PHYSICS 642 - SPRING 2004
HOMEWORK 1

Due date: Mon., Feb. 2

Problem 1.1

(a) Suppose that $S_{\alpha\beta}$ is a symmetric tensor and $T^{\alpha\beta}$ is an anti-symmetric tensor. Show that $S_{\alpha\beta}T^{\alpha\beta} = 0$.

(b) Show that any tensor $T^{\alpha\beta}$ may be expressed as the sum of a symmetric and an anti-symmetric tensor.

(c) You are given a symmetric tensor $T^{\alpha\beta}$ and a vector V^α. Suppose that they satisfy

$$T^{\alpha\beta}V^\gamma + T^{\beta\gamma}V^\alpha + T^{\gamma\alpha}V^\beta = 0$$

If V^α is not zero, show that $T^{\alpha\beta} = 0$.
[*Hint:* Take V^α to be the first vector of a basis.]

(d) Suppose that the tensor $T_{\alpha\beta\gamma\delta}$ satisfies

$$T_{\alpha\beta\gamma\delta}V^\alpha W^\beta V^\gamma W^\delta = 0$$

for arbitrary contravariant vectors V^α and W^α. Show that

$$T_{\alpha\beta\gamma\delta} + T_{\gamma\delta\alpha\beta} + T_{\alpha\delta\gamma\beta} + T_{\gamma\beta\alpha\delta} = 0$$

Problem 1.2

Let $\{\vec{e}_0, \vec{e}_1\}$ be an orthonormal basis in two-dimensional Minkowski space, so

$$\vec{e}_0 \cdot \vec{e}_0 = -1 \quad , \quad \vec{e}_1 \cdot \vec{e}_1 = +1 \quad , \quad \vec{e}_0 \cdot \vec{e}_1 = 0$$

Let $\vec{e}_+ = \vec{e}_0 + \vec{e}_1$, $\vec{e}_- = \vec{e}_0 - \vec{e}_1$. Evaluate

(a) $\hat{\theta}^+$ and $\hat{\theta}^-$ in terms of $\hat{\theta}^0$ and $\hat{\theta}^1$.

(b) Components of $\mathbf{g}$ in $\pm$ basis: $g_{\alpha\beta}$, ${g_\alpha}^\beta$, $g^{\alpha\beta}$.

(c) for any vector $\vec{A}$: A_+ and A_- in terms of A^+ and A^-.

(d) The components ω_+, ω_- of the 1-form ω which corresponds to the vector $\vec{A} = \vec{e}_+$.

Problem 1.3
Let $\vec{e}_{\hat{1}}, \vec{e}_{\hat{2}}, \vec{e}_{\hat{3}}$ be a right-handed orthonormal basis in the three-dimensional Euclidean space, so that

$$g_{\hat{i}\hat{j}} = \delta_{\hat{i}\hat{j}}$$

(a) Show that in a non-orthonormal basis,

$$\epsilon_{ijk} = \pm\sqrt{g}\,\epsilon_{\hat{i}\hat{j}\hat{k}}$$

where g is the determinant of g_{ij}. It is $+$ for a right-handed basis and $-$ for a left-handed basis.

Hint: Let $L_i^{\hat{j}}$ be the transformation matrix from the orthonormal basis to the non-orthonormal basis. Show that

$$\epsilon_{ijk} = L\epsilon_{\hat{i}\hat{j}\hat{k}}$$

where L is the determinant of $L_i^{\hat{j}}$ and then show that

$$g = L^2\hat{g}$$

where $\hat{g} = 1$ is the determinant of $g_{\hat{i}\hat{j}}$.

(b)

$$\epsilon_{ijk}\epsilon^{ilm} = \delta_{jk}^{lm} = \delta_j^l\delta_k^m - \delta_j^m\delta_k^l$$

(c)

$$\epsilon_{ijk}\epsilon^{ijl} = \delta_k^l$$

(d)

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Problem 1.4
In four-dimensional spacetime, in a Lorentz basis (*i.e.*, a basis where $g_{\alpha\beta} = \eta_{\alpha\beta}$), denote the components of the electromagnetic field tensor by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

where $\vec{E}$, $\vec{B}$ are the electric and magnetic fields, respectively (three-dimensional vectors).
$F^{\mu\nu}$ is an antisymmetric tensor satisfying Maxwell's equations

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

where $\vec{J}$ is the current-density four-vector.
The dual of $F^{\mu\nu}$ is defined by

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

and the electromagnetic stress-energy tensor is given by

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

(a) Show that Maxwell's equations imply conservation of current,

$$\partial_\mu J^\mu = 0$$

(b) Express $*F^{\mu\nu}$ and $T^{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$.

(c) Show that

$$F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2) \quad , \quad *F^{\mu\nu}F_{\mu\nu} = 2\vec{B}\cdot\vec{E}$$

These are the only independent scalars constructible from **F**.

(d) Show that

$$\partial_\mu T^{\mu\nu} = \frac{1}{4\pi}\left(\partial_\mu F^{\nu\rho}F^{\mu}{}_{\rho}F^{\nu\rho}\partial_\mu F^{\mu}{}_{\rho} - \frac{1}{2}\partial_\mu F_{\rho\sigma}F^{\rho\sigma}\right)$$

Manipulate this into the form

$$\partial_\mu T^{\mu\nu} = \frac{1}{4\pi}\left(F_{\nu\rho}\partial_\mu F^{\mu\rho} - \frac{1}{2}F^{\rho\sigma}(\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho})\right)$$

Use Maxwell's equations to conclude that

$$\partial_\mu T^{\mu\nu} = F^{\mu\nu}J_\mu$$

Show that in an inertial frame this says

$$\frac{\partial}{\partial t}T^{00} + \frac{\partial}{\partial x^j}T^{0j} = -\vec{E}\cdot\vec{J}$$

which is the rate of joule heating, and

$$\left(\frac{\partial}{\partial t}T^{k0} + \frac{\partial}{\partial x^j}T^{kj}\right)\vec{e}_k = -(J^0\vec{E} + \vec{J}\times\vec{B})$$

which is the Lorentz force per unit volume.

Problem 1.5

(a) Consider the wave equation,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{1}$$

Show that under the Galilean transformation

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t \ \ , \ \ t' = t \tag{2}$$

the wave equation becomes

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\mathbf{v}\cdot\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}(\mathbf{v}\cdot\nabla)^2\right)\phi = 0 \tag{3}$$

What happens under a Lorentz transformation?

(b) Consider the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{4}$$

Show that under a Galilean transformation, its form does not change, provided we change ψ by a phase:

$$\psi \to e^{i\theta}\psi \ \ , \ \ \theta = \frac{1}{\hbar}(Et - \mathbf{p}\cdot\mathbf{x}) \tag{5}$$

where $E = \frac{1}{2}mv^2$, $\mathbf{p} = m\mathbf{v}$.

What happens under a Lorentz transformation?

Problem 1.6

(Doppler shift) Light of frequency ω is emitted from a source moving with velocity $\mathbf{v}$. Show that a stationary observer will measure a frequency

$$\omega' = \omega\sqrt{\frac{1 - v/c}{1 + v/c}} \tag{6}$$