PHYSICS 642 - SPRING 2004
HOMEWORK 2

Due date: Mon., Feb. 23

Problem 2.1
In three-dimensional Euclidean space with Cartesian coordinates (x, y, z) consider the upper half sphere

$$x = a\sin\theta\cos\phi \quad , \quad y = a\sin\theta\sin\phi \quad , \quad z = a\cos\theta$$

where $\theta \in [0, \pi/2]$, $\phi \in [0, 2\pi]$. Call this two-dimensional surface $\mathcal{S}_2$. Consider the vector field

$$\vec{A} = (x^2 + y^2)\vec{e}_z$$

Compute using θ, ϕ as integration variables on $\mathcal{S}_2$:

$$\int_{\mathcal{S}_2} \vec{A} \cdot d^2\vec{\Sigma}$$

Problem 2.2
You are given a curve $x^\alpha(t)$. If we change the parameter to $t' = f(t)$, we obtain new tangent vectors, dx^α/dt'. Show that they are proportional to the old tangent vectors.
Problem 2.3

(a) Show that under a change of coordinates, the connection coefficients transform as follows

$$\Gamma^{\alpha'}_{\beta'\gamma'} = L^{\alpha'}_{\alpha} L^{\beta'}_{\beta} L^{\gamma}_{\gamma'} \Gamma^{\alpha}_{\beta\gamma} + L^{\alpha'}_{\alpha} L^{\alpha}_{\beta'\gamma'}$$

where $L^{\alpha}_{\beta'\gamma'} = \partial_{\beta'}\partial_{\gamma'}x^\alpha$.

(b) Define a coordinate transformation in a neighborhood around the origin by

$$x^{\alpha'} = x^\alpha + \tfrac{1}{2}\Gamma^{\alpha}_{\beta\gamma}(0)x^\beta x^\gamma$$

where $\Gamma^{\alpha}_{\beta\gamma}(0)$ are the connection coefficients at the origin. Show that at the origin,

$$\frac{\partial x^{\alpha'}}{\partial x^\alpha}(0) = \delta^{\alpha}_{\alpha'} \quad , \quad \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\gamma}(0) = \Gamma^{\alpha}_{\beta\gamma}(0)$$

Deduce from part *(a)* that the new connection coefficients $\Gamma^{\alpha'}_{\beta'\gamma'}$ vanish at the origin. It follows that the derivatives of $g_{\alpha\beta}$ also vanish there.

Problem 2.4

(a) Show that $\nabla_\gamma \delta^\alpha_\beta = 0$.

(b) Show that for a scalar field Φ, $\nabla_\alpha\nabla_\beta\Phi = \nabla_\beta\nabla_\alpha\Phi$.

(c) Show that $\nabla_\alpha g^{\beta\gamma} = 0$, using $\nabla_\alpha g_{\beta\gamma} = 0$, or otherwise.

(d) Suppose that V^α is a field of vectors of constant length (i.e., $V^\alpha V_\alpha = \text{const.}$). Show that $V^\alpha \nabla_\beta V_\alpha = 0$.

(e) Suppose that a manifold is filled with geodesics whose unit tangent vectors V^α ($V^\alpha V_\alpha = 1$) form a vector field. Show that $V^\alpha \nabla_\alpha V^\beta = 0$.

Problem 2.5

(a) The *curl* of a covariant vector field V_α is the anti-symmetric tensor field

$$A_{\alpha\beta} = \nabla_\alpha V_\beta - \nabla_\beta V_\alpha$$

Show that $A_{\alpha\beta} = \partial_\alpha V_\beta - \partial_\beta V_\alpha$.

(b) Suppose that $A_{\alpha\beta}$ is an anti-symmetric tensor field. Show that

$$B_{\alpha\beta\gamma} = \partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta}$$

is a tensor.
[*Hint:* Write $\partial_\gamma A_{\alpha\beta}$ in terms of $\nabla_\gamma A_{\alpha\beta}$, etc.]

Problem 2.6

(a) Prove the cyclic identity and the Bianchi identity for $R^\alpha{}_{\beta\gamma\delta}$.
[*Hint*: do the calculations in a local Lorentz frame.]

(b) Show that the covariant curvature tensor $R_{\alpha\beta\gamma\delta}$ may be written as

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = {} & \tfrac{1}{2}(\partial_\delta\partial_\alpha g_{\beta\gamma} - \partial_\delta\partial_\beta g_{\alpha\gamma} + \partial_\gamma\partial_\beta g_{\alpha\delta} - \partial_\gamma\partial_\alpha g_{\beta\delta}) \\ & - g_{\epsilon\zeta}(\Gamma^\epsilon_{\alpha\gamma}\Gamma^\zeta_{\beta\delta} - \Gamma^\epsilon_{\alpha\delta}\Gamma^\zeta_{\beta\gamma}) \end{aligned}$$

Hence deduce the symmetry properties of $R_{\alpha\beta\gamma\delta}$.

(c) Use the cyclic identity to show that the Ricci tensor $R_{\alpha\beta}$ is symmetric.

Problem 2.7
Consider two neighboring geodesics (great circles) on a sphere of radius a, one the equator and the other a geodesic slightly displaced from the equator and parallel to it at $\phi = 0$. Let $\vec{\xi}$ be the displacement vector between the two geodesics, and note that at $\phi = 0$, $\vec{\xi} = \xi^\theta \vec{e}_\theta$. Let ℓ be the proper distance along the equatorial geodesic, so $u^\alpha = dx^\alpha/d\ell$ is the unit tangent to the equatorial geodesic.

(a) Show that $\ell = a\phi$ along the equatorial geodesic.

(b) Show that the equation of geodesic deviation,

$$\left(\frac{D^2\vec{\xi}}{D\ell^2}\right)^\alpha + R^\alpha{}_{\beta\gamma\delta}u^\beta\xi^\gamma u^\delta = 0$$

reduces to

$$\frac{d^2\xi^\theta}{d\phi^2} = -\xi^\theta \quad , \quad \frac{d^2\xi^\phi}{d\phi^2} = 0$$

(c) Solve this to obtain

$$\xi^\theta = (\xi^\theta)_{\phi=0}\ \cos\phi$$

(d) From this, infer that the geodesic near the equator is the curve

$$\theta = b\,\cos\phi\,[1 + o(b/a)]$$

where b is a constant.

Problem 2.7

For the Killing vector K^μ, show that

(a)

$$\nabla_\mu\nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu}K^\nu$$

(b)

$$K^\lambda\nabla_\lambda R = 0$$