based on $SU(3)$, we have the Standard Model of *all* fundamental interactions in Nature excluding gravity (Nobel prize to Glashow, Salam and Weinberg).

This model has received truely remarkable experimental confirmation. It is unsatisfactory, though, because it contains a large number of arbitrary parameters:

- gauge couplings, g, g' (or e, θ_W) and g_{QCD}

- fermion masses for leptons and quarks
- the Kobayashi-Maskawa matrix with 4 parameters and similarly for leptons
- the Higgs mass

One can only hope that one day one will be able to understand the origin of all these parameters.

4.5 Chiral anomalies

4.5.1 $\pi^0 \to \gamma\gamma$

The decay of the neutral pion (π^0) to two photons is an interesting process because it provides a direct measurement of the number of colors ($N = 3$) and gives us a glimpse of the *chiral anomalies*.
It is clear that classically the decay $\pi^0 \to \gamma\gamma$ is forbidden because π^0 is neutral. It is, however, allowed quantum mechanically because the pion couples to the axial current

$$J_A^{\mu 3} = \bar{\psi}\gamma^\mu\gamma_5 \frac{\sigma^3}{2}\psi = \frac{1}{2}\left(\bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d\right) \tag{4.5.1}$$

with an amplitude

$$\tilde{G}^\mu(p) = \langle 0|\tilde{J}_A^{\mu 3}(p)|\pi^0\rangle = -if_\pi p^\mu \quad , \quad f_\pi = 93 \text{ MeV} \tag{4.5.2}$$

The axial current consists of charged quarks which can couple to two photons via electromagnetic currents

$$J^\mu = Q_u e\bar{u}\gamma^\mu u + Q_d e\bar{d}\gamma^\mu d \tag{4.5.3}$$

where $Q_u = \frac{2}{3}$, $Q_d = -\frac{1}{3}$ with an amplitude

$$\begin{aligned} \tilde{G}_{\mu\nu\lambda}(p, p_1, p_2) &= \langle 0|\tilde{J}_A^{\mu 3}(p)\tilde{J}^\nu(p_1)\tilde{J}^\lambda(p_2)|0\rangle \\ &= \frac{1}{2}Ne^2(Q_u^2 - Q_d^2)\tilde{\mathcal{G}}_{\mu\nu\lambda}(p, p_1, p_2) \end{aligned} \tag{4.5.4}$$

where $\tilde{\mathcal{G}}$ is the fourier transform of

$$\mathcal{G}_{\mu\nu\lambda}(x, x', x'') = \langle 0|T(\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)\bar{\psi}(x')\gamma^\nu\psi(x')\bar{\psi}(x'')\gamma^\lambda\psi(x''))|0\rangle \tag{4.5.5}$$

and ψ is any massless Dirac field (the masses of the quarks may be ignored). The momenta p_1 and p_2 are those of the two photons the respective electromagnetic currents couple to.
The total amplitude for the decay $\pi^0(p) \to \gamma(p_1)\gamma(p_2)$ is proportional to

$$\tilde{G}^\mu(p)\tilde{G}_{\mu\nu\lambda} \sim p^\mu\tilde{\mathcal{G}}_{\mu\nu\lambda}(p, p_1, p_2) \tag{4.5.6}$$

Notice that it is proportional to N so a measurement of the decay rate provides a measurement of the number of colors ($N = 3$).

Also notice that it should vanish because of conservation of the axial current ($\partial_\mu J_A^\mu = 0$, if the quark masses vanish). However, it does not vanish due to quantum effects *(chiral anomaly)*.
Conservation of the electromagnetic current leads to two more constraints,

$$p_1^\nu \tilde{\mathcal{G}}_{\mu\nu\lambda}(p,p_1,p_2) = p_2^\lambda \tilde{\mathcal{G}}_{\mu\nu\lambda}(p,p_1,p_2) = 0 \tag{4.5.7}$$

Recall that this conservation law is a consequence of gauge invariance, so quantum effects had better not spoil it, otherwise gauge invariance will be violated (*cf.* with the violation of coservation of axial current which is an interesting but not earth shattering effect).
To calculate $\mathcal{G}_{\mu\nu\lambda}$, we may expand the Dirac field in creation and annihilation operators. After some algebra, we obtain

$$\mathcal{G}_{\mu\nu\lambda}(x,x',x'') = \mathrm{tr}[\gamma^\mu\gamma_5 S_F(x-x')\gamma^\nu S_F(x'-x'')\gamma^\lambda S_F(x''-x)] + (\nu, x' \leftrightarrow \lambda, x'') \tag{4.5.8}$$

Taking fourier transforms, i.e.,

$$\int d^4x d^4x' d^4x''\, e^{i(p\cdot x - p_1\cdot x' - p_2\cdot x'')}\mathcal{G}_{\mu\nu\lambda}(x,x',x'') = (2\pi)^4\delta^4(p+p_1+p_2)\tilde{\mathcal{G}}_{\mu\nu\lambda}(p,p_1,p_2) \tag{4.5.9}$$

we obtain

$$\tilde{\mathcal{G}}_{\mu\nu\lambda}(p,p_1,p_2) = \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[\gamma^\mu\gamma_5\frac{i\gamma\cdot(k+p_1)}{(k+p_1)^2}\gamma^\nu\frac{i\gamma\cdot k}{k^2}\gamma^\lambda\frac{i\gamma\cdot(k-p_2)}{(k-p_2)^2}\right] + (\nu,p_1 \leftrightarrow \lambda,p_2) \tag{4.5.10}$$

Taking divergence, we obtain

$$\begin{aligned} p_1^\nu\tilde{\mathcal{G}}_{\mu\nu\lambda} &= -i\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[\gamma^\mu\gamma_5\frac{\gamma\cdot k}{k^2}\gamma^\lambda\frac{\gamma\cdot(k-p_2)}{(k-p_2)^2} - \gamma^\mu\gamma_5\frac{\gamma\cdot(k+p_1)}{(k+p_1)^2}\gamma^\lambda\frac{\gamma\cdot(k-p_2)}{(k-p_2)^2}\right] \\ &\quad -i\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[\gamma^\mu\gamma_5\frac{\gamma\cdot(k+p_2)}{(k+p_2)^2}\gamma^\lambda\frac{\gamma\cdot(k-p_1)}{(k-p_1)^2} - \gamma^\mu\gamma_5\frac{\gamma\cdot(k+p_2)}{(k+p_2)^2}\gamma^\lambda\frac{\gamma\cdot k}{k^2}\right] \end{aligned} \tag{4.5.11}$$

This is seen to vanish if on the first line we let $k \to k + p_2$ in the 1st term and $k \to k + p_2 - p_1$ in the 2nd term. Unfortunately, the above argument fails, because individual terms are infinite (so we just argued $\infty - \infty = 0$). In fact, the answer is ambiguous ($\infty - \infty$ can be anything!). To exhibit the ambiguity, we shall shift $k \to k + A$ in the first line (A being arbitrary), switch to *Euclidean space* by rotating $k^0 \to ik^0$ and then regulate the integrals by including a factor e^{-k^2/Λ^2}. At the end of the day, we shall let $\Lambda \to \infty$ (so $e^{-k^2/\Lambda^2} \to 1$).
We arrive at

$$\begin{aligned} p_1^\nu\tilde{\mathcal{G}}_{\mu\nu\lambda} &= -4i\epsilon_{\mu\alpha\lambda\beta}\int\frac{d^4k}{(2\pi)^4}e^{-k^2/\Lambda^2}\left[\frac{(k+A)^\alpha(k+A-p_2)^\beta}{(k+A)^2(k+A-p_2)^2} - \frac{(k+A+p_1)^\alpha(k+A-p_2)^\beta}{(k+A+p_1)^2(k+A-p_2)^2}\right] \\ &\quad -4i\epsilon_{\mu\alpha\lambda\beta}\int\frac{d^4k}{(2\pi)^4}e^{-k^2/\Lambda^2}\left[\frac{(k+p_2)^\alpha(k-p_1)^\beta}{(k+p_2)^2(k-p_1)^2} - \frac{(k+p_2)^\alpha k^\beta}{(k+p_2)^2k^2}\right] \end{aligned} \tag{4.5.12}$$

where we used $\mathrm{tr}(\gamma_\mu\gamma_\alpha\gamma_\lambda\gamma_\beta\gamma_5) = 4i\epsilon_{\mu\alpha\lambda\beta}$.
If we now shift $k \to k - A + p_2$ in the 1st term, it does not cancel the last term; instead we obtain a factor

$$e^{-(k-A+p_2)^2/\Lambda^2} - e^{-k^2/\Lambda^2} = e^{-k^2/\Lambda^2}\left[\frac{2k\cdot(A-p_2)}{\Lambda^2} + \dots\right] \qquad (4.5.13)$$

UNIT 5

Spin 2 (gravity)

5.1 Curved spaces

5.1.1 The metric

So far we have been working in flat Minkowski spacetime where the distance between two events (proper time) was given by

$$d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{x}^2 \tag{5.1.1}$$

where $\eta_{\mu\nu}$ is the matrix (1.1.10). We made sure all inertial observers agreed on our results; they were connected via Lorentz transformations which left proper time invariant, or, equivalently, $\Lambda^T \eta \Lambda = \eta$ (eq. (1.1.11)).
As Einstein first realized, gravity creates curvature. To include gravity, we need to modify our assumption of flatness (Minkowski space) and work in a more general *curved* space. Proper time is then given by

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{5.1.2}$$

where $g_{\mu\nu}$ is a symmetric 4×4 matrix (the metric). It has to be non-singular so that g^{-1} exists everywhere. We have certain freedom in defining $g_{\mu\nu}$, because we may always change coordinates

$$x^\mu \to \bar{x}^\mu = x^\mu + \omega^\mu \tag{5.1.3}$$

where ω^μ is itself a function of x^μ. I shall always assume ω^μ is small and drop terms of second order in ω^μ. By the chain rule,

$$dx^\mu \to d\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} dx^\nu = (\delta^\mu_\nu + \partial_\nu \omega^\mu) dx^\nu \tag{5.1.4}$$

Since

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu \tag{5.1.5}$$

it follows that the metric in the new coordinates is given by

$$\bar{g}_{\mu\nu} = (\delta^\rho_\mu - \partial_\mu \omega^\rho)(\delta^\sigma_\nu - \partial_\nu \omega^\sigma) g_{\rho\sigma} \tag{5.1.6}$$

Locally, we may always choose coordinates so that

$$g_{\mu\nu} = \eta_{\mu\nu} + \dots \tag{5.1.7}$$

where the dots represent *second-order* corrections.
EXAMPLE 1: Consider the Euclidean plane on which the line element may be written as

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \tag{5.1.8}$$

in Cartesian and polar coordinates respectively. We have $g_{xx} = g_{yy} = 1$, $g_{xy} = 0$ and $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{r\theta} = 0$, in the two coordinate systems. In polar coordinates, $g_{\mu\nu}$ may look complicated, but it still describes a boring (flat) plane.
EXAMPLE 2: The line element on the surface of a unit sphere is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{5.1.9}$$

so $g_{\theta\theta} = 1$, $g_{\phi\phi} = \sin^2\theta$, $g_{\phi\theta} = 0$. This surface is not flat. Near the north pole ($\theta = 0$), we have $\sin^2\theta = \theta^2 - \frac{1}{3}\theta^4 + \dots$, so

$$ds^2 \approx d\theta^2 + \theta^2 d\phi^2 \tag{5.1.10}$$

which is a plane in polar coordinates (with θ playing the rôle of distance from the origin) and corrections are of second order.
EXAMPLE 3: Our Universe at *cosmic scales* looks almost flat (in *space*). Proper time is given by

$$d\tau^2 = dt^2 - a^2(t)d\vec{x}^2 \ \ , \ \ d\vec{x}^2 = dx^2 + dy^2 + dz^2 \tag{5.1.11}$$

so $g_{tt} = 1$, $g_{xx} = g_{yy} = g_{zz} = -a^2(t)$. If $a =$ const., this is just Minkowski spacetime. However, a increases with time giving our Universe spacetime (but *not spatial*) curvature. $a(t)$ is a cosmic scale factor and not a measurable distance between any two objects. Galaxies are at fixed $\vec{x}$ but appear to be flying apart, because the distance between them that one measures is

$$\Delta s = a(t)|\Delta\vec{x}| \tag{5.1.12}$$

which is increasing with time. As we go back in time, a decreases. It turns out (from the equations of Einstein's General Theory of Relativity) that $a \to 0$ in finite time, which implies that spacetime started as a singularity (the Big Bang).

5.1.2 Scalars, vectors, tensors

Scalars are invariant under coordinate transformations. E.g., the Klein-Gordon field

$$\phi(x^\mu) \to \phi(\bar{x}^\mu) \tag{5.1.13}$$

changes only through its argument ($x^\mu \to \bar{x}^\mu$).
A vector V^μ transforms in the same way as dx^μ (eq. (5.1.4)),

$$V^\mu \to (\delta^\mu_\nu + \partial_\nu\omega^\mu)V^\nu \tag{5.1.14}$$

EXAMPLE 1: The velocity

$$v^\mu = \frac{dx^\mu}{d\tau} \tag{5.1.15}$$

clearly transforms as dx^μ, since $d\tau$ is invariant.
EXAMPLE 2: The vector potential A^μ (a vector field).
We can take dot products of vectors. This produces a scalar, which is a physical quantity and can therefore be measured. In Minkowski space we defined for vectors A^μ, B^μ,

$$A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A} \cdot \vec{B} \tag{5.1.16}$$

This is generalized in curved space to

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu \tag{5.1.17}$$

That this is a scalar follows from the transformation properties of vectors (5.1.14) and the metric (5.1.6). For the velocity (5.1.15), we have

$$v^2 \equiv v \cdot v = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 1 \tag{5.1.18}$$

where we used (5.1.2), clearly a scalar.
We may write

$$A \cdot B = A_\mu B^\mu \ \ , \ \ A_\mu = g_{\mu\nu} A^\nu \tag{5.1.19}$$

in terms of the covariant vector A_μ which is obtained by *lowering the index* of the contravariant vector A^μ with $g_{\mu\nu}$. In Minkowski space, this was kind of trivial ($A_0 = A^0$, $B_i = -B^i$), but in curved space it isn't, for $g\mu\nu$ may be complicated. The transformation properties of V_μ is deduced from those of V^μ (eq. (5.1.14) and $g_{\mu\nu}$ (eq. (5.1.6)). We obtain

$$V_\mu \to (\delta_\mu^\nu - \partial_\mu \omega^\nu) V_\nu \tag{5.1.20}$$

Notice the minus sign - it was plus for a contravariant vector. Other than the sign, the form of the factor is similar with indices easily guessed at.
EXAMPLE: The gradient of a scalar,

$$\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu} \tag{5.1.21}$$

is a covariant vector. This follows from the chain rule,

$$\frac{\partial \phi}{\partial \bar{x}^\mu} = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \frac{\partial \phi}{\partial x^\nu} = (\delta_\mu^\nu - \partial_\mu \omega^\nu) \frac{\partial \phi}{\partial x^\nu} \tag{5.1.22}$$

where we used (5.1.3).
Multiplying two vectors, we produce a two-index object (tensor)

$$T^{\mu\nu} = A^\mu B^\nu \tag{5.1.23}$$

whose transformation properties follow from those of the vectors (5.1.14),

$$T^{\mu\nu} \to (\delta_\rho^\mu + \partial_\rho \omega^\mu)(\delta_\sigma^\nu + \partial_\sigma \omega^\nu) T^{\rho\sigma} \tag{5.1.24}$$

Any object which transforms in this manner is a *tensor* with two indices upstairs. Similarly, $T_{\mu\nu} = A_\mu B_\nu$ is a tensor but with two indices *downstairs*. Generalizing, the object

$$T^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n} \tag{5.1.25}$$

is a tensor if it transforms in the same manner as

$$A_1^{\mu_1} \dots A_m^{\mu_m} B_{1\nu_1} \dots B_{n\nu_n} \tag{5.1.26}$$

Notice that each index transforms independently of the rest - an upstairs index transforms contravariantly (eq. (5.1.14)) whereas a downstairs index transforms covariantly (eq. (5.1.20)).
It is also convenient to define $g^{\mu\nu}$ as the inverse matrix g^{-1},

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \tag{5.1.27}$$

i.e., $g^{-1}g = I$. It transforms as a tensor. From (5.1.19), we deduce

$$A^\mu = g^{\mu\nu} A_\nu \tag{5.1.28}$$

i.e., $g^{\mu\nu}$ *raises an index*.
Another special tensor is δ^ν_μ whose components do *not* change under coordinate transformations. It is the only tensor with this property.

5.1.3 Derivatives

We have already established (eq. (5.1.22)) that the gradient of a scalar, $\partial_\mu \phi$ is a covariant vector (with an index downstairs). What about the gradient of a vector, $\partial_\mu V^\alpha$ - is it a tensor? Let's transform it to see. Using the chain rule and (5.1.14), we obtain

$$\begin{aligned} \partial_\mu V^\alpha &\to (\delta^\nu_\mu - \partial_\mu \omega^\nu)\partial_\nu \{(\delta^\alpha_\beta + \partial_\beta \omega^\alpha) V^\beta\} \\ &= (\delta^\nu_\mu - \partial_\mu \omega^\nu)(\delta^\alpha_\beta + \partial_\beta \omega^\alpha)\partial_\nu V^\beta + (\partial_\mu \partial_\beta \omega^\alpha) V^\beta \end{aligned} \tag{5.1.29}$$

If $\partial_\mu V^\alpha$ were a tensor, the last term would be absent. We can correct this by replacing the partial derivative ∂_μ with the covariant derivative ∇_μ. Its action on a scalar should coincide with the partial derivative,

$$\nabla_\mu \phi = \partial_\mu \phi \tag{5.1.30}$$

since $\partial_\mu \phi$ transforms as a vector. On a vector, we define[1]

$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta} V^\beta \tag{5.1.31}$$

The $\Gamma^\alpha_{\mu\beta}$ are the Christoffel symbols (connection coefficients). They should *not* form a tensor, if we want $\nabla_\mu V^\alpha$ to be one. If they formed a tensor, then they would transform as

$$\Gamma^\alpha_{\mu\beta} \to \bar{\Gamma}^\alpha_{\mu\beta} = (\delta^\alpha_\gamma + \partial_\gamma \omega^\alpha)(\delta^\nu_\mu - \partial_\mu \omega^\nu)(\delta^\lambda_\beta - \partial_\beta \omega^\lambda)\Gamma^\gamma_{\nu\lambda} \tag{5.1.32}$$

[1] Other notation often used: $V^\alpha{}_{,\mu} = \partial_\mu V^\alpha$, $V^\alpha{}_{;\mu} = \nabla_\mu V^\alpha$.

To take care of the unwanted piece in (5.1.29), we demand that the connection coefficients transform as

$$\Gamma^\alpha_{\mu\beta} \to \bar{\Gamma}^\alpha_{\mu\beta} - \partial_\mu \partial_\beta \omega^\alpha \tag{5.1.33}$$

Combining (5.1.29) and (5.1.33), we deduce

$$\nabla_\mu V^\alpha \to (\delta^\nu_\mu - \partial_\mu \omega^\nu)(\delta^\alpha_\beta + \partial_\beta \omega^\alpha)\nabla_\nu V^\beta \tag{5.1.34}$$

establishing that $\nabla_\mu V^\alpha$ is indeed a tensor.
An explicit expression for the connection coefficients is given by

$$\Gamma^\alpha_{\mu\beta} = \tfrac{1}{2} g^{\alpha\lambda}(\partial_\mu g_{\lambda\beta} + \partial_\beta g_{\lambda\mu} - \partial_\lambda g_{\mu\beta}) \tag{5.1.35}$$

Thusly defined, they are symmetric (*torsionless*),

$$\Gamma^\alpha_{\mu\beta} = \Gamma^\alpha_{\beta\mu} \tag{5.1.36}$$

This definition is by no means unique, but no physical insight is gained by a different definition (which is usually lacking). It follows from the transformation properties of the metric tensor that the connection coefficients defined by (5.1.35) transform as desired (eq. (5.1.33)).
The covariant derivative on a product of vectors (tensor) is obtained by the product rule,

$$\begin{aligned} \nabla_\mu (A^\alpha B^\beta) &= (\nabla_\mu A^\alpha) B^\beta + A^\alpha \nabla_\mu B^\beta \\ &= \partial_\mu (A^\alpha B^\beta) + \Gamma^\alpha_{\mu\gamma} A^\gamma B^\beta + \Gamma^\beta_{\mu\gamma} A^\alpha B^\gamma \end{aligned} \tag{5.1.37}$$

Similarly, for any tensor with two indices upstairs, we have

$$\nabla_\mu T^{\alpha\beta} = \partial_\mu T^{\alpha\beta} + \Gamma^\alpha_{\mu\gamma} T^{\gamma\beta} + \Gamma^\beta_{\mu\gamma} T^{\alpha\gamma} \tag{5.1.38}$$

To define the action of ∇_μ on a covariant vector, note that the inner product is a scalar on which ∇_μ coincides with ∂_μ,

$$\nabla_\mu (A_\alpha B^\alpha) = \partial_\mu (A_\alpha B^\alpha) \tag{5.1.39}$$

Using the product rule, we deduce after some rearrangements,

$$(\nabla_\mu A_\alpha) B^\alpha + \Gamma^\beta_{\mu\alpha} A_\beta B^\alpha = (\partial_\mu A_\alpha) B^\alpha \tag{5.1.40}$$

Since B^α is an arbitrary vector, it follows that

$$\nabla_\mu A_\alpha = \partial_\mu A_\alpha - \Gamma^\beta_{\mu\alpha} A_\beta \tag{5.1.41}$$

This is similar to the action (5.1.31) on a contravariant vector with an obvious placement of indices but with a minus sign. Generalizing to an arbitrary tensor (5.1.25) is now straightforward: ∇_μ acts on (5.1.25) as it does on the product of vectors (5.1.26) using (5.1.31), (5.1.41) and the product rule.
Special tensors:

- δ^ν_μ. It is the only tensor that satisfies

$$\partial_\mu \delta^\beta_\alpha = \nabla_\mu \delta^\beta_\alpha = 0 \tag{5.1.42}$$

- $g_{\mu\nu}$ is also *covariantly* constant,

$$\nabla_\mu g_{\alpha\beta} = 0 \tag{5.1.43}$$

and so is its inverse, $\nabla_\mu g^{\alpha\beta} = 0$.

5.1.4 Curvature

Curvature is best captured by going around a small square with sides along two coordinate axes, x^μ and x^ν, say, holding a vector V_α. Along each side, the change is given by the corrsponding covariant derivative of v_α, so the total change around the square is proportional to the second derivative. More precisely, it is given by the commutator $[\nabla_\mu, \nabla_\nu]$ acting on the vector, which vanishes in flat space. We define the curvature (Riemann) tensor by

$$[\nabla_\mu, \nabla_\nu] V_\alpha = -R^\beta{}_{\alpha\mu\nu} V_\beta \tag{5.1.44}$$

Using the definition of a covariant derivative, after some algebra, we arrive at an explicit expression in terms of the connection coefficients,

$$R^\beta{}_{\alpha\mu\nu} = \partial_\mu \Gamma^\beta_{\nu\alpha} - \partial_\nu \Gamma^\beta_{\mu\alpha} + \Gamma^\beta_{\mu\gamma}\Gamma^\gamma_{\nu\alpha} - \Gamma^\beta_{\nu\gamma}\Gamma^\gamma_{\mu\alpha} \tag{5.1.45}$$

To see the analogy with gauge theories, let us introduce the notation

$$(\mathbb{A}_\mu)^\beta_\alpha = i\Gamma^\beta_{\mu\alpha} \ , \ (\mathbb{F}_{\mu\nu})^\beta_\alpha = iR^\beta{}_{\alpha\mu\nu} \tag{5.1.46}$$

where $\mathbb{A}_\mu$ and $\mathbb{F}_{\mu\nu}$ are matrices. The action of the covariant derivative (5.1.41) on the covariant vector V_α may be written as

$$\nabla_\mu V = \partial_\mu V + i\mathbb{A}_\mu V \tag{5.1.47}$$

which is of the same form as the *gauge derivative*. The definition of the Riemann tensor (5.1.44) reads

$$[\nabla_\mu, \nabla_\nu] V = i\mathbb{F}_{\mu\nu} V \tag{5.1.48}$$

which shows that $\mathbb{F}$ is a “field strength” (in electromagnetism it led to the Aharonov-Bohm effect by selecting spatial directions and integrating over a finite surface). Finally the explicit expression (5.1.45) becomes

$$\mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu - i[\mathbb{A}_\mu, \mathbb{A}_\nu] \tag{5.1.49}$$

which is similar to the expression of the field strength in terms of the vector potential in a gauge theory.
Another useful result which applies directly to both gravity and gauge theories is the Bianchi identity,

$$[\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + [\nabla_\nu, [\nabla_\rho, \nabla_\mu]] + [\nabla_\rho, [\nabla_\mu, \nabla_\nu]] = 0 \tag{5.1.50}$$

which is obvious for any three objects, not just ∇_μ. It leads to

$$\nabla_\mu R^\beta{}_{\alpha\nu\rho} + \nabla_\nu R^\beta{}_{\alpha\rho\mu} + \nabla_\rho R^\beta{}_{\alpha\mu\nu} = 0 \tag{5.1.51}$$

which may also be written as

$$\nabla_\mu \mathbb{F}_{\nu\rho} + \nabla_\nu \mathbb{F}_{\rho\mu} + \nabla_\rho \mathbb{F}_{\mu\nu} = 0 \tag{5.1.52}$$

The latter is of the same form as the *homogeneous Maxwell equations* in a gauge theory (derived similarly by replacing $\nabla_\mu \to D_\mu$). In electromagnetism, they are: $\vec{\nabla} \cdot \vec{B} = 0$ and Faraday's Law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.
By contracting indices, we construct the Ricci tensor

$$R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta} \tag{5.1.53}$$

which is a symmetric tensor, and the Ricci scalar,

$$R = R^\alpha{}_\alpha = g^{\alpha\beta} R_{\alpha\beta} \tag{5.1.54}$$

EXAMPLE: For a two-dimensional sphere of radius a,

$$R = \frac{2}{a^2} \tag{5.1.55}$$

These curvature tensors have a lot of interesting properties, but I shall resist the temptation to dwell upon them, because we ought to get on with our program: field theory.

5.2 Fields in curved spacetime

5.2.1 Scalars

We wish to generalize to curved spacetime the action (1.2.30) with Lagrangian density $\mathcal{L}$ given by (1.2.32) whose variation led to the Klein-Gordon equation (1.2.33). Since $\partial_\mu \phi$ is a vector even in curved spacetime, $\mathcal{L}$ is still a scalar. On the other hand, the measure d^4x in the action is not invariant under coordinate transformations. This is true even in flat space. Unless we use cartesian coordinates, we ought to include a *Jacobian* in the definition of the volume element. To find a general expression, recall that under a coordinate transformation, dx^μ transforms as in (5.1.4). Therefore, d^4x gets multiplied by the Jacobian

$$\mathcal{J} = \det(\delta_\mu^\nu + \partial_\mu \omega^\nu) \tag{5.2.1}$$

Comparing with the transformation of the metric tensor (5.1.6), which yields

$$\det g \to [\det(\delta_\mu^\nu - \partial_\mu \omega^\nu)]^2 \det g = \mathcal{J}^{-2} \det g \tag{5.2.2}$$

we immediately deduce that

$$d^4x \sqrt{|\det g|} \tag{5.2.3}$$

is invariant under coordinate transformations. In spacetime, $\det g < 0$, so $|\det g| = -\det g$. We shall keep the same definition for the action (1.2.30) as in flat spacetime (in terms of a non-scalar measure) and absorb the Jacobian into $\mathcal{L}$ instead. The new $\mathcal{L}$ is no longer a scalar. In curved spacetime, there is one more term we can add which is a scalar quadratic in ϕ and has two derivatives: $R\phi^2$, where R is the Ricci scalar (5.1.54). Adding it and including the Jacobian, we obtain the general form of the Lagrangian density for a scalar field in curved spacetime,

$$\mathcal{L} = \frac{1}{2}\sqrt{|\det g|}\left\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right\} \tag{5.2.4}$$

in terms of input parameters m and ξ. Applying the field equation (1.2.31), we obtain the generalized Klein-Gordon equation

$$\frac{1}{\sqrt{|\det g|}}\partial_\mu\left(\sqrt{|\det g|}g^{\mu\nu}\partial_\nu\phi\right) + m^2\phi + \xi R\phi = 0 \tag{5.2.5}$$

In the presence of $g_{\mu\nu}$, there is no translational invariance and the Hamiltonian (total energy) is not conserved, in general. We can no longer obtain the stress-energy tensor as a Noether current as in (1.5.16). Fortunately, we now have a new cool way of calculating $T_{\mu\nu}$ (which will, of course, not be conserved in general). As Einstein taught us, matter creates curvature, so $g^{\mu\nu}$ couples to $T_{\mu\nu}$. Just like in electromagnetism we defined the electric current by varying the vector potential in the Lagrangian (eq. (3.2.46)), we shall define $T_{\mu\nu}$ by varying $g^{\mu\nu}$ in (5.2.4),

$$T_{\mu\nu} = \frac{2}{\sqrt{|\det g|}}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} \tag{5.2.6}$$

The $1/\sqrt{|\det g|}$ factor is needed because $\mathcal{L}$ is not a scalar due to the inclusion of $\sqrt{|\det g|}$ in its definition. The numerical factor of 2 can be obtained by comparing with earlier flat-space results. Eq. (5.2.6) gives a superior method of calculating $T_{\mu\nu}$ even in flat space. It always gives a physical answer. Recall that this was not the case with the Noether current calculation of $T_{\mu\nu}$ in electromagnetism which gave an expression that was not gauge-invariant (and therefore not physical).
Let us calculate the stress-energy tensor in flat space. To use (5.2.6), let

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \ , \ |h^{\mu\nu}| \ll 1 \tag{5.2.7}$$

We may drop terms which are of higher order than 1 in $h^{\mu\nu}$. We have

$$g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + \dots \ , \ \sqrt{|\det g|} = 1 - \tfrac{1}{2}h^\mu_\mu + \dots \tag{5.2.8}$$

where indices are raised and lowered by η. Also, the connection coefficients (5.1.35) are

$$\Gamma^\alpha_{\mu\beta} = -\tfrac{1}{2}(\partial_\mu h^\alpha_\beta + \partial_\beta h^\alpha_\mu - \partial^\alpha h_{\mu\beta}) + \dots \tag{5.2.9}$$

leading to the Ricci scalar

$$\begin{aligned} R &= g^{\mu\nu}R^\alpha{}_{\mu\alpha\nu} \\ &= \eta^{\mu\nu}(\partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha}) + \dots \\ &= \partial_\alpha\partial_\mu h^{\alpha\mu} - \partial_\alpha\partial^\alpha h^\mu_\mu + \dots \end{aligned} \tag{5.2.10}$$

The Lagrangian density (5.2.4) becomes

$$\mathcal{L} = \tfrac{1}{2}\left\{(1-\tfrac{1}{2}h^\mu_\mu)(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2) + h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \xi(\partial_\alpha\partial_\mu h^{\alpha\mu} - \partial_\alpha\partial^\alpha h^\mu_\mu)\phi^2\right\} + \dots \tag{5.2.11}$$

The zeroth-order term is the Klein-Gordon Lagrangian density in flat space (1.2.32). The first-order terms contribute to the derivative in (5.2.6). We obtain

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}\eta_{\mu\nu}(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2) - \xi(\partial_\mu\partial_\nu\phi^2 - \eta_{\mu\nu}\partial_\alpha\partial^\alpha\phi^2) \\ &= (1-2\xi)\partial_\mu\phi\partial_\nu\phi + (2\xi - \tfrac{1}{2})\eta_{\mu\nu}(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2) - 2\xi\phi\partial_\mu\partial_\nu\phi \end{aligned} \tag{5.2.12}$$

where in the second step I used the Klein-Gordon equation (1.2.33). Notice that ξ survived the flat-space limit! If $\xi = 0$, then we recover the Noether current result. For an arbitrary ξ, the stress-energy tensor (5.2.12) is conserved,

$$\partial^\mu T_{\mu\nu} = 0 \tag{5.2.13}$$

Their origin is evident when we place the system in a curved background. It is less straightforward to derive these expressions working in flat space. An interesting quantity is the trace, which is a scalar. Using the Klein-Gordon equation, we obtain from (5.2.12)

$$T^\mu_\mu = (6\xi - 1)\partial_\mu\phi\partial^\mu\phi - 2(3\xi - 1)m^2\phi^2 \tag{5.2.14}$$

This vanishes in the special case

$$\xi = \frac{1}{6} \;,\quad m = 0 \tag{5.2.15}$$

Since $T^\mu_\mu = 0$, there is *no scale* in the system and the theory has conformal symmetry. The stress-energy tensor (5.2.12) is given by

$$T_{\mu\nu} = \frac{2}{3}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}\eta_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi - \frac{1}{3}\phi\partial_\mu\partial_\nu\phi \tag{5.2.16}$$

Sometimes this is referred to as the new improved stress-energy tensor.

5.2.2 The Casimir effect

We shall now calculate a vacuum effect that has actually been confirmed experimentally. The vacuum energy is part of the vacuum expectation value of the stress-energy tensor,

$$\langle 0|T_{\mu\nu}|0\rangle \tag{5.2.17}$$

since T_{00} is the energy density. Let us first calculate it in flat space with $\xi = 0$. We already know the answer, because this is the Klein-Gordon field we have already studied in detail. We shall do the calculation in a slightly different way with an eye toward generalization. Since $T_{\mu\nu}$ is quadratic in ϕ, its expectation value involves expressions of the form $\langle\phi(x)\phi(x)\rangle$, together with various derivatives. These are singular expressions involving fields at coincident points. This is the reason why we obtained a divergent result for the energy density before (eq. (1.3.7)). To render $\langle T_{\mu\nu}\rangle$ finite, we shall separate the arguments of the fields, thus replacing

$$\langle\phi(x)\phi(x)\rangle \to \langle\phi(x)\phi(y)\rangle \tag{5.2.18}$$

with the understanding that at the end of the day we ought to take the limit $y \to x$. Thus, in flat Minkowski space with $\xi = 0$, we may write

$$\langle 0|T_{\mu\nu}|0\rangle = \left\{-\partial_\mu\partial_\nu + \tfrac{1}{2}\eta_{\mu\nu}(\partial_\alpha\partial^\alpha + m^2)\right\} D(x-y) \tag{5.2.19}$$

where $D(x-y)$ is the propagator (1.4.4) which is only a function of the distance $(x-y)^\mu$ by translational invariance. It is a solution of the Klein-Gordon equation,

$$(\partial_\mu\partial^\mu + m^2)D(x-y) = 0 \tag{5.2.20}$$

Only positive-energy solutions contribute to this propagator (which fixes the boundary conditions).
For the energy density, we obtain from (5.2.19), using the integral representation of the propagator (1.4.4),

$$\langle 0|T_{00}|0\rangle = \int \frac{d^3k}{(2\pi)^3}\ \tfrac{1}{2}\omega_k\ e^{-k\cdot(x-y)} \tag{5.2.21}$$

Letting $y \to x$, we recover our earlier result (1.3.7).
Now suppose that ϕ is confined in the 3-direction (z-axis) between two infinite planes at $x^3 = 0, L$, where it has to vanish (Dirichlet boundary conditions). Then, instead of $e^{ik_3x^3}$ in the wavefunction, we ought to choose

$$\sin(k_3x^3)\ ,\ \ k_3 = \frac{n\pi}{L}\ \ (n \in \mathbb{Z}) \tag{5.2.22}$$

The calculation of the propagator changes. Since there is no translational invariance in the 3-direction, the propagator is not a function of the distance only. We shall compute it in the massless case ($m = 0$). In this case, the Minkowski propagator (1.4.4) is

$$D(x-y) = -\frac{1}{4\pi^2(x-y)^2}\ ,\ \ (x-y)^2 = (x^0-y^0-i\epsilon)^2 - (\vec{x}-\vec{y})^2 \tag{5.2.23}$$

where $\epsilon > 0$ for the integral (1.4.4) to exist. If we were in a four-dimensional *Euclidean* space, this propagator would be the electrostatic potential at y^μ due to a unit charge at x^μ. To change the boundary conditions and introduce the two surfaces at $x^3 = 0, L$, we may follow what we already know from electrostatics: use the *method of images*.
We need two infinite series of image charges of alternating signs. The new potential (propagator) is

$$D(x,y) = -\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\left\{\frac{1}{(x-y-2nL\hat{e}_3)^2} - \frac{1}{(x-y-2(nL-y^3)\hat{e}_3)^2}\right\} \tag{5.2.24}$$

where $\hat{e}_3$ is a unit vector in the 3-direction. This propagator satisfies the massless Klein-Gordon equation, for the same reason $D(x-y)$ (eq. (5.2.23)) did. It also satisfies the boundary conditions,

$$D(x,y) = 0\ ,\ \ x^3 = 0, L\ \text{ or }\ y^3 = 0, L \tag{5.2.25}$$

The vacuum energy density can be written as

$$\langle 0|T_{00}|0\rangle = \tfrac{1}{2}\left\{\partial_0^x\partial_0^y + \vec{\nabla}^x\cdot\vec{\nabla}^y\right\} D(x,y) \tag{5.2.26}$$

to be compared with the infinite space expression (5.2.19) with $\mu = \nu = 0$ and $m = 0$ (to which it reduces in the limit $L \to \infty$). To calculate this, use

$$\partial_0^x \partial_0^y \frac{1}{\mathcal{D}_a} = \frac{2}{\mathcal{D}_a^2} + \dots \quad , \quad \partial_i^x \partial_i^y \frac{1}{\mathcal{D}_a} = -\frac{2}{\mathcal{D}_a^2} + \dots \quad (i = 1, 2) \tag{5.2.27}$$

where $\mathcal{D}_a = (x - y - 2a\hat{e}_3)^2$ with $a = nL, nL - y^3$, and

$$\begin{aligned} \partial_3^x \partial_3^y \frac{1}{\mathcal{D}_{nL}} &= -\frac{2}{\mathcal{D}_{nL}^2} - \frac{8(x^3 - y^3 - 2nL)^2}{\mathcal{D}_{nL}^3} + \dots \\ \partial_3^x \partial_3^y \frac{1}{\mathcal{D}_{nL-y^3}} &= \frac{2}{\mathcal{D}_{nL-y^3}^2} + \frac{8(x^3 + y^3 - 2nL)^2}{\mathcal{D}_{nL-y^3}^3} + \dots \end{aligned} \tag{5.2.28}$$

where I omitted terms that vanish as $y \to x$. Putting everything together, we obtain

$$\begin{aligned} \langle 0|T_{00}|0\rangle = -\frac{1}{8\pi^2} \sum_{n=-\infty}^{\infty} & \left\{ \frac{2(1-3)}{\mathcal{D}_{nL}^2} - \frac{8(x^3 - y^3 - 2nL)^2}{\mathcal{D}_{nL}^3} \right. \\ & \left. - \frac{2(1-2+1)}{\mathcal{D}_{nL-y^3}^2} - \frac{8(x^3 + y^3 - 2nL)^2}{\mathcal{D}_{nL-y^3}^3} \right\} \end{aligned} \tag{5.2.29}$$

This is an infinite quantity, but the infinity is entirely due to the contribution of the original charge at x^μ (first two terms in series with $n = 0$). All the other terms which are due to the image charges are finite as $y \to x$. After subtracting the contribution of the original charge, i.e., the energy density (5.2.21) of the infinite space, and letting $y \to x$, we obtain the *normal-ordered* density

$$\langle 0| : T_{00} : |0\rangle = -\frac{1}{32\pi^2} \sum_{n \neq 0} \frac{1}{(nL)^4} - \frac{1}{16\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(x^3 - nL)^4} \tag{5.2.30}$$

which is a physical quantity that can be measured. It is finite in the interior, but diverges as one approaches the boundaries ($x^3 \to 0, L$). The divergence is due to the $n = 0, 1$ terms in the second series. We obtain

$$\langle 0| : T_{00} : |0\rangle \sim -\frac{1}{16\pi^2 (x^3)^4} \quad \text{as} \quad x^3 \to 0 \tag{5.2.31}$$

and similarly for $x^3 \to L$.

Our subtracting of infinity in the interior did not guarantee finiteness at the boundary. We need to work harder to achieve that. Instead, we turn to a case where no extra work is needed: $\xi = \frac{1}{6}$ which possesses *conformal symmetry* ($T_\mu^\mu = 0$). Using the *new improved stress-energy tensor* (5.2.16), instead of (5.2.26), we have

$$\langle 0|T_{00}|0\rangle = \tfrac{1}{6} \left\{ 5\partial_0^x \partial_0^y + \vec{\nabla}^x \cdot \vec{\nabla}^y \right\} D(x, y) \tag{5.2.32}$$

Working as before, this leads to

$$\begin{aligned} \langle 0|T_{00}|0\rangle = -\frac{1}{24\pi^2} \sum_{n=-\infty}^{\infty} & \left\{ \frac{2(5-3)}{\mathcal{D}_{nL}^2} - \frac{8(x^3 - y^3 - 2nL)^2}{\mathcal{D}_{nL}^3} \right. \\ & \left. - \frac{2(5-2+1)}{\mathcal{D}_{nL-y^3}^2} - \frac{8(x^3 + y^3 - 2nL)^2}{\mathcal{D}_{nL-y^3}^3} \right\} \end{aligned} \tag{5.2.33}$$

instead of (5.2.29). This time, the last two terms in the series cancel each other. After normal-ordering and taking the limit $y \to x$, we obtain instead of (5.2.30),

$$\langle 0| : T_{00} : |0\rangle = -\frac{1}{24\pi^2} \sum_{n\neq 0} \frac{12}{(2nL)^4} = -\frac{\pi^2}{1440 L^4} \tag{5.2.34}$$

where I used

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{5.2.35}$$

This vacuum energy density is finite and constant everywhere.
The conformal field mimics the electromagnetic vector potential which also has traceless stress-energy tensor (no scale). The latter has two degrees of freedom, so the vacuum electromagnetic energy density should be *double* the expression (5.2.34),

$$\rho_{em} = -\frac{\pi^2}{720 L^4} \tag{5.2.36}$$

This has actually been measured. It is a *negative* energy density showing that there is an *attractive* force between the two plates at $x^3 = 0, L$ (one *gains* energy by bringing them from infinity). To find the force, let each plate have (large) area A (with negligible edge effects). The total energy is

$$E_{em} = \rho_{em}(AL) = -\frac{\pi^2 A}{720 L^3} \tag{5.2.37}$$

The force is

$$F = -\frac{dE_{em}}{dL} = -\frac{\pi^2 A}{240 L^4} \tag{5.2.38}$$

In everyday units, the force per unit area is

$$\frac{F}{A} = -\frac{\pi^2 \hbar c}{240 L^4} \tag{5.2.39}$$

a tiny but finite *vacuum* effect which has been observed experimentally.

5.2.3 A toy Universe

The metric in our Universe looks like (5.1.11). To simplify the calculation, we shall get rid of two spatial coordinates. Similar results are obtained if we reinstate them, but the calculation becomes considerably more involved. The metric (5.1.11) simplifies to

$$d\tau^2 = d\mathbf{T}^2 - a^2(\mathbf{T}) dx^2 \tag{5.2.40}$$

where $\mathbf{T}$ is the time coordinate (measured by our clocks) and x is the single spatial coordinate. I used $\mathbf{T}$ for our time because I wanted to reserve t for a different time variable which is physically important (even though it is not what our clocks read) defined by

$$\frac{d\mathbf{T}}{dt} = a(\mathbf{T}) \quad \Rightarrow \quad t = \int \frac{d\mathbf{T}}{a(\mathbf{T})} \tag{5.2.41}$$

The metric in terms of t reads

$$d\tau^2 = a^2(\mathbf{T})\{dt^2 - dx^2\} \tag{5.2.42}$$

therefore

$$g_{\mu\nu} = a^2\eta_{\mu\nu} \ , \ \sqrt{|\det g|} = a^2 \tag{5.2.43}$$

The wave equation (5.2.5) with $\xi = 0$ (another simplification) reads

$$\partial_t^2\phi - \partial_x^2\phi + a^2m^2\phi = 0 \tag{5.2.44}$$

With $a = 1$, we would obtain the standard Minkowski spacetime positive-energy solutions (1.2.34),

$$u_k = \frac{1}{\sqrt{2\omega_k}}e^{-i(\omega_k t - kx)} \ , \ \omega_k = \sqrt{k^2 + m^2} \tag{5.2.45}$$

where I included a normalization factor for convenience. The general solution may be expanded as in (1.2.39), keeping in mind that we have one and not three spatial dimensions,

$$\phi(x,t) = \int \frac{dk}{2\pi}\left\{a(k)u_k(x,t) + a^\dagger(k)u_k^*(x,t)\right\} \tag{5.2.46}$$

With a time-varying a, plane waves are no longer solutions. Separating variables,

$$\phi(x,t) = e^{ikx}\mathcal{T}(t) \tag{5.2.47}$$

we obtain

$$\ddot{\mathcal{T}} + (k^2 + a^2m^2)\mathcal{T} = 0 \tag{5.2.48}$$

For an explicit calculation, choose

$$a^2 = 1 + \frac{\lambda}{1 + e^{-2Ht}} \qquad (\lambda > 0) \tag{5.2.49}$$

This is an increasing function of t. The parameter H represents the rate of expansion, like the *Hubble parameter* in our Universe. Let us choose units so that

$$H = 1 \tag{5.2.50}$$

In the infinite past ($t \to -\infty$), $a^2 \to 1$, so we have Minkowski space and $t \approx \mathbf{T}$ (our time).
In the infinite future ($t \to +\infty$), $a^2 \to 1 + \lambda$, so the Universe is Minkowski again,

$$d\tau^2 \approx d\bar{t}^2 - d\bar{x}^2 \ , \ \bar{t} = \mathbf{T} \approx \sqrt{1+\lambda}t \ , \ \bar{x} \approx \sqrt{1+\lambda}x \tag{5.2.51}$$

but different from the Minkowski space of the infinite past due to scaling: Galaxies at $x =$const. have moved apart. The wave equation admits plane-wave positive-energy solutions

$$\bar{u}_k = \frac{1}{\sqrt{2\bar{\omega}_k}}e^{-i(\bar{\omega}_k t - kx)} \ , \ \bar{\omega}_k = \sqrt{k^2 + (1+\lambda)m^2} \tag{5.2.52}$$

The general solution may be expanded as

$$\phi(x,t) = \int \frac{dk}{2\pi} \left\{ b(k)\bar{u}_k(x,t) + b^\dagger(k)\bar{u}_k^*(x,t) \right\} \tag{5.2.53}$$

To relate this expansion to the expansion (5.2.46) in the infinite past, we need to solve the wave equation for arbitrary t. The solution to (5.2.48) may be written in terms of a *hypergeometric function* as

$$\mathcal{T}(t) = \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t}(1+e^{2t})^{-i\omega_-} F(1+i\omega_-; i\omega_-; 1-i\omega_k; 1/(1+e^{-2t})) \tag{5.2.54}$$

where $\omega_\pm = \frac{1}{2}(\bar{\omega}_k \pm \omega_k)$. Another linearly independent solution of (5.2.48) is the complex conjugate of (5.2.54). The normalization constant is chosen so that in the infinite past,

$$e^{ikx}\mathcal{T}(t) \approx u_k(x,t) \qquad (t \to -\infty) \tag{5.2.55}$$

where u_k is given by (5.2.45) and we used $F(0) = 1$.[2]
To find what happens in the infinite future, we need the hypergeometric identity

$$\begin{aligned} F(a,b;c;z) \;=\; & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z) \\ + \; & \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F(c-a,c-b;c-a-b+1;1-z) \end{aligned}$$

In our case, $z = 1/(1+e^{-2t})$, so $1 - z = 1/(1+e^{2t}) \to 0$ as $t \to +\infty$. For $z \approx 1$, we have

$$F(a,b;c;z) \approx \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b}$$

so as $t \to +\infty$,

$$e^{ikx}\mathcal{T}(t) \approx \alpha(k)\bar{u}_k + \beta(k)\bar{u}_k^* \tag{5.2.56}$$

where $\bar{u}_k$ is given by (5.2.52) and the coefficients are

$$\alpha(k) = \sqrt{\frac{\bar{\omega}_k}{\omega_k}} \frac{\Gamma(1-i\omega_k)\Gamma(-i\bar{\omega}_k)}{\Gamma(1-i\omega_+)\Gamma(-i\omega_+)} \;,\; \beta(k) = \sqrt{\frac{\bar{\omega}_k}{\omega_k}} \frac{\Gamma(1-i\omega_k)\Gamma(-i\bar{\omega}_k)}{\Gamma(1-i\omega_-)\Gamma(-i\omega_-)} \tag{5.2.57}$$

It is disturbing that $\beta \neq 0$ for it implies that a positive-energy solution in the infinite past evolves into a mixture of positive- and negative-energy solutions in the future! [3] In Minkowski space such a mixture is not possible: a Lorentz transformation preserves the sign of the energy k^0 and maps a creation operator to a creation operator (eq. (1.3.40)). In our Universe, it seems that what looks like a creation operator for one observer may

[2]The hypergeometric function $F(a,b;c;z)$ satisfies the second-order equation

$$z(1-z)F'' + [c - (a+b+1)z]F' - abF = 0$$

and is regular at $z = 0$, so one sets $F(0) = 1$.

[3]This is, of course, not the case if there is no expansion. Then $\omega_- = 0$ and $\beta(k) \propto 1/\Gamma(0) = 0$.

be a mixture of a creation and an annihilation operator for another. Two observers may not agree on the very concept of a *particle*.
Using the results (5.2.55) and (5.2.56) and their complex conjugates, we deduce that the field (5.2.46) in the infinite past evolves into (5.2.53) in the infinite future, where

$$b(k) = \alpha(k)a(k) + \beta^*(k)\alpha^\dagger(k) \ , \ b^\dagger(k) = \beta(k)a(k) + \alpha^*(k)a^\dagger(k) \qquad (5.2.58)$$

This is a Bogolubov transformation and α, β are the Bogolubov coefficients.
If a and $a^\dagger$ satisfy the commutation relations

$$[a(k), a^\dagger(k')] = 2\pi\delta(k - k') \qquad (5.2.59)$$

then we have

$$[b(k), b^\dagger(k')] = 2\pi\{|\alpha(k)|^2 - |\beta(k)|^2\}\delta(k - k') \qquad (5.2.60)$$

Using the *Gamma function identities*

$$|\Gamma(1 + ix)|^2 = \frac{\pi x}{\sinh \pi x} \quad (x \in \mathbb{R}) \ , \ \Gamma(1 + z) = z\Gamma(z) \quad (z \in \mathbb{C})$$

we obtain from (5.2.57)

$$|\alpha(k)|^2 = \frac{\sinh^2 \pi\omega_+}{\sinh \pi\omega_k \sinh \pi\bar{\omega}_k} \ , \ |\beta(k)|^2 = \frac{\sinh^2 \pi\omega_-}{\sinh \pi\omega_k \sinh \pi\bar{\omega}_k} \qquad (5.2.61)$$

After a little algebra, we obtain

$$|\alpha(k)|^2 - |\beta(k)|^2 = 1 \qquad (5.2.62)$$

We deduce from (5.2.60) that b ($b^\dagger$) is an annihilation (creation) operator.
Suppose that the Universe evolves from the vacuum state $|0\rangle$ in the infinite past, where

$$a(k)|0\rangle = 0 \qquad (5.2.63)$$

An inertial observer in the infinite future will define a *different* vacuum state $|\bar{0}\rangle$, where

$$b(k)|\bar{0}\rangle = 0 \qquad (5.2.64)$$

For him, $|0\rangle$ is not a vacuum. To see what the (empty!) Universe looks like in the infinite future, let us introduce the *number operator*

$$N(k) = b^\dagger(k)b(k) \qquad (5.2.65)$$

It counts the number of particles with momentum k. If a state has n particles with momenta $p_1, \dots, p_n$, then

$$N(k)|p_1, \dots, p_n\rangle = 2\pi \sum_{i=1}^{n} \delta(k-p_i)|p_1, \dots, p_n\rangle \ , \ |p_1, \dots, p_n\rangle = b^\dagger(p_1)\cdots b^\dagger(p_n)|\bar{0}\rangle \qquad (5.2.66)$$

If m of the momenta equal k ($m \leq n$), then

$$N(k)|p_1, \ldots, p_n\rangle = 2\pi m\delta(0)|p_1, \ldots, p_n\rangle \tag{5.2.67}$$

so this is an eigenstate of $N(k)$ with eigenvalue $2\pi m\delta(0)$. For a general state $|\psi\rangle$, the *average* number of particles with momentum k is given by

$$\langle N(k)\rangle \sim \langle\psi|N(k)|\psi\rangle \tag{5.2.68}$$

This is not an equality because of the infinite factor $\delta(0)$. To do this properly, put the system in a box. This will not alter the results; it will only complicate your life.
Now if $|\psi\rangle = |0\rangle$ (the vacuum in the infinite past), then, using (5.2.58)

$$\begin{aligned}\langle 0|N(k)|0\rangle &= \langle 0|b^\dagger(k)b(k)|0\rangle \\ &= |\beta(k)|^2\langle 0|a(k)a^\dagger(k)|0\rangle \\ &= 2\pi\delta(0)|\beta(k)|^2 \end{aligned} \tag{5.2.69}$$

where in the last step we used (5.2.59) and (5.2.63). Therefore $|\beta(k)|^2$ (eq. (5.2.61)) gives the *spectrum* of particles an observer in the infinite future will see. The future Universe is hardly empty. Where did all these particles come from? Well, energy is *not* conserved since there is no time translation invariance. On the other hand, do not forget that this is hardly a closed system due to the presence of gravity. The gravitational field also contains energy and the *total* energy of the gravitational field and ϕ may be conserved (under a proper definition). It may even be *zero*. An everyday analogy is a ball rolling down a hill - its kinetic energy keeps increasing but its potential (gravitational) energy decreases; the sum is constant (assuming no friction) and may be set to zero. This is arbitrary for the rolling ball, but not so for the Universe because, according to Einstein, energy gravitates. It is an interesting issue.

5.3 The Unruh effect

5.3.1 Rindler space

A similar effect (production of particles from the vacuum) is also encountered in plain flat Minkowski space in which

$$d\tau^2 = dt^2 - d\vec{x}^2 \tag{5.3.1}$$

The vacuum is a concept on which all inertial observers agree, so if a stationary observer (feeling no forces) sees nothing, then so does an observer moving with constant velocity $\vec{v}$ with respect to the stationary observer, because the two are related by a Lorentz transformation (boost).
It is a different ball game if $\vec{v}$ is changing. Then the moving observer is no longer inertial. We shall study the simplest case of <u>uniform acceleration</u>. Let us choose axes so that the accelerating observer is moving in the x-direction. To simplify the discussion, we shall ignore the other two directions and write

$$d\tau^2 = dt^2 - dx^2 \tag{5.3.2}$$

We shall reinstate y and z in the next section.
How do we describe uniform acceleration? In your non-relativistic (everyday) life you learned

$$x = x_0 + \tfrac{1}{2}at^2 \tag{5.3.3}$$

if you start counting seconds when your speed $v = 0$. Then $v = at$ which cannot be true forever, because $v < c = 1$. This is reminiscent of the expression for the kinetic energy, $E = \frac{p^2}{2m}$ which is modified to $E^2 = m^2 + p^2$. The latter reads $E = m + \frac{p^2}{2m} + \ldots$, in the non-relativistic limit $p \ll E$. Similarly,

$$x^2 = t^2 + \frac{1}{a^2} \tag{5.3.4}$$

leads to $x = \frac{1}{a} + \frac{1}{2}at^2$ for $v = at \ll 1$. To see that this leads to uniform acceleration in the relativistic sense, parametrize

$$x(\tau) = \frac{1}{a}\cosh(a\tau)\ ,\ \ t(\tau) = \frac{1}{a}\sinh(a\tau) \tag{5.3.5}$$

The velocity is

$$v^\mu = \begin{pmatrix} \frac{dt}{d\tau} & \frac{dx}{d\tau} \end{pmatrix} = \begin{pmatrix} \cosh(a\tau) & \sinh(a\tau) \end{pmatrix} \tag{5.3.6}$$

which gives $v_\mu v^\mu = 1$ showing that τ is indeed proper time, satisfying (5.3.2). The acceleration is

$$a^\mu = \frac{dv^\mu}{d\tau} = a\begin{pmatrix} \sinh(a\tau) & \cosh(a\tau) \end{pmatrix} \tag{5.3.7}$$

which gives $a_\mu a^\mu = -a^2$, a constant. The trajectory is a *hyperbola* with asymptotes $x = \pm t$ (the observer approaches the speed of light in the infinite past and future). We shall pick the segment with $x > 0$; the other segment leads to equivalent results. The two asymptotes $x = \pm t$ divide Minkowski space into four regions which may be conveniently described in terms of the *light-cone coordinates*

$$x_\pm = x \pm t \tag{5.3.8}$$

as

Region	x_+	x_-
I	$+$	$+$
II	$+$	$-$
III	$-$	$+$
IV	$-$	$-$

The line element (5.3.1) in these coordinates is

$$d\tau^2 = -dx_+ dx_- \tag{5.3.9}$$

To describe the world in a language the accelerating observer will use, we need to define appropriate coordinates. One coordinate should be proper time which parametrizes the hyperbola. Different values of the acceleration will lead to different hyperbolae. These hyperbolae will define one set of coordinate axes. The other set is conveniently chosen as the straight lines through the origin. This is similar to the Euclidean plane which may be described in terms of Cartesian (x, y) or polar (r, θ) coordinates. The line element is given by (5.1.8). Lines of constant $r^2 = x^2 + y^2$ are *circles* whereas lines of constant θ are straight lines through the origin. We may introduce "polar coordinates" in our Minkowski space by defining

$$x = \frac{1}{a}\rho\cosh(a\eta) \ , \ t = \frac{1}{a}\rho\sinh(a\eta) \quad (\rho > 0 \ , \ -\infty < \eta < +\infty) \tag{5.3.10}$$

Proper time (5.3.1) reads

$$d\tau^2 = -\frac{1}{a^2}d\rho^2 + \rho^2 d\eta^2 \tag{5.3.11}$$

where η (ρ) is timelike (spacelike). This is Rindler space. It is just region I of Minkowski space parametrized differently. Constant ρ curves are hyperbolae, $x^2 - t^2 = \rho^2/a^2$, and constant η curves are straight lines through the origin, $t/x = \tanh(a\eta)$.
It is also convenient to rewrite the metric as

$$d\tau^2 = e^{2a\xi}(d\eta^2 - d\xi^2) \ , \ \xi = \frac{1}{a}\ln\rho \tag{5.3.12}$$

and in terms of the *light-cone coordinates*

$$\xi_\pm = \xi \pm \eta = \frac{1}{a}\ln(ax_\pm) \tag{5.3.13}$$

as

$$d\tau^2 = -e^{a(\xi_+ + \xi_-)}d\xi_+ d\xi_- \tag{5.3.14}$$

Now consider a massless scalar field ϕ. According to the inertial observer, ϕ satisfies the Klein-Gordon equation

$$\partial_t^2\phi - \partial_x^2\phi = 0 \tag{5.3.15}$$

which admits positive-energy plane-wave solutions

$$u_k(x,t) = \frac{1}{\sqrt{2\omega}}e^{-i(\omega t - kx)} \ , \ \omega = |k| \tag{5.3.16}$$

where I included the standard convenient normalization factor. Negative-energy solutions are u_k^*. If $k > 0$, we have a *right-moving* (positive-energy) wave,

$$u_k(x,t) = \frac{1}{\sqrt{2\omega}}e^{i\omega x_-} \tag{5.3.17}$$

whereas if $k < 0$, we have a *left-moving* wave,

$$u_k(x,t) = \frac{1}{\sqrt{2\omega}}e^{-i\omega x_+} \tag{5.3.18}$$

The former is an analytic function of x_- and *bounded* in the upper-half x_--plane:

$$|e^{i\omega x_-}| = e^{-\omega\Im x_-} \leq 1 \ , \ \Im x_- \geq 0 \tag{5.3.19}$$

The latter is an analytic function of x_+ and *bounded* in the lower-half x_+-plane:

$$|e^{-i\omega x_+}| = e^{\omega \Im x_+} \le 1 \ , \ \Im x_+ \le 0 \tag{5.3.20}$$

The significance of this observation will be seen shortly.
The general solution of the wave equation may be expanded as

$$\phi = \int \frac{dk}{2\pi} \left(a(k) u_k + a^\dagger(k) u_k^* \right) \tag{5.3.21}$$

Upon quantization, $a(k)$ ($a^\dagger(k)$) becomes an annihilation (creation) operator. The vacuum state for the inertial observer is defined, as usual, by

$$a(k)|0\rangle = 0 \tag{5.3.22}$$

Turning to the Rindler (accelerating) observer, the wave equation is deduced from the metric (5.3.12),

$$\partial_\eta^2 \phi - \partial_\xi^2 \phi = 0 \tag{5.3.23}$$

which is of the same form as the wave equation for the inertial observer (5.3.15) but with a crucial difference: unlike (t,x), (η,ξ) only cover region I which is 1/4 of Minkowski space. The Rindler wave equation leads to positive-energy solutions

$$\bar{u}_k(\eta,\xi) = \frac{1}{\sqrt{2\omega}} e^{-i(\omega\eta - k\xi)} \ , \ \omega = |k| \tag{5.3.24}$$

and negative-energy solutions v_k^* in region I. Again, we distinguish between *right-moving* waves ($k > 0$),

$$\bar{u}_k(\eta,\xi) = \frac{1}{\sqrt{2\omega}} e^{i\omega\xi_-} \tag{5.3.25}$$

and *left-moving* waves ($k < 0$),

$$\bar{u}_k(\eta,\xi) = \frac{1}{\sqrt{2\omega}} e^{-i\omega\xi_+} \tag{5.3.26}$$

These plane waves may be analytically continued beyond region I by expressing $\xi_\pm$ in terms of the Minkowski light-cone coordinates $x_\pm$ (5.3.13). We have

$$e^{\mp i\omega\xi_\pm} = (a x_\pm)^{\mp i\omega/a} \ , \ x_\pm > 0 \tag{5.3.27}$$

Thus, $e^{i\omega\xi_-}$ may be analytically continued into region II where $x_- < 0$, but *not* into III and IV, where $x_+ < 0$ and this wave has no dependence on x_+.
Similarly, $e^{-i\omega\xi_+}$ may be analytically continued into region III where $x_+ < 0$, but *not* into II and IV, where $x_- < 0$.
It follows that there is *no* solution that can be continued into region IV. Plane waves with support in region I *vanish* in region IV. Therefore, these plane waves do not form a complete set of solutions. To find the missing set, we may simply reverse the sign of $(t,x) \to (-t,-x)$ and repeat the above discussion. This interchanges regions I $\leftrightarrow$ IV and II $\leftrightarrow$ III.

We obtain two sets of modes (positive-energy solutions). One set is the analytic continuation of (5.3.24),

$$\bar{u}_k^{(1)} = \begin{cases} \frac{1}{\sqrt{2\omega}} e^{-i(\omega\eta - k\xi)} & , \text{ in region I} \\ 0 & , \text{ in region IV} \end{cases} \tag{5.3.28}$$

the other set has support in region IV,

$$\bar{u}_k^{(2)} = \begin{cases} 0 & , \text{ in region I} \\ \frac{1}{\sqrt{2\omega}} e^{i(\omega\eta - k\xi)} & , \text{ in region IV} \end{cases} \tag{5.3.29}$$

where in both cases $\omega = |k|$. Notice that the positive-energy wave with support in region IV contains a factor $e^{i\omega\eta}$ instead of $e^{-i\omega\eta}$. This is because after flipping the sign $t \to -t$, the time coordinate η (defined through (5.3.10)) points toward the *past* in region IV. It should also be pointed out that the coordinates (η, ξ) in region IV are distinct from their counterparts in region I. I used the same notation in the two regions for economy, at the risk of confusing you.

We may now expand the general solution of the Rindler wave equation as

$$\phi = \int \frac{dk}{2\pi} \left(b^{(1)}(k)\bar{u}_k^{(1)} + b^{(2)}(k)\bar{u}_k^{(2)} + b^{(1)\dagger}(k)\bar{u}_k^{(1)*} + b^{(2)\dagger}(k)\bar{u}_k^{(2)*} \right) \tag{5.3.30}$$

Upon quantization, $b^{(r)}(k)$ ($b^{(r)\dagger}(k)$) become annihilation (creation) operators ($r = 1, 2$). The vacuum state of the Rindler observer is defined by

$$b^{(r)}(k)|\bar{0}\rangle = 0 \ , \ \ r = 1, 2 \tag{5.3.31}$$

It is really awkward to derive the Bogoliubov transformation relating the Rindler modes $b^{(r)}$ ($r = 1, 2$) to the creation and annihilation operators, $a, a^\dagger$ (5.3.21), of the inertial observer. To avoid hard labor, we shall follow an argument due to Unruh.

Consider the right-moving waves first. As we have already discussed, the solution (5.3.28) with support in region I is extended to region II taking the form (5.3.27), i.e., for $k > 0$,

$$\bar{u}_k^{(1)} = \begin{cases} \frac{1}{\sqrt{2\omega}} (ax_-)^{i\omega/a} & , \text{ in regions I, II} \\ 0 & , \text{ in regions III, IV} \end{cases} \tag{5.3.32}$$

Similarly, the solution (5.3.29) with support in region IV is extended to region III. After taking its complex conjugate, we obtain

$$\bar{u}_k^{(2)*} = \begin{cases} 0 & , \text{ in regions I, II} \\ \frac{1}{\sqrt{2\omega}} (-ax_-)^{i\omega/a} & , \text{ in regions III, IV} \end{cases} \tag{5.3.33}$$

The factor $(-)^{i\omega/a}$ is ambiguous. To define it, we shall place the *branch cut* in the *lower-half* x_--plane so that the function remains analytic in the *upper-half* plane. We have $x_- < 0$ in region IV, so by our demand its argument must be π, i.e.,

$$x_- = (-x_-)e^{i\pi} \ \ \Rightarrow \ \ -x_- = x_- e^{-i\pi} \tag{5.3.34}$$

and similarly in region III. We obtain

$$\bar{u}_k^{(2)*} = \begin{cases} 0 & , \text{ in regions I, II} \\ \frac{1}{\sqrt{2\omega}} e^{\omega\pi/a} (ax_-)^{i\omega/a} & , \text{ in regions III, IV} \end{cases} \quad (5.3.35)$$

It follows that the solution

$$U_k^{(1)} = \bar{u}_k^{(1)} + e^{-\omega\pi/a} \bar{u}_k^{(2)*} = \frac{1}{\sqrt{2\omega}} (ax_-)^{i\omega/a} \quad (5.3.36)$$

in the entire Minkowski space (all four regions). This is an analytic function which is bounded in the upper-half plane. Indeed,

$$|(ax_-)^{i\omega/a}| = e^{-(\omega/a)\arg x_-} \leq 1 \quad (5.3.37)$$

because $0 \leq \arg x_- \leq \pi$ in the upper-half plane having placed the branch cut in the lower-half plane. This property is shared with the $k > 0$ modes of the Minkowski space (5.3.17) as we have already shown. It follows that these solutions form just as good a set as the plane-wave solutions u_k for the inertial observer. U_k may be expanded in terms of u_k. Thus, U_k form a set of *positive-energy* solutions of the Minkowski wave equation (with $k > 0$).

Similarly, we obtain

$$U_k^{(2)} = \bar{u}_k^{(2)} + e^{-\omega\pi/a} \bar{u}_k^{(1)*} = \frac{1}{\sqrt{2\omega}} (ax_+)^{-i\omega/a} \quad (5.3.38)$$

in the entire Minkowski space. These also form a set of *positive-energy* solutions of the Minkowski wave equation (with $k < 0$).

Therefore, the inertial observer may expand a general solution as

$$\phi = \int \frac{dk}{2\pi} \left(C^{(1)}(k) U_k^{(1)} + C^{(2)}(k) U_k^{(2)} + C^{(1)\dagger}(k) U_k^{(1)*} + C^{(2)\dagger}(k) U_k^{(2)*} \right) \quad (5.3.39)$$

instead of (5.3.21). The vacuum state of the inertial observer defined by (5.3.22) may also be defined by

$$C^{(r)}(k)|0\rangle = 0 \ , \ r = 1, 2 \quad (5.3.40)$$

The two definitions (5.3.22) and (5.3.40) are equivalent.

We may easily relate these modes to those of the Rindler observer (5.3.30) by using (5.3.36) and (5.3.38). We obtain

$$b^{(1)}(k) = C^{(1)}(k) + e^{-\pi\omega/a} C^{(2)\dagger}(k) \ , \ b^{(2)}(k) = C^{(2)}(k) + e^{-\pi\omega/a} C^{(1)\dagger}(k) \quad (5.3.41)$$

The C-modes are not properly normalized. From the commutation relations

$$[b^{(r)}(k), b^{(s)\dagger}(k')] = \delta^{rs} (2\pi) \delta(k - k') \quad (5.3.42)$$

we deduce

$$[C^{(r)}(k), C^{(s)\dagger}(k')] = \frac{e^{\pi\omega/a}}{2\sinh(\pi\omega/a)} \delta^{rs} (2\pi) \delta(k - k') \quad (5.3.43)$$

For creation and annihilation operators, define

$$c^{(r)}(k) = e^{-\pi\omega/2a}\sqrt{2\sinh(\pi\omega/a)}\, C^{(r)}(k) \tag{5.3.44}$$

Then

$$[c^{(r)}(k), c^{(s)\dagger}(k')] = \delta^{rs}(2\pi)\delta(k-k') \tag{5.3.45}$$

The Rindler modes may be written as

$$\begin{aligned} b^{(1)}(k) &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left\{e^{\pi\omega/2a}c^{(1)}(k) + e^{-\pi\omega/2a}c^{(2)\dagger}(k)\right\} \\ b^{(2)}(k) &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left\{e^{\pi\omega/2a}c^{(2)}(k) + e^{-\pi\omega/2a}c^{(1)\dagger}(k)\right\} \end{aligned} \tag{5.3.46}$$

This is the *Bogoliubov transformation* relating Rindler and inertial observer modes. Now suppose the system is in the vacuum state of Minkowski space $|0\rangle$. The number operator for the Rindler observer is

$$N(k) = b^{(1)\dagger}(k)b^{(1)}(k) \tag{5.3.47}$$

since $b^{(2)\dagger}$ excites modes which vanish in region I and are therefore not accessible to the Rindler observer whose trajectory is in region I. Using the Bogoliubov transformation (5.3.46) and the definition (5.3.40) of the vacuum state, we obtain the expectation value of the number operator

$$\begin{aligned} \langle 0|N(k)|0\rangle &= \frac{e^{-\pi\omega/a}}{2\sinh(\pi\omega/a)}\langle 0|c^{(2)}(k)c^{(2)\dagger}(k)|0\rangle \\ &= \frac{1}{e^{2\pi\omega/a}-1}(2\pi)\delta(0) \end{aligned} \tag{5.3.48}$$

which is a *blackbody spectrum* (recall the Planck factor $(e^{\hbar\omega/k_BT}-1)^{-1}$) of temperature

$$T = \frac{\hbar a}{2\pi c k_B} \tag{5.3.49}$$

where I restored the physical constants for an expression in degrees Kelvin. Thus, the Rindler observer is traveling through a thermal bath. Somehow she got hot feeling friction from the vacuum!

This is puzzling at first sight, because the inertial observer still thinks that $\langle 0|T_{\mu\nu}|0\rangle = 0$, so who supplied the energy for all these particles in the thermal bath to be created? Moreover, the inertial observer should see no particles since he is in the vacuum. But if the Rindler detector beeps, how can the inertial observer disagree? It either beeps or it doesn't! To answer these questions, let us take a closer look at the Rindler detector.

5.3.2 The detector

Consider a detector traveling along the trajectory $x^\mu(\tau)$ (not necessarily Rindler) in the Minkowski vacuum $|0\rangle$. To detect a particle, it must couple to the field ϕ. Suppose that the interaction is linear.[4] When the detector sees something, it gets excited and there is a transition in the surrounding field $|0\rangle \to |\psi\rangle$. The amplitude for this transition is

$$\mathcal{A}(\tau) \sim \langle\psi|\phi(x(\tau))|0\rangle \tag{5.3.50}$$

where the constant of proportionality depends on the details of the detector. Suppose that the detector gets switched on and off *adiabatically* at times $\pm T/2$ where T is large. By fourier transforming, we obtain the amplitude for the detector to see a particle of frequency ω and at the same time for the field to transition $|0\rangle \to |\psi\rangle$,

$$\tilde{\mathcal{A}}(\omega) \sim \int_{-T/2}^{T/2} e^{i\omega\tau}\mathcal{A}(\tau) \tag{5.3.51}$$

The total probability per unit time for the detector to see a particle of frequency ω is

$$\frac{P(\omega)}{T} \sim \mathcal{F}(\omega) = \frac{1}{T}\sum_\psi |\tilde{\mathcal{A}}(\omega)|^2 \tag{5.3.52}$$

where $\mathcal{F}(\omega)$ is the *detector response function* and we are interested in the large-T limit ($T \to \infty$). Using the completeness of the states $|\psi\rangle$, we obtain

$$\mathcal{F}(\omega) = \frac{1}{T}\int_{-T/2}^{T/2} d\tau \int_{-T/2}^{T/2} d\tau' e^{-i\omega(\tau-\tau')} D(x(\tau), x(\tau')) \tag{5.3.53}$$

where $D(x,y)$ is the propagator (1.4.4). If D depends on the time difference $\tau - \tau'$ only (as is often the case), then in the limit $T \to \infty$, we obtain

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t}\mathcal{D}(t) \;\;,\;\; \mathcal{D}(t) = D(x(\tau+t), x(\tau)) \tag{5.3.54}$$

Let us calculate this for a Rindler detector. To simplify the discussion earlier, we ignored the two directions transverse to the Rindler trajectory. It is now time to reinstate them and work in four dimensions. For a massless scalar field in four dimensions, the propagator is given by (5.2.23). For the Rindler trajectory (5.3.5),

$$x^0(\tau) = \frac{1}{a}\sinh(a\tau) \;\;,\;\; x^1(\tau) = \frac{1}{a}\cosh(a\tau) \;\;,\;\; x^2(\tau) = x^3(\tau) = 0 \tag{5.3.55}$$

we obtain after some algebra,

$$(x(\tau) - x(\tau'))^2 = \frac{4}{a^2}\sinh^2\frac{a(\tau-\tau')}{2} \tag{5.3.56}$$

therefore

$$\begin{aligned} \mathcal{D}(t) &\equiv D(x(\tau+t), x(\tau)) \\ &= -\frac{1}{4\pi^2(x(\tau+t)-x(\tau))^2} \\ &= -\frac{a^2}{16\pi^2\sinh^2(at/2)} \end{aligned} \tag{5.3.57}$$

[4]Such is the case if the field is the vector potential A_μ. Then the interaction Hamiltonian is $\sim A_\mu J^\mu$, where J^μ is the current representing the detector.

Its fourier transform is the detector response function (5.3.54). To perform the requisite integral, notice that the propagator has poles along the *imaginary* t-axis located at $t = -2\pi in/a$ ($n \in \mathbb{Z}$). Near a pole the propagator behaves as

$$\mathcal{D}(t) \sim -\frac{1}{4\pi^2(t+2\pi i/a)^2} \ , \ t \to -2\pi in/a \quad (n \in \mathbb{Z}) \tag{5.3.58}$$

which is of the same form as the massless propagator (5.2.23). We deduce

$$\mathcal{D}(t) = -\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(t+2\pi in/a)^2} \tag{5.3.59}$$

We may now use contour integration to calculate the fourier transform. Due to the $e^{-i\omega t}$ factor in (5.3.54), we ought to close the contour in the lower-half plane ($\Im t < 0$). Then only the poles with $n > 0$ contribute.[5] Using

$$\oint_{\mathcal{C}}\frac{dt}{2\pi i}\frac{e^{-i\omega t}}{(t+2\pi in/a)^2} = -i\omega\, e^{-2\pi n\omega/a} \tag{5.3.60}$$

for a contour $\mathcal{C}$ surrounding the pole $t = -2\pi i/a$, we obtain

$$\mathcal{F}(\omega) = \frac{\omega}{2\pi}\sum_{n=1}^{\infty}e^{-2\pi n\omega/a} = \frac{\omega}{2\pi}\,\frac{1}{e^{2\pi\omega/a}-1} \tag{5.3.61}$$

Once again (*cf.* eq. (5.3.48)), we obtain a Planckian distribution with temperature $T = \frac{a}{2\pi}$ (eq. (5.3.49)).

Having understood the response of the Rindler detector, we are now in a position to address the questions that remained unanswered at the end of the previous section. According to the inertial observer, the Rindler observer is accelerating, therefore she is *emitting* radiation. If ϕ is part of the vector potential, then this is the known effect of electromagnetic radiation by an accelerating charge. Whoever is accelerating the Rindler observer and her detector is supplying the energy for the emission of these particles (radiation). Part of this energy goes to the detector which also absorbs particles (so yes, it beeps) and gets excited (possibly through atomic transitions). The Rindler observer is oblivious to all these nuances relevant to the inertial observer and only sees the net effect which is a thermal bath of particles.

We would like to understand the origin of these thermal effects a little better including the intriguing emergence of poles along the imaginary time axis (eq. (5.3.58)) which appear to be responsible for a *physical* effect: the response of the detector.

5.3.3 Thermodynamics

[5] The $n = 0$ pole at $t = 0$ looks tricky because it lies on the real axis, but recall that the expression for the massless propagator makes sense only for $t - i\epsilon$ ($\epsilon > 0$) so the $n = 0$ pole is at $t = i\epsilon$, slightly above the real axis and does *not* contribute. There is actually a good *physical* reason for that: The $n = 0$ term corresponds to the Minkowski propagator. For the inertial observer, the system is in the vacuum state which contains no particles. Therefore, the detector response function ought to *vanish* for him. The $t - i\epsilon$ prescription ensures that it does.

We start with a (short) review of thermodynamics. It is curious that thermodynamics provides an appropriate description of the systems we have been discussing because finite temperature is associated with *lack* of information about the state of the system. How did we lose information? It seems that once gravity is included, this occurs inevitably. The ultimate quantum theory of gravity will probably have a deep connection with thermodynamics. At the moment, we can only glimpse that connection.

Consider a system with conserved Hamiltonian H and *number operator* N. Assuming H and N commute with each other, $[H, N] = 0$, they can be simultaneously diagonalized. Let $|\psi_i\rangle$ form a complete set of normalized states which are eigenfunctions of both H and N,

$$H|\psi_i\rangle = E_i|\psi_i\rangle \ , \ N|\psi_i\rangle = N_i|\psi_i\rangle \ , \ \langle\psi_i|\psi_i\rangle = 1 \tag{5.3.62}$$

EXAMPLE: The Klein-Gordon field in Minkowski space has (eq. (1.3.24))

$$H = \int \frac{d^3k}{(2\pi)^3}\, \omega_k a^\dagger(\vec{k})a(\vec{k}) \ , \ N = \int \frac{d^3k}{(2\pi)^3}\, a^\dagger(\vec{k})a(\vec{k}) \tag{5.3.63}$$

Their common eigenstates are the N-particle states (1.3.47) with energy eigenvalue given by (1.3.49) and number eigenvalue N.

If the system is in a thermal bath of temperature T and *chemical potential* μ, the probability that the system will be in the state $|\psi_i\rangle$ is

$$p_i = \frac{1}{\mathcal{Z}} e^{-(E_i - \mu N_i)/T} \tag{5.3.64}$$

where

$$\mathcal{Z} = \sum_i e^{-(E_i - \mu N_i)/T} = e^{-\Omega/T} \tag{5.3.65}$$

is the grand partition function of the system and Ω is the corresponding thermodynamical potential (*Gibbs free energy*). Evidently,

$$\sum_i p_i = 1 \tag{5.3.66}$$

Let us introduce the density matrix

$$\rho = \frac{1}{\mathcal{Z}} e^{-(H-\mu N)/T} \ , \ \mathcal{Z} = \mathrm{tr}\, e^{-(H-\mu N)/T} \tag{5.3.67}$$

satisfying $\mathrm{tr}\rho = 1$ (the equivalent of (5.3.66)). Its eigenfunctions are the states $|\psi_i\rangle$,

$$\rho|\psi_i\rangle = p_i|\psi_i\rangle \tag{5.3.68}$$

For an operator $\mathcal{O}$, its *average* value in the state $|\psi_i\rangle$ is

$$\langle\mathcal{O}\rangle_i = \langle\psi_i|\mathcal{O}|\psi_i\rangle \tag{5.3.69}$$

The ensemble average is

$$\langle\mathcal{O}\rangle_{T,\mu} = \sum_i p_i \langle\mathcal{O}\rangle_i = \mathrm{tr}(\rho\mathcal{O}) \tag{5.3.70}$$

An important example of an operator is the product $\phi(x)\phi(y)$ whose ensemble average is the thermal propagator

$$D_{T,\mu}(x,y) \equiv \langle \phi(x)\phi(y)\rangle_{T,\mu} = \mathrm{tr}(\rho\phi(x)\phi(y)) \tag{5.3.71}$$

From now on we shall set the chemical potential

$$\mu = 0 \tag{5.3.72}$$

to simplify the discussion. The ensemble average (5.3.70) reduces to the vacuum expectation value $\langle 0|\mathcal{O}|0\rangle$ in the zero-temperature limit ($T \to 0$). Indeed, let E_0 be the vacuum energy ($H|0\rangle = E_0|0\rangle$). Then we have

$$\langle \mathcal{O}\rangle_T = \frac{\sum_i e^{-(E_i-E_0)/T}\langle \mathcal{O}\rangle_i}{\sum_i e^{-(E_i-E_0)/T}} \tag{5.3.73}$$

As $T \to 0$, all factors $e^{-(E_i-E_0)/T} \to 0$ since $E_i \geq E_0$, except $E_i = E_0$. It follows that in this limit only one term survives in the series: the vacuum contribution,

$$\lim_{T\to 0}\langle \mathcal{O}\rangle_T = \langle 0|\mathcal{O}|0\rangle \tag{5.3.74}$$

Thus we recover the field theory we have been studying as the zero-temperature limit of a thermodynamical system. In particular, the thermal propagator (5.3.71) reduces to the standard propagator (1.4.4) as $T \to 0$. To find an explicit expression for the thermal propagator, start with

$$D_T(x,y) = \frac{\mathrm{tr}\left\{e^{-H/T}\phi(x)\phi(y)\right\}}{\mathrm{tr}\; e^{-H/T}} \tag{5.3.75}$$

The operator $e^{-H/T}$ is similar to the evolution operator e^{iHt} (eq. (1.3.29)) except it shifts time by an *imaginary* amount,

$$e^{-H/T}\phi(\vec{x},x^0)e^{H/T} = \phi(\vec{x},x^0+i/T) \tag{5.3.76}$$

Commuting $e^{-H/T}$ through $\phi(x)$ and using the cyclic property of the trace ($\mathrm{tr}(AB) = \mathrm{tr}(BA)$), we obtain

$$D_T(\vec{x},x^0;\vec{y},y^0) = D_T(\vec{y},y^0;\vec{x},x^0+i/T) \tag{5.3.77}$$

where I separated space from time in the arguments of the propagator. It follows that the *symmetrized* propagator,

$$D_T^{(S)}(x,y) = D_T(x,y) + D_T(y,x) = \langle\{\phi(x),\phi(y)\}\rangle_T \tag{5.3.78}$$

is periodic in time with *imaginary* period i/T,

$$D_T^{(S)}(\vec{x},x^0;\vec{y},y^0) = D_T^{(S)}(\vec{x},x^0+i/T;\vec{y},y^0) \tag{5.3.79}$$

The 3 properties of the thermal propagator:

(a) periodicity

(b) it satisfies the wave equation (since ϕ does)

(c) it reduces to the symmetrized propagator

$$D^{(S)}(x,y) = D(x,y) + D(y,x) = \langle 0|\{\phi(x),\phi(y)\}|0\rangle \tag{5.3.80}$$

in the limit $T \to 0$

uniquely define it. An explicit expression is not hard to find by thinking of the propagator as a potential due to a unit charge. To satisfy the periodic boundary conditions we need to add an infinite series of images. We obtain

$$D_T^{(S)}(x,y) = \sum_{n=-\infty}^{\infty} D^{(S)}(\vec{x}, x^0 + ni/T; \vec{y}, y^0) \tag{5.3.81}$$

This expression satisfies all 3 conditions *(a)-(c)*.
In the massless case, using (5.2.23), we obtain

$$D^{(S)}(x,y) = -\frac{1}{2\pi^2(x-y)^2} \tag{5.3.82}$$

For an inertial observer in his rest frame, $x^0 = \tau$, $\vec{x} = \vec{0}$, so

$$D^{(S)} = -\frac{1}{2\pi^2(\tau-\tau')^2} \tag{5.3.83}$$

The corresponding thermal propagator is

$$D_T^{(S)} = -\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(\tau-\tau'+ni/T)^2} \tag{5.3.84}$$

Remarkably, this expression agrees with the *Rindler* propagator (5.3.59) [6] if $T = \frac{a}{2\pi}$, which is the Rindler temperature (5.3.49). The latter was found by plugging the Rindler trajectory (5.3.55) into the massless (zero-temperature) propagator (5.2.23) whereas (5.3.84) is the thermal propagator for the trajectory of an *inertial observer*.

5.3.4 Energy density

Let us also calculate the difference of the two vacuum energy densities (an observable)

$$\Delta\rho = \langle\bar{0}|T_{00}|\bar{0}\rangle - \langle 0|T_{00}|0\rangle \tag{5.3.85}$$

where $|0\rangle$ ($|\bar{0}\rangle$) is the vacuum state of the inertial (Rindler) observer. T_{00} is the Hamiltonian density (1.2.37) with $m=0$,

$$T_{00} = \tfrac{1}{2}(\partial_0\phi)^2 + \tfrac{1}{2}(\vec{\nabla}\phi)^2 \tag{5.3.86}$$

[6] The factor of 2 discrepancy is due to symmetrization.

To calculate its vacuum expectation value, we split the arguments. Then

$$\langle 0|T_{00}|0\rangle = -\tfrac{1}{4}(\partial_0^2 + \nabla^2)D^{(S)}(x-y) \tag{5.3.87}$$

given in terms of the symmetrized propagator (5.3.82) and the limit $y \to x$ is understood. The propagator satisfies the Klein-Gordon eq. (1.2.33) with $m = 0$, because ϕ does. It follows that

$$\langle 0|T_{00}|0\rangle = -\tfrac{1}{2}\partial_0^2 D^{(S)}(x-y) \tag{5.3.88}$$

Since only time derivatives appear, we may already take the limit $\vec{y} \to \vec{x}$ and write

$$\langle 0|T_{00}|0\rangle = \frac{1}{4\pi^2}\,\partial_0^2\,\frac{1}{t^2} \quad , \qquad t = x^0 - y^0 \tag{5.3.89}$$

The Rindler contribution is obtained by using the thermal propagator (5.3.84), instead. We obtain similarly

$$\langle \bar{0}|T_{00}|\bar{0}\rangle = \frac{1}{4\pi^2}\,\partial_0^2 \sum_{n=-\infty}^{\infty} \frac{1}{(t + ni/T)^2} \tag{5.3.90}$$

The derivatives are straightforward. After subtracting the two expressions and taking the limit $t \to 0$, we obtain

$$\Delta\rho = \frac{3T^4}{2\pi^2}\sum_{n\neq 0}\frac{1}{n^4} = \frac{\pi^2}{30}\,T^4 \tag{5.3.91}$$

where I used (5.2.35). Notice $\rho \propto T^4$, as expected for a gas of massless particles (e.g., electromagnetic radiation).
The same result holds for a *conformal field* (eq. (5.2.16) with $\xi = \frac{1}{6}$). For a photon gas, we simply need to double the result because the photon has two degrees of freedom,

$$\rho_{em} = \frac{\pi^2}{15}\,\frac{(k_B T)^4}{(c\hbar)^3} \tag{5.3.92}$$

where I restored all the physical constants. This agrees with standard results from statistical mechanics.

5.4 The Universe

5.4.1 Scalars

Earlier we considered an expanding Universe that started and ended its life as flat Minkowski space. In its lifetime, galaxies flew apart, so the two Minkowski spaces did not coincide. Let us now consider models which are closer to reality. The metric is given by (5.1.11),

$$d\tau^2 = d\mathbf{T}^2 - a^2(\mathbf{T})d\vec{x}^2 \;\; , \;\; d\vec{x}^2 = dx^2 + dy^2 + dz^2 \tag{5.4.1}$$

$\mathbf{T}$ is the time of our clocks, but I'll use t for the time parameter defined by

$$\frac{d\mathbf{T}}{dt} = a(\mathbf{T}) \;\; \Rightarrow \;\; t = \int \frac{d\mathbf{T}}{a(\mathbf{T})} \tag{5.4.2}$$

In terms of t,

$$d\tau^2 = a^2(dt^2 - d\vec{x}^2) \ , \ g_{\mu\nu} = a^2\eta_{\mu\nu} \tag{5.4.3}$$

The metric is *conformally equivalent* to the Minkowski metric $\eta_{\mu\nu}$, so t is called conformal time. We also have

$$\sqrt{|\det g_{\mu\nu}|} = a^4 \tag{5.4.4}$$

and the Ricci scalar (5.1.54) is found to be

$$R = \frac{6\ddot{a}}{a^3} \ , \ \dot{a} \equiv \frac{da}{dt} \tag{5.4.5}$$

The wave eq. (5.2.5) reads

$$\frac{1}{a^4}\partial_t(a^2\partial_t\phi) - \frac{1}{a^2}\nabla^2\phi + \left(m^2 + 6\xi\frac{\ddot{a}}{a^3}\right)\phi = 0 \tag{5.4.6}$$

To solve it, we shall separate arguments thusly,

$$\phi = e^{i\vec{k}\cdot\vec{x}}\frac{1}{a}\,\mathcal{T}(t) \tag{5.4.7}$$

We obtain

$$\ddot{\mathcal{T}} + \left\{\vec{k}^2 + m^2a^2 + (6\xi - 1)\frac{\ddot{a}}{a}\right\}\mathcal{T} = 0 \tag{5.4.8}$$

For a *conformal field* ($m = 0$, $\xi = \frac{1}{6}$ (5.2.15), so that the stress-energy tensor is *traceless*), we obtain the positive-energy solution $\mathcal{T} = e^{-i\omega_k t}$, where $\omega_k = |\vec{k}|$, so the positive-energy solutions of the wave equation (5.4.6) with the usual normalization factor are

$$\bar{u}_k(t,\vec{x}) = \frac{1}{\sqrt{2\omega_k}}\,\frac{1}{a}\,e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} \quad , \qquad \omega_k = |\vec{k}| \tag{5.4.9}$$

This is not exactly a plane wave because of the time-dependent $1/a$ factor and, perhaps more importantly, it is not in a language that we would use, since t is not our time; $\mathbf{T}$ is. Notice that I named these plane waves $\bar{u}_k$ in analogy with the wavefunctions of the Rindler observer (5.3.24) who also used a *conformal time*, η (5.3.12). This analogy will be made more precise in the next section.

5.4.2 The Milne Universe

To do an explicit calculation, suppose a is linear in *our* time $\mathbf{T}$ (Milne Universe),

$$a(\mathbf{T}) = H\mathbf{T} \tag{5.4.10}$$

This captures the essential features of our Universe, in which $a \sim \mathbf{T}^p$, for some p. The parameter H is the *Hubble constant.* Notice that as $\mathbf{T} \to 0$, $a \to 0$, so $\mathbf{T} = 0$ is a singularity (the Big Bang!). To avoid it, let $\mathbf{T} > 0$. $\mathbf{T} = 0$ is when the Universe is born. This means that we have no idea what came *before* $\mathbf{T} = 0$. In fact, this may be a meaningless question, because time probably loses its meaning near the singularity due to quantum effects.

To avoid doing any extra work, let us also get rid of two of the spatial dimensions (say, y and z) and write

$$d\tau^2 = d\mathbf{T}^2 - H^2\mathbf{T}^2dx^2 \tag{5.4.11}$$

This is eerily similar to the *Rindler metric* (5.3.11) with an important twist: the signs are wrong! This implies that we may use the conclusions we reached in the Rindler case if we are careful with the interpretation and interchange space and time.
We may define Minkowski coordinates (*cf.* eq. (5.3.10)),

$$y^0 = \mathbf{T}\cosh(Hx) \ , \ y^1 = \mathbf{T}\sinh(Hx) \tag{5.4.12}$$

in terms of which proper time reads

$$d\tau^2 = (dy^0)^2 - (dy^1)^2 \tag{5.4.13}$$

showing that this Universe is part of a larger Minkowski space. The coordinates $(\mathbf{T}, x)$ cover what we called *region II* in the Rindler case. In terms of conformal time (5.4.2), we have

$$\mathbf{T} = \frac{1}{H}e^{Ht} \ , \ a = e^{Ht} \tag{5.4.14}$$

where we chose the arbitrary integration constant conveniently. The metric reads

$$d\tau^2 = e^{2Ht}(dt^2 - dx^2) \tag{5.4.15}$$

showing that (t, x) play the rôle of (ξ, η) in the Rindler case (*cf.* eq. (5.3.12)).
The trajectory of a Rindler (accelerated) observer was a ξ =const. curve (hyperbola). This translates to t =const., or equivalently, $\mathbf{T}$ =const., so the hyperbolae are snapshots of the Universe.
The η =const. lines (straight lines through the origin) are now x =const. lines, therefore they represent the *worldlines* of galaxies. They are straight lines

$$\frac{y^1}{y^0} = \tanh(Hx) \tag{5.4.16}$$

through the origin.
At fixed time $\mathbf{T}$, distance in the Universe is measured by

$$ds^2 = -d\tau^2 = H^2\mathbf{T}^2dx^2 \tag{5.4.17}$$

where we used (5.4.11), so the distance between galaxies is $\Delta s = H\mathbf{T}\Delta x$, increasing with time. In terms of $s = H\mathbf{T}x$, we may write the Minkowski coordinates (5.4.12) as

$$y^0 = \mathbf{T}\cosh\frac{s}{\mathbf{T}} \ , \ y^1 = \mathbf{T}\sinh\frac{s}{\mathbf{T}} \tag{5.4.18}$$

Upon comparison with (5.3.5), it follows that $1/\mathbf{T}$ plays the rôle of the acceleration of the Rindler observer. Thus, the coordinates (t, x) are associated with a thermal bath of temperature (*cf.* eq. (5.3.49))

$$T = \frac{\hbar}{2\pi k_B\mathbf{T}} \tag{5.4.19}$$

As this Universe expands, it is cooling down with temperature inversely proportional to time. This bath is experienced by the *inertial observers* (galactic detectors which are at fixed x and therefore move along straight lines (5.4.16) in Minkowski space) if the Universe is in the vacuum state $|0\rangle$ associated with the Minkowski coordinates (y^0, y^1). The vacuum of the inertial observers $|\bar{0}\rangle$ is analogous to the Rindler vacuum. The energy density is found as before (eq. (5.3.91)),

$$\rho = -\frac{\pi^2}{30}\frac{(k_B T)^4}{(c\hbar)^3} = -\frac{\hbar}{480\pi^2 c^3 \mathbf{T}^4} \tag{5.4.20}$$

This is a four-dimensional result. In two dimensions, we obtain $\rho \propto 1/\mathbf{T}^2$. This negative energy consists of *spurious* particles. One ought to rethink the definition of a particle.

Which vacuum state is "preferred"? One may argue that the Minkowski vacuum $|0\rangle$ is special because *all* inertial observers in Minkowski space agree on its definition (being connected to each other via Lorentz transformations). That may be so, but the Milne Universe can be embedded in a Minkowski space only in two dimensions. In four dimensions or for a more realistic metric no such embedding is possible.

In a curved space inertial observers are *freely falling observers.* [7] Unfortunately, they are not connected with each other with Lorentz transformations (hence the curvature), so it is hard to justify promoting their vacuum state (which changes depending on who you ask) to a "preferred" status.

5.4.3 The conformal vacuum

In the general case, the vacuum corresponding to *our* coordinates $(T, \vec{x})$ (galactic detector) is hard to define. One resorts to an *adiabatic approximation*, hence its name: adiabatic vacuum.

On the other hand, the vacuum state correspoding to conformal time, $|\bar{0}\rangle$ (conformal vacuum), is much easier to define and plays a special rôle. Indeed, in the conformal case, the positive-energy solutions of the wave equation (5.4.8) are given by (5.4.9). A general field may be expanded, as usual,

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3}\left(b(\vec{k})u_k(\vec{x}, t) + b^\dagger(\vec{k})u_k^*(\vec{x}, t)\right) \tag{5.4.21}$$

This expression is valid throught the history of the Universe! This implies that annihilation and creation operators never rotate into each other via a Bogoliubov transformation, so the conformal vacuum $|\bar{0}\rangle$ *always* remains the vacuum and no particles are ever produced.

On the other hand, t is not our time; we use $\mathbf{T}$. For us, $|\bar{0}\rangle$ doesn't necessarily represent a vacuum state and we are likely to see particle production due to a Bogoliubov transformation. To see that this is indeed the case, let us calculate the response function of a galactic detector. We need the propagator

$$\bar{D}(\vec{x}, t; \vec{y}, t') = \langle\bar{0}|\phi(\vec{x}, t)\phi(\vec{y}, t')|\bar{0}\rangle \tag{5.4.22}$$

[7] E.g., on a *satellite* orbiting the earth. One feels no forces, no energy is needed to keep the satellite in orbit and Newton's laws obviously hold: throw a ball in any direction and it will keep moving along a straight line with constant velocity.

Using the expansion (5.4.21), we deduce

$$\bar{D}(\vec{x},t;\vec{y},t') = \frac{1}{a(t)a(t')} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \, e^{-i(\omega_k(t-t') - \vec{k}\cdot(\vec{x}-\vec{y}))} \tag{5.4.23}$$

given in terms of the Minkowski propagator (1.4.4). Using the explicit expression (5.2.23) in the massless case, we obtain

$$\bar{D}(\vec{x},t;\vec{y},t') = -\frac{1}{4\pi^2 a(t)a(t')[(t-t'-i\epsilon)^2 - (\vec{x}-\vec{y})^2]} \tag{5.4.24}$$

The response function (5.3.53) is

$$\mathcal{F}(\omega) \sim \int d\mathbf{T} \int d\mathbf{T}' e^{-i\omega(\mathbf{T}-\mathbf{T}')} \bar{D}(\vec{x},t;\vec{y},t') \tag{5.4.25}$$

where $\vec{x} = \vec{y}$ (galaxies are at $\vec{x}$ =const.). Using the explicit expression (5.4.24), we obtain

$$\mathcal{F}(\omega) \sim -\frac{1}{4\pi^2} \int dt \int dt' \frac{e^{-i\omega \int_{t'}^{t} dt'' a(t'')}}{(t-t'-i\epsilon)^2} \tag{5.4.26}$$

where we also used (5.4.2) to express $\mathbf{T}$ in terms of t.
In Minkowski space ($a = 1$), $\mathcal{F} = 0$, as expected.[8] In general, $\mathcal{F} \neq 0$ and a galactic detector detects particles in the conformal vacuum.
In the more general (non-conformal) case, the wave eq. (5.4.8),

$$\ddot{\mathcal{T}} + (\vec{k}^2 + M^2)\mathcal{T} = 0 \ , \ M^2(t) = m^2 a^2 + (6\xi - 1)\frac{\ddot{a}}{a} \tag{5.4.27}$$

may be solved approximately. Notice that it is possible for $M^2 < 0$, if, e.g., $m = 0$, $\xi = 0$ and $\ddot{a} > 0$ (which is the case in our Universe). Then we have tachyons! This signals an *instability*. The harmonic oscillator potential has a maximum at zero (unstable equilibrium). If the Hamiltonian is bounded from below (as it ought to in a physical system), then there must be minima of the potential where equilibrium will be stable. These minima correspond to lower energy levels. The Universe will want to roll from the symmetric point (local maximum) down to one of those minima. This leads to *symmetry breaking*!

5.5 The Hawking effect

5.5.1 Schwarzschild black hole

5.5.2 Vacuum states

5.5.3 Entropy

[8]To see this, note that the pole is at $t - t' = i\epsilon$ in the upper-half complex plane whereas the contour is in the lower-half plane (due to the $e^{-i\omega(t-t')}$ factor).