PHYSICS 642 - SPRING 2004
HOMEWORK 4

Due date: Mon., Apr. 26

Problem 4.1

Gravitational-wave displacements

Consider the local Lorentz frame of a freely falling point mass. Because the mass is at its origin, the frame and the mass move together forever and as seen by the mass, the frame remains locally Lorentz forever. Thus, for all t, we have

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + k_{\mu\nu} dx^\mu dx^\nu$$

where

$$k_{\mu\nu} = 0 \quad , \quad \partial_\rho k_{\mu\nu} = 0 \quad \text{at} \quad (x, y, z) = 0$$

(a) Show that this means that

$$\partial_0\, \partial_0 k_{\mu\nu} = \partial_0\, \partial_j k_{\mu\nu} = 0 \quad \text{at} \quad (x, y, z) = 0 \quad \text{for all } t.$$

(b) Consider the geodesic-deviation equation

$$\frac{D^2 \xi^\alpha}{D\tau^2} \equiv u^\gamma \nabla_\gamma (u^\beta \nabla_\beta \xi^\alpha) = -R^\alpha{}_{\beta\gamma\delta} u^\beta \xi^\gamma u^\delta$$

where $\vec{u}$ is the 4-velocity of a geodesic at the origin of the local Lorentz frame $(x, y, z) = 0$, and ξ is the vector reaching from that geodesic to any neighboring geodesic. Show that this geodesic-deviation equation reduces to

$$\frac{\partial^2 \xi^\alpha}{\partial t^2} + R^\alpha{}_{0\gamma 0} \xi^\gamma = 0$$

in the local Lorentz frame.

(c) Let $\xi^0 = \frac{\partial \xi^0}{\partial t} = 0$ initially, and note that $R_{\alpha\beta\gamma\delta}$ is antisymmetric in $\alpha\beta$ and also antisymmetric in $\gamma\delta$. Then show that

$$\xi^0 = 0$$

always (purely spatial separation always), and

$$\frac{\partial^2 \xi^i}{\partial t^2} + R_{i0j0} \xi^j = 0$$

in the local Lorentz frame.

(d) Suppose that the wave propagates in the z-direction and has $xx - yy$ polarization so that

$$R_{x0x0} = -R_{y0y0} = -\frac{1}{2} \ddot{h}_{xx}$$

and all others vanish (in the *trasverse traceless gauge*).

Show that the test particle displacements $\delta \xi^j$ produced by this wave in the local Lorentz coordinates exhibit a *quadrapole* pattern. Draw a picture of these displacements.

Problem 4.2

Bar detector for gravitational waves

Joseph Weber pioneered gravitational-wave detection using the fundamental normal mode of an aluminum cylinder as his antenna. Idealize this normal mode as two compact masses m connected by a spring of length $2\ell_0$. In flat spacetime and in the absence of any external forces, the outward displacement ξ of each mass from its equilibrium position obeys the harmonic oscillator equation of motion

$$m\ddot{\xi} + \frac{m}{\tau_0}\,\dot{\xi} + m\omega_0^2\xi = 0$$

where ω_0 is the frequency and τ_0 is the frictional damping time associated with the spring. Now orient the detector along the x-axis of a local inertial frame, and let gravitational waves propagate past it in the z-direction. The detector is idealized as out in interplanetary space, falling freely. As the waves pass, the center of the detector continues to fall freely, *i.e.,* it moves on a geodesic of the now curved spacetime. The masses m also want to move on geodesics; but they are restrained by the spring, which exerts a restoring force

$$F = -m\omega_0^2\xi$$

as measured in the local inertial frame, whenever the proper length separating the masses is changed from its equilibrium value of $2\ell_0$ to $2\ell_0 + 2\xi$.

(a) Explain in words and equations why the equation of motion for the proper length $2\ell_0 + 2\xi$ separating the masses is

$$m\,\frac{d^2\xi}{dt^2} + \frac{m}{\tau_0}\,\frac{d\xi}{dt} + m\omega_0^2\xi + mR_{x0x0}\ell_0 = 0$$

(b) The energy flux carried by the gravitational waves (energy per unit area per unit time), as measured in the rest frame of the detector, is given by

$$T^{0z} = \frac{1}{16\pi G}\,\langle(\partial_t h_{xx})^2 + (\partial_t h_{xy})^2\rangle$$

where $h_{xx} = -h_{yy}$ and $h_{xy} = h_{yx}$ are the components of the metric perturbation in the *transverse traceless gauge*, and $\langle\cdots\rangle$ denotes an average over several oscillations of the waves. (The energy is not localizable more accurately than a few wavelengths; one cannot say whether the energy rides in the crests of the waves or in their troughs.) Suppose that the waves are monochromatic with frequency ω:

$$h_{xy} = 0 \quad , \quad h_{xx} = -h_{yy} = A\,e^{i\omega(t-z)}$$

(Take real parts of all complex numbers.) Assume that the wavelength is very small,

$$\lambda = \frac{2\pi}{\omega} \ll \ell_0$$

What is the energy flux of the waves?

(c) Show that these waves drive the detector into steady-state oscillations with

$$\xi = \frac{\frac{1}{2}\,\omega^2 A\ell_0\, e^{-i\omega t}}{\omega^2 - \omega_0^2 + i\omega/\tau_0}$$

(d) Show that for $\tau_0 \gg 1/\omega_0$ and $|\omega - \omega_0| \ll \omega_0$ (near-resonance waves and weakly damped detector), the detector's steady-state vibration energy is

$$E_{\text{vibration}} = \frac{\frac{1}{16} m\ell_0^2\omega_0^4 A^2}{(\omega - \omega_0)^2 + 1/(2\tau_0)^2}$$

(e) Show that the rate of frictional damping of this vibration energy, and thus also the rate that the detector absorbs energy from the waves, is

$$\frac{dE}{dt} = \frac{1}{\tau_0}\, E_{\text{vibration}}$$

(f) By equating this to [energy flux of waves] $\times$ [cross-section], show that the detector's cross-section for waves of frequency ω is

$$\sigma(\omega) = \frac{2\pi G m\ell_0^2\omega_0^2/\tau_0}{(\omega - \omega_0)^2 + 1/(2\tau_0)^2}$$

(g) Show that, although $\sigma(\omega)$ depends on τ_0, its integral over frequency (a measure of the detector's ability to detect waves with a spread of frequencies $\Delta\omega \gg 1/\tau_0$) is independant of τ_0:

$$\int \sigma(\omega)\, d\omega = 4\pi^2 G m\ell_0^2\omega_0^2$$

(h) Weber's original detector had $m \simeq 1$ ton, $2\ell_0 \simeq 1$ meter, $\omega_0 \simeq 10^4$ rad/s, $\tau_0 \simeq 20$ s. What was the value of σ at the center of resonance, $\sigma(\omega_0)$, in cm^2? What was $\int \sigma\, d\omega$?

Problem 4.3

The Bertotti-Robinson Magnetic Universe.

B. Bertotti, Phys. Rev. **116** (1959) 1331;
I. Robinson, Bill. Acad. Polon. Sci. **7** (1959) 351

The solution to the Einstein field equations for a universe endowed with a uniform magnetic field is

$$ds^2 = Q^2[-dt^2 + \sin^2 t\, dz^2 + d\theta^2 + \sin^2\theta d\phi^2]$$

where Q is a constant and $t \in [0,\pi]$, $z \in (-\infty, +\infty)$, $\theta \in [0,\pi]$, $\phi \in [0, 2\pi]$.

(a) Which coordinate increases in a timelike direction and which coordinates in spacelike directions?

(b) Is this universe spherically symmetric?
[Hint: Consider the 2-surfaces of constant t and z.]

(c) Is this universe cylindrically symmetric?
[Hint: Consider the 2-surfaces of constant t and ϕ.]

(d) Is this universe asymptotically flat?

(e) How does the geometry of this universe change as t ranges from 0 to π?
[Hint: Show that the curves $\{(z, \theta, \phi) = \text{const.}, t = \tau/Q\}$ are timelike geodesics.]

(f) Give as complete a characterization as you can of the coordinates t, z, θ, ϕ.

Problem 4.4
Geodesic motion in a Robertson-Walker Universe (Cosmological redshift)
Consider a particle which moves along the χ direction in a Robertson-Walker Universe, so its four-momentum p^α has as its only nonzero components

$$p^0 = \frac{dt}{d\lambda} \quad , \quad p^\chi = \frac{d\chi}{d\lambda}$$

(a) Explain why symmetry guarantees that if the particle starts out moving in the χ-direction, it will always continue moving in that direction (p^θ and p^ϕ will remain zero).

(b) Show from the geodesic equation

$$p^\beta \nabla_\beta p_\alpha = p^\beta (\partial_\beta p_\alpha - \Gamma^\gamma{}_{\alpha\beta} p_\gamma) = 0$$

that p_χ is conserved along the geodesic:

$$p^\alpha \partial_\alpha p_\chi = 0$$

(c) Consider the observers who move along geodesics orthogonal to the homogeneous hypersurfaces (at rest relative to the mean local motion of galaxies). Explain why such an observer at (t, χ, θ, ϕ) has as the basis vectors of his local Lorentz frame,

$$\vec{e}_{\hat{0}} = \vec{e}_t \quad , \quad \vec{e}_{\hat{1}} = \frac{1}{a}\,\vec{e}_\chi \quad , \quad \vec{e}_{\hat{2}} = \frac{1}{a\Sigma}\,\vec{e}_\theta \quad , \quad \vec{e}_{\hat{3}} = \frac{1}{a\Sigma\sin\theta}\,\vec{e}_\phi$$

and find Σ.

(d) Show that these observers measure the freely moving particle to have energy E and 3-momentum p given by

$$E = \sqrt{p^2 + m^2} \quad , \quad p(t) = p(t_e)\,\frac{a(t_e)}{a(t)}$$

Thus the momentum is *redshifted* by the expansion of the universe, $p \propto 1/a$.
Note: Your derivation is valid for particles with $m = 0$ as well as $m \neq 0$.

(e) From this result, show that photons emitted by galaxies when the Universe had scale factor a_e, and received at Earth today (scale factor a_o) have a redshift

$$z = \frac{\Delta\lambda}{\lambda} = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{a_o}{a_e} - 1$$

(f) Show that if the photons were emitted recently, $a_o - a_e \ll a_o$, and if the distance between the emitting galaxy and earth is ℓ, then (Hubble formula)

$$z = \frac{\Delta\lambda}{\lambda} = H_o \ell \quad , \quad H = \frac{\dot{a}}{a}$$

(g) Using the result of part *(d)*, show that a particle moving at speed $v_e \ll 1$ as measured by nearby galaxies at time t_e will have slowed, at time t_o, to

$$v_o = \frac{v_e}{1+z} = \frac{v_e a_e}{a_o}$$

Problem 4.5

Consider de Sitter space in coordinates where the metric takes the form

$$ds^2 = -dt^2 + e^{Ht}(dx^2 + dy^2 + dz^2)$$

Solve the geodesic equation

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\gamma_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

for a photon to find the affine parameter λ as a function of t. Show that the geodesics reach $t = -\infty$ in a finite λ, demonstrating that these coordinates fail to cover the entire manifold.

Problem 4.6

Observations of cosmic microwave radiation

(a) As measured in the local Lorentz frame of a "comoving observer" (one who sees the Universe as homogeneous and isotropic in the large), the cosmic microwave radiation is thermal, with zero chemical potential:

$$\overline{N}_{occ} = \frac{1}{e^{E/kT_o} - 1}$$

where $E = h\nu$ is the photon energy and $T_o = 2.7$ K.

Isotropy corresponds to $\overline{N}_{occ}$ being independent of the direction on the sky from which the photon comes. Show that the specific intensity of this radiation is

$$I_\nu = \frac{2h\nu^3}{e^{h\nu/kT_o} - 1}$$

This is the Planck spectrum; *i.e.*, the radiation is *blackbody*.

(b) Show that $\overline{N}_{occ}$ can be rewritten in the frame-independent form

$$\overline{N}_{occ} = \frac{1}{\exp(-\vec{p} \cdot \vec{u}_o / kT_o) - 1}$$

where $\vec{p}$ is the four-momentum of the photon and $\vec{u}_o$ is the four-velocity of a "comoving observer" (*i.e.,* of the mean rest frame of the Universe).

(c) Suppose that the Earth moves in the z-direction with speed $v = \frac{dz}{dt}$ relative to the mean rest frame of the Universe. An observer on Earth points his microwave receiver in a direction that makes an angle θ with the z-direction (as measured in the Earth's frame). Show that the radiation received, I_ν, has a precisely Planck spectrum, but with Doppler shifted temperature

$$T = T_o \, \frac{(1 + v\cos\theta)}{\sqrt{1 - v^2}}$$

Note that the doppler shift of T is precisely the same as the doppler shift of frequency!