PHYSICS 642 - SPRING 2004
HOMEWORK 3

Due date: Mon., Mar. 29

Problem 3.1

The light-cone structure suggests that the in-falling particle's world line approaches $(t, \theta, \phi) =$ const. as $r \to 0$. (Recall: r is a *time* coordinate inside the horizon.)

(a) Show that $\{(t, \theta, \phi) = \text{const.}, r \text{ variable}\}$ is a time-like geodesic for $r < 2GM$. For this geodesic, what is the proper time τ in terms of the coordinate time r?

(b) A straightforward calculation gives for the components of the Riemann curvature tensor in the Schwarzschild coordinate frame $\{\vec{e}_t \ , \ \vec{e}_r \ , \ \vec{e}_\theta \ , \ \vec{e}_\phi\}$

$$R_{trtr} = -\frac{2GM}{r^3} \quad , \quad R_{t\theta t\theta} = \frac{R_{t\phi t\phi}}{\sin^2\theta} = \left(1 - \frac{2GM}{r}\right)\frac{GM}{r}$$

$$R_{\theta\phi\theta\phi} = 2GMr\sin^2\theta \quad , \quad R_{r\theta r\theta} = \frac{R_{r\phi r\phi}}{\sin^2\theta} = -\frac{GM}{r(1-2GM/r)}$$

Show that the basis vectors of the in-falling observer's local Lorentz frame are related to the Schwarzschild coordinate basis by

$$\vec{e}_{\hat{0}} = -\left(\frac{2GM}{r} - 1\right)^{1/2}\vec{e}_r \quad , \quad \vec{e}_{\hat{1}} = \frac{1}{r}\,\vec{e}_\theta \quad , \quad \vec{e}_{\hat{2}} = \frac{1}{r\sin\theta}\,\vec{e}_\phi \quad , \quad \vec{e}_{\hat{3}} = \left(\frac{2GM}{r} - 1\right)^{-1/2}\vec{e}_t$$

What are the components of the Riemann tensor in this local Lorentz frame?

(c) Your answer to part *(b)* should be

$$R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = \text{const.}\ \frac{GM}{r^3}$$

which diverges as $r \to 0$. Show that your answer corresponds to geodesic deviation *(tidal gravitational forces)* that *stretch* the in-falling observer in the $\vec{e}_{\hat{3}}$ direction and *squeeze* him in the $\vec{e}_{\hat{1}}$ and $\vec{e}_{\hat{2}}$ directions, and that the stretching and squeezing forces become infinitely strong as $r \to 0$.

(d) Idealize the body of an in-falling observer to consist of a head of mass $\mu \simeq 20$ kg and feet of mass $\mu \simeq 20$ kg separated by $\vec{\xi} = \ell\vec{e}_{\hat{3}}$, $\ell \simeq 2$ meters. Compute the stretching force between head and feet as a function of proper time τ, as the observer falls into the singularity. Assume that the hole has mass $m = 5 \times 10^9$ solar masses (*e.g.*, the hole in the center of the galaxy M87). How long before hitting the singularity (at what proper time τ) does the observer die, if he is a human being made of flesh and blood?

Problem 3.2
Consider a photon emitted from the surface of a static star of mass M, and radius R.

(a) An observer on the star's surface and momentarily at rest with respect to the emitter, but freely falling so he will soon be zooming into the star, measures the photon's frequency to be ω_{em}. This observer has local Lorentz basis vectors

$$\vec{e}_{\hat{r}} = \left(1 - \frac{2GM}{R}\right)^{1/2} \vec{e}_r \quad , \quad \vec{e}_{\hat{\theta}} = \frac{1}{R}\,\vec{e}_\theta \quad , \quad \vec{e}_{\hat{\phi}} = \frac{1}{R\sin\theta}\,\vec{e}_\phi \quad , \quad \vec{e}_{\hat{0}} = -\left(1 - \frac{2GM}{R}\right)^{-1/2} \vec{e}_t$$

Show that

$$g_{\hat{\alpha}\hat{\beta}} \equiv \vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}} = \eta_{\alpha\beta}$$

as is necessary for a local Lorentz frame.

(b) Explain why $\hbar\omega_{em} = -p_{\hat{0}} \equiv -\vec{p}\cdot\vec{e}_{\hat{0}}$ at the surface of the star.

(c) Show that

$$p_0 \equiv \vec{p}\cdot\vec{e}_t = -\sqrt{1 - 2GM/R}\,\hbar\omega_{em}$$

at the surface of the star.

(d) Show that as the photon flies out (radially or non-radially) toward $r = \infty$, p_0 is conserved.

(e) Show that when received by an observer at rest relative to the star and very far from it, the photon is measured to have frequency

$$\omega_{rec} = -p_0/\hbar$$

(f) Show that the photon is red-shifted by an amount

$$\frac{\omega_{em} - \omega_{rec}}{\omega_{em}} = 1 - \sqrt{1 - 2GM/R} \simeq \frac{GM}{R}$$

Give the numerical value for our Sun.

Problem 3.3
Solve Einstein's equation in vacuum ($T_{\alpha\beta} = 0$) assuming cylindrical symmetry. Use cylindrical coordinates (t, r, θ, z) and show that the solution may be written in the form

$$ds^2 = -dt^2 + \frac{1}{1-\alpha}\,dr^2 + r^2 d\theta^2 + dz^2$$

where $\alpha > 0$ is a constant.
Describe the two-dimensional surfaces $t, z =$ const.
[Hint: Show that the circle $t, z, r =$ const. has radius $R = r/\sqrt{1-\alpha}$ and circumference $2\pi R\sqrt{1-\alpha}$.]

Problem 3.4
Consider Einstein's equation in vacuum, but with a cosmological constant,

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = -\Lambda g_{\alpha\beta}$$

(a) Find the most general spherically symmetric solution using spherical coordinates (t, r, θ, ϕ) which reduces to the Schwarzschild solution when $\Lambda = 0$.

(b) Write down the equation of motion for a radial geodesic in terms of a potential, as we did in the Schwarzschild case. Sketch the potential for a massive particle.

Problem 3.5
Consider the orbit of a photon in the equatorial plane of a Kerr black hole.

(a) Show that

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{\Sigma^2}{\rho^4}(E - LW_+(r))(E - LW_-(r))$$

where E (L) is the conserved energy (angular momentum) of the photon, and find $W_\pm(r)$.

(b) Assuming $\Sigma^2 > 0$ everywhere, show that the orbit cannot have a turning point inside the outer horizon r_+. This means that ingoing light rays cannot escape once they cross r_+.